Indian J. pure appl. Math., 32(1) : 125-141, January 2001

EXISTENCE OF LIBRATION POINTS IN THE RESTRICTED THREE BODY PROBLEM WHEN BOTH THE PRIMARIES ARE TRIAXIAL RIGID BODIES

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(*Received* 22 *December* 1999; *accepted* 26 *June* 2000)

This paper deals with the stationary solutions of the planar restricted three body problem when both the primaries are triaxial rigid bodies. The primaries are having axes of symmetry but their equatorial planes are not coinciding with the plane of motion. The principal axes of the primaries are oriented to the synodic axes with the help of Euler's angles. The rigid bodies move around their centre of mass without rotation. It is seen that there are five libration points, two triangular and three collinear.

Key Words : Restricted Three Body Problem; Libration Points; Rigid Bodies

1. INTRODUCTION

It is well known that the classical planar restricted three body problem possesses five libration points, two triangular and three collinear.

In recent times many perturbing forces i.e., oblateness and radiation forces of the primaries, coriolis and centrifungal forces, variation of the masses of the primaries and of the infinitesimal mass etc., have been included in the study of the restricted three body problem. In the case of restricted three body problem, where both the primaries are oblate spheroids whose equatorial plane coincides with the plane of motion, the location of libration points and their stability in the Liapunov sense has been studied by Vidyakin[1]. For the case, where the bigger primary is an oblate spheroid whose equatorial plane coincides with the plane of motion, Subba Rao and Sharma[2] have studied the stability of the libration points. A similar problem has been studied by El-Shaboury[3]. Khanna and Bhatnagar have studied the problem when the smaller primary is a triaxial rigid body (1998[4], 1999[5]).

In all the above papers authors have taken one of the axis of the primaries as the axis of symmetry and its equatorial plane coinciding with the plane of motion whereas in this paper though the primaries are having axes of symmetry but their equatorial planes are not coinciding with the plane of motion. Further we assume that both the primaries are moving without rotation about their centre of mass in circular orbits. An attempt is made to study the existence of libration points.

2. EQUATIONS OF MOTION

We shall adopt the notation and terminology of Szebehely[6]. As a consequence, the distance between the primaries does not change and is taken equal to one; the sum of the masses of the primaries is also taken as one. The unit of time is so chosen so as to make the gravitational constant unity. Besides this the principal axes of the primaries are oriented to the synodic axes by Euler's angles v_i, ϕ_i, ψ_i ($i = 1, 2$). Since the axes are supposed to rotate with the same angular velocity as that of the rigid bodies and the bodies are moving around their centre of mass without rotation, the Euler's angles remain constant throughout the motion. Using dimensionless variables, the equations of motion of the infinitesimal mass m_3 in a synodic coordinate system (x, y) are

$$\ddot{x} - 2n\,\dot{y} = \frac{\partial \Omega}{\partial x}$$

and

$$\ddot{y} + 2n\,\dot{x} = \frac{\partial \Omega}{\partial y}, \qquad \ldots (1)$$

where

$$\Omega = \frac{n^2}{2}[\{1 - \mu\}\, r_1^2 + \mu\, r_2^2] + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}$$

$$+ \frac{1-\mu}{2m_1 r_1^3}[I_1 + I_2 + I_3 - 3I]$$

$$+ \frac{\mu}{2m_2 r_2^3}[I'_1 + I'_2 + I'_3 - 3I'], \text{ (McCusky[7])} \qquad \ldots (2)$$

$$r_1^2 = (x - \mu)^2 + y^2$$

and

$$r_2^2 = (x + 1 - \mu)^2 + y^2. \qquad \ldots (3)$$

Here μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 < \mu \le \frac{1}{2}$, i.e., $\mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2}$ with $m_1 \ge m_2$ being the masses of the primaries.

I_1, I_2, I_3 are the principal moments of inertia of the triaxial rigid body of mass m_1 at its centre of mass, with a, b, c as its axes. I is the moment of inertia about a line joining the centre of the rigid body of mass m_1 and the infinitesimal body of mass m_3 and is given by

$$I = I_1\, l_1'^2 + I_2\, m_1'^2 + I_3\, n_1'^2,$$

where l'_1, m'_1 and n'_1 are the direction cosines of the line with respect to its principal axes.

I'_1, I'_2, I'_3 are the principal moments of inertia of the triaxial rigid body of mass m_2 at its centre of mass, with a', b', c' as its axes. I' is the moment of inertia about a line joining the centre of the rigid body of mass m_2 and the infinitesimal body of mass m_3 and is given by

$$I' = I'_1\, l_2'^2 + I'_2\, m_2'^2 + I'_3\, n_2'^2,$$

where l'_2, m'_2, n'_2 are the direction cosines of the line with respect to its principal axes.

We denote the unit vectors along the principal axes at p_1 (or p_2) by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and the unit vectors parallel to the synodic axes by $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$ with the help of Euler's angles v_i, ϕ_i, ψ_i $(i = 1, 2)$. They are connected by[8]

$$\boldsymbol{I} = a_{1i}\,\boldsymbol{i} + b_{1i}\boldsymbol{j} + c_{1i}\,\boldsymbol{k}$$

$$\boldsymbol{J} = a_{2i}\,\boldsymbol{i} + b_{2i}\boldsymbol{j} + c_{2i}\,\boldsymbol{k}$$

and

$$\boldsymbol{K} = a_{3i}\,\boldsymbol{i} + b_{3i}\boldsymbol{j} + c_{3i}\,\boldsymbol{k},$$

$(i = 1, 2)$,

where

$$a_{1i} = -\operatorname{Sin}\phi_i \operatorname{Sin}\psi_i + \operatorname{Cos} v_i \cos\phi_i \operatorname{Cos}\psi_i,$$

$$a_{2i} = \operatorname{Cos}\phi_i \operatorname{Sin}\psi_i + \operatorname{Cos} v_i \operatorname{Sin}\phi_i \operatorname{Cos}\psi_i,$$

$$a_{3i} = -\operatorname{Sin} v_i \operatorname{Cos}\psi_i,$$

$$b_{1i} = -\operatorname{Sin}\phi_i \operatorname{Cos}\psi_i - \operatorname{Cos} v_i \operatorname{Cos}\phi_i \operatorname{Sin}\psi_i,$$

$$b_{2i} = \operatorname{Cos}\phi_i \operatorname{Cos}\psi_i - \operatorname{Cos} v_i \operatorname{Sin}\phi_i \operatorname{Sin}\psi_i,$$

$$b_{3i} = \operatorname{Sin}\phi_i \operatorname{Sin}\psi_i,$$

$$c_{1i} = \operatorname{Sin} v_i \operatorname{Cos}\phi_i,$$

$$c_{2i} = \operatorname{Sin} v_i \operatorname{Sin}\phi_i$$

and

$$c_{3i} = \operatorname{Cos} v_i,$$

$(i = 1, 2)$.

The axes O (xyz) have been defined by Szebehely[6].

Now, Ω in eq. (2) can be written as

$$\Omega = \frac{1}{2} n^2 \left[(1-\mu)\, r_1^2 + \mu\, r_2^2 \right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

$$+ \frac{1}{2} \frac{1-\mu}{r_1^3} \Big[2\,(A_1 + A_2 + A_3) - 3 \frac{1}{r_1^2} \; \{(A_2 + A_3)\,(a_{11}\,(x-\mu)$$

$$+ a_{21}\, y)^2 + (A_3 + A_1)\,(b_{11}\,(x-\mu) + b_{21}\, y)^2$$

$$+ (A_1 + A_2)\,(c_{11}\,(x-\mu) + c_{21}\, y)^2\}\Big]$$

$$+ \frac{1}{2}\frac{\mu}{r_2^3}\Bigg[2(A'_1 + A'_2 + A'_3) - 3\frac{1}{r_2^2} \{(A'_2 + A'_3)(a_{12}(x+1-\mu)$$

$$+ a_{22}y)^2 + (A'_3 + A_1')(b_{12}(x+1-\mu) + b_{22}y)^2$$

$$+ (A'_1 + A'_2)(c_{12}(x+1-\mu) + c_{22}y)^2\}\Bigg], \quad \text{... (4)}$$

where

$$A_1 = \frac{a^2}{5R^2}, A_2 = \frac{b^2}{5R^2}, A_3 = \frac{c^2}{5R^2},$$

$$A'_1 = \frac{a'^2}{5R^2}, \; A'_2 = \frac{b'^2}{5R^2}, A'_3 = \frac{c'^2}{5R^2}. \quad \text{... (5)}$$

and R is the distance between the primaries.

The mean motion, n, is given by

$$n^2 = 1 + \frac{3}{2}[2(A_1 + A_2 + A_3 - 3a_{11}^2(A_2 + A_3) - 3b_{11}^2(A_3 + A_1)$$

$$- 3c_{11}^2(A_1 + A_2)] + \frac{3}{2}[2(A'_1 + A'_2 + A'_3) - 3a_{12}^2(A'_2 + A'_3)$$

$$- 3b_{12}^2(A'_3 + A'_1) - 3c_{12}^2(A'_1 + A'_2)]. \quad \text{... (6)}$$

3. Location of Libration Points

Equation (1) permit an integral analogous to Jacobi integral

$$\dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0.$$

The libration points are the singularities of the manifold

$$F(x, y, \dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 - 2\,\Omega + C = 0.$$

Therefore, these points are the solutions of the equations

$$\Omega_x = 0, \Omega_y = 0.$$

Therefore, we have

$$\Omega_x = n^2 x - \frac{1-\mu}{r_1^3}(x-\mu) - \frac{\mu}{r_2^3}(x+1-\mu)$$

$$+ \frac{3(1-\mu)}{r_1^5}[- \{a_{11}(A_2+A_3)(a_{11}(x-\mu)+a_{21}y)$$

$$+ b_{11}(A_3+A_1)(b_{11}(x-\mu)+b_{21}y)$$

$$+ c_{11}(A_1+A_2)(c_{11}(x-\mu)+c_{21}y)\}$$

$$+ \frac{1}{r_1^2}(x-\mu)\ \{(A_2+A_3)(a_{11}(x-\mu)+a_{21}y)^2$$

$$+ (A_3+A_1)(b_{11}(x-\mu)+b_{21}y)^2$$

$$+ (A_1+A_2)(c_{11}(x-\mu)+c_{21}y)^2\}]$$

$$- \frac{3(1-\mu)}{2r_1^5}(x-\mu)[2(A_1+A_2+A_3)$$

$$- \frac{3}{r_1^2}\ \{(A_2+A_3)(a_{11}(x-\mu)+a_{21}y)^2$$

$$+ (A_3+A_1)(b_{11}(x-\mu)+b_{21}y)^2$$

$$+ (A_1+A_2)(c_{11}(x-\mu)+c_{21}y)^2\}]$$

$$+ \frac{3\mu}{r_2^5}\ [\ -\ \{a_{12}(A'_2+A'_3)(a_{12}(x+1-\mu)+a_{22}y)$$

$$+ b_{12}(A'_3+A'_1)(b_{12}(x+1-\mu)+b_{22}y)$$

$$+ c_{12}(A'_1+A'_2)(c_{12}(x+1-\mu)+c_{22}y)\}$$

$$+ \frac{1}{r_2^2}(x+1-\mu)\ \{(A'_2+A'_3)(a_{12}(x+1-\mu)+a_{22}y)^2$$

$$+ (A'_3+A'_1)(b_{12}(x+1-\mu)+b_{22}y)^2$$

$$+ (A'_1+A'_2)(c_{12}(x+1-\mu)+c_{22}y)^2\}]$$

$$- \frac{3\mu}{2r_2^5}(x+1-\mu)[2(A'_1+A'_2+A'_3)$$

$$- \frac{3}{r_2^2}\{(A'_2+A'_3)(a_{12}(x+1-\mu)+a_{22}y)^2$$

$$+ (A'_3 + A'_1)(b_{12}(x+1-\mu) + b_{22}y)^2$$

$$+ (A'_1 + A'_2)(c_{12}(x+1-\mu) + c_{22}y)^2\}] = 0$$

and

$$\Omega_y = n^2 y - \frac{1-\mu}{r_1^3} y - \frac{\mu}{r_2^3} y$$

$$+ \frac{3(1-\mu)}{r_1^5} [- \{a_{21}(A_2 + A_3)(a_{11}(x-\mu) + a_{21}y)$$

$$+ b_{21}(A_3 + A_1)(b_{11}(x-\mu) + b_{21}y)$$

$$+ C_{21}(A_1 + A_2)(c_{11}(x-\mu) + c_{21}y)\}$$

$$+ \frac{1}{r_1^2} y \; \{(A_2 + A_3)(a_{11}(x-\mu) + a_{21}y)^2$$

$$+ (A_3 + A_1)(b_{11}(x-\mu) + b_{21}y)^2$$

$$+ (A_1 + A_2)(c_{11}(x-\mu) + c_{21}y)^2\}]$$

$$- \frac{3(1-\mu)}{2r_1^5} y \, [2(A_1 + A_2 + A_3)$$

$$- \frac{3}{r_1^2} \{(A_2 + A_3)(a_{11}(x-\mu) + a_{21}y)^2$$

$$+ (A_3 + A_1)(b_{11}(x-\mu) + b_{21}y)^2$$

$$+ (A_1 + A_2)(c_{11}(x-\mu) + c_{21}y)^2\}]$$

$$+ \frac{3\mu}{r_2^5} [- \{a_{22}(A'_2 + A'_3)(a_{12}(x+1-\mu) + a_{22}y)$$

$$+ b_{22}(A'_3 + A'_1)(b_{12}(x+1-\mu) + b_{22}y)$$

$$+ c_{22}(A'_1 + A'_2)(c_{12}(x+1-\mu) + c_{22}y)\}$$

$$+ \frac{1}{r_2^2} y \; \{(A'_2 + A'_3)(a_{12}(x+1-\mu) + a_{22}y)^2$$

$$+ (A'_3 + A'_1)(b_{12}(x+1-\mu) + b_{22}y)^2$$

$$+ (A'_1 + A'_2)(c_{12}(x+1-\mu) + c_{22}y)^2\}]$$

$$- \frac{3\mu}{2r_2^5} y \, [2\,(A'_1 + A'_2 + A'_3)$$

$$- \frac{3}{r_2^2} \{(A'_2 + A'_3)\,(a_{12}\,(x + 1 - \mu) + a_{22}\,y)^2$$

$$+ \, (A'_3 + A'_1)\,(b_{12}\,(x + 1 - \mu) + b_{22}\,y)^2$$

$$+ \, (A'_1 + A'_2)\,(c_{12}\,(x + 1 - \mu) + c_{22}\,y)^2\}] \; = \; 0. \; \text{(using eq. 4.)} \qquad \ldots (7)$$

The solution of $\Omega_x = 0 = \Omega_y$ (From Eq. (7)) gives us the required libration points.

We shall now discuss some special cases in the next few sections.

SECTION 1

The primaries have axes of symmetry and their equatorial planes are coincident with the plane of motion, then $a_{1i}, b_{2i}, c_{3i}\;(i = 1, 2)$ are equal to one and $a_{2i}, a_{3i}, b_{1i}, b_{3i}, c_{1i}, c_{2i}\;(i = 1, 2)$ are equal to zero and therefore eqs. (7) become

$$\Omega_x = n^2 x - \frac{1-\mu}{r_1^3}(x - \mu) - \frac{\mu}{r_2^3}(x + 1 - \mu)$$

$$+ \; \frac{3\,(1-\mu)}{r_1^5}(x - \mu)\left[-A_2 + \frac{1}{r_1^2}\left\{ A_2\,(x - \mu)^2 + A_1\,y^2 \right\} \right]$$

$$- \; \frac{3(1-\mu)}{2r_1^5}(x - \mu)\left[2A_1 + 2A_2 - A_3 - \frac{3}{r_1^2}\left\{ A_2\,(x - \mu)^2 + A_1\,y^2 \right\} \right]$$

$$+ \; \frac{3\mu}{r_2^5}(x + 1 - \mu)\left[-A'_2 + \frac{1}{r_2^2}\left\{ A'_2\,(x + 1 - \mu)^2 + A'_1\,y^2 \right\} \right]$$

$$- \; \frac{3\mu}{2r_2^5}(x + 1 - \mu)\left[2A'_1 + 2A'_2 - A'_3 - \frac{3}{r_2^2}\left\{ A'_2\,(x + 1 - \mu)^2 + A'_1\,y^2 \right\} \right] = 0$$

and

$$\Omega_y = n^2 y - \frac{1-\mu}{r_1^3} y - \frac{\mu}{r_2^3} y$$

$$+ \; \frac{3(1-\mu)}{r_1^5} y \left[-A_1 + \frac{1}{r_1^2}\left\{ A_2\,(x - \mu)^2 + A_1\,y^2 \right\} \right]$$

$$- \; \frac{3(1-\mu)}{2r_1^5} y \, [2A_1 + 2A_2 - A_3$$

$$- \frac{3}{r_1^2}\left\{ A_2 (x-\mu)^2 + A_1 y^2 \right\} \Bigg]$$

$$+ \frac{3\mu}{r_2^5} y \left[-A'_1 + \frac{1}{r_2^2}\left\{ A'_2 (x+1-\mu)^2 + A'_1 y^2 \right\} \right]$$

$$- \frac{3\mu}{2r_2^5} y \left[2A'_1 + 2A'_2 - A'_3 - \frac{3}{r_2^2}\left\{ A'_2 (x+1-\mu)^2 + A'_1 y^2 \right\} \right] = 0 \qquad \ldots (8)$$

Case (i) — Both the primaries are Spheres[6]

In this case, we have $A_i = A'_i = 0$ (i = 1, 2, 3) and then eqs. (8) are in confirmity with those of classical problem thereby giving the same libration points. The co-ordinates of triangular libration points are

$$x = \mu - \frac{1}{2}, \; y = \pm \frac{3^{1/2}}{2}.$$

Case (ii) — Bigger Primary an Oblate Body and the Smaller Primary a Sphere (Bhatnagar and Hallan[9]

In this case we have $A_1 = A_2, A'_i = 0$ (i = 1, 2, 3), putting $\sigma = A_1 - A_3 = \frac{a^2 - c^2}{5R^2}$ = oblateness factor and then eqs. (8) are in confirmity with those of Bhatnagar and Hallan thereby giving the same libration points. The co-ordinates of triangular libration points are

$$x = \mu - \frac{1}{2} - \frac{\sigma}{2}, \; y = \pm \frac{3^{1/2}}{2}\left\{ 1 - \frac{\sigma}{3} \right\}.$$

Case (iii) — Both the Primaries are Oblate Bodies (Bhatnagar and Hallan[9] —

In this case we have $A_1 = A_2, A'_1 = A'_2$, putting $\sigma_1 = A_1 - A_3$, $\sigma_2 = A'_1 - A'_3$ and then eqs. (8) are in conformity with those of Bhatnagar and Hallan thereby giving the same libration points. The co-ordinates of triangular libration points are

$$x = \mu - \frac{1}{2} + \frac{\sigma_2 - \sigma_1}{2}, \; y = \pm \frac{3^{1/2}}{2}\left\{ 1 - \frac{\sigma_1 + \sigma_2}{3} \right\}.$$

Case (iv) — Bigger Primary a Sphere and the Smaller One a Triaxial Rigid Body (Khanna and Bhatnagar[4]

In this case, we have $A_i = 0$ (i = 1, 2, 3), putting $\sigma'_1 = A'_1 - A'_3$, $\sigma'_2 = A'_2 - A'_3$ and then eqs. (8) are in confirmity with those of Khanna and Bhatnagar thereby giving the same libration points. The coordinates of triangular libration points are

$$x = \mu - \frac{1}{2} + \left\{ -\frac{3}{8} - \frac{\mu}{2(1-\mu)} \right\} \sigma'_1 + \left\{ \frac{7}{8} + \frac{\mu}{2(1-\mu)} \right\} \sigma'_2,$$

and $$y=\pm\frac{3^{1/2}}{2}\left[1+\frac{2}{3}\left\{\left(-\frac{19}{8}+\frac{\mu}{2(1-\mu)}\right)\sigma'_1+\left(\frac{15}{8}-\frac{\mu}{2(1-\mu)}\right)\sigma'_2\right\}\right].$$

Case (v) — Bigger Primary An Oblate Spheroid and the Smaller One A Triaxial Rigid Body (Khanna and Bhatnagar[5])

In this case we have $A_1=A_2$, putting $\sigma_1=A_1-A_3, \sigma'_1=A'_1-A'_3, \sigma'_2=A'_2-A'_3$ and then eqs. (8) are in confirmity with those of Khanna and Bhatnagar thereby giving the same libration points. The co-ordinates of triangular libration points are

$$x=\mu-\frac{1}{2}-\frac{\sigma_1}{2}+\left\{-\frac{3}{8}-\frac{\mu}{2(1-\mu)}\right\}\sigma'_1+\left\{\frac{7}{8}+\frac{\mu}{2(1-\mu)}\right\}\sigma'_2$$

and $$y=\pm\frac{3^{1/2}}{2}\left[1+\frac{2}{3}\left\{-\frac{\sigma_1}{2}+\left(-\frac{19}{8}+\frac{\mu}{2(1-\mu)}\right)\sigma'_1+\left(\frac{15}{8}-\frac{\mu}{2(1-\mu)}\right)\sigma'_2\right\}\right].$$

Case (vi) — Both the Primaries are Triaxial Rigid Bodies (Sharma, Taqvi and Bhatnagar)[10]

In this case we put $\sigma_1=A_1-A_3, \sigma_2=A_2-A_3, \sigma'_1=A'_1-A'_3, \sigma'_2=A'_2-A'_3$ and then eqs. (8) are in conformity with those of Sharma, Taqvi and Bhatnagar thereby giving the same libration points. The coordinates of triangular libration points are

$$x=\mu-\frac{1}{2}+\left(\frac{1}{2\mu}-\frac{1}{8}\right)\sigma_1+\left(-\frac{1}{2\mu}-\frac{3}{8}\right)\sigma_2$$

$$+\left\{-\frac{3}{8}-\frac{\mu}{2(1-\mu)}\right\}\sigma'_1+\left\{\frac{7}{8}+\frac{\mu}{2(1-\mu)}\right\}\sigma'_2$$

and $$y=\pm\frac{3^{1/2}}{2}\left[1+\frac{2}{3}\left\{\left(-\frac{23}{8}+\frac{1}{2\mu}\right)\sigma_1+\left(\frac{19}{8}-\frac{1}{2\mu}\right)\dot{\sigma}_2\right.\right.$$

$$\left.\left.+\left(-\frac{19}{8}+\frac{\mu}{2(1-\mu)}\right)\sigma'_1+\left(\frac{15}{8}-\frac{\mu}{2(1-\mu)}\right)\sigma'_2\right]\right].$$

SECTION 2

The primaries have axes of symmetry but their equatorial planes are not coincident with the plane of motion. Further we suppose that v_i, ϕ_i, ψ_i $(i = 1, 2)$ are small quantities, then eqs. (7) become

$$\Omega_x=n^2x-\frac{1-\mu}{r_1^3}(x-\mu)-\frac{\mu}{r_2^3}(x+1-\mu)$$

$$+\frac{3(1-\mu)}{r_1^5}[-A_2(x-\mu)+(A_1-A_2)(\phi_1+\psi_1)y$$

$$+\frac{1}{r_1^2}(x-\mu)\ \{A_2(x-\mu)^2+A_1y^2$$

$$+ \; 2\,(A_2 - A_1)\,(x-\mu)\,(\phi_1 + \psi_1)\,y\}]$$

$$- \; \frac{3(1-\mu)}{2r_1^5}\,(x-\mu)\,[2A_1 + 2A_2 - A_3$$

$$- \; \frac{3}{r_1^2}\,\{A_2\,(x-\mu)^2 + A_1\,y^2$$

$$+ \; 2\,(A_2 - A_1)\,(x-\mu)\,(\phi_1 + \psi_1)\,y\}]$$

$$+ \; \frac{3\mu}{r_2^5}\,[-A'_2\,(x+1-\mu) + (A'_1 - A'_2)\,(\phi_2 + \psi_2)\,y$$

$$+ \; \frac{1}{r_2^2}\,(x+1-\mu)\;\;\{A'_2\,(x+1-\mu)^2 + A'_1\,y^2$$

$$+ \; 2\,(A'_2 - A'_1)\,(x+1-\mu)\,(\phi_2 + \psi_2)\,y\}]$$

$$- \; \frac{3\mu}{2r_2^5}\,(x+1-\mu)\,[2A'_1 + 2A'_2 - A'_3$$

$$- \; \frac{3}{r_2^2}\;\{A'_2\,(x+1-\mu)^2 + A'_1\,y^2$$

$$+ \; 2\,(A'_2 - A'_1)\,(x+1-\mu)\,(\phi_2 + \psi_2)\,y\}] \; = \; 0$$

and

$$\Omega_y \; = \; n^2 y - \frac{1-\mu}{r_1^3}\,y - \frac{\mu}{r_2^3}\,y$$

$$+ \; \frac{3(1-\mu)}{r_1^5}\,[(A_1 - A_2)\,(\phi_1 + \psi_1)\,(x-\mu) - A_1\,y$$

$$+ \; \frac{1}{r_1^2}\,y\;\;\{A_2\,(x-\mu)^2 + A_1\,y^2$$

$$+ \; 2\,(A_2 - A_1)\,(x-\mu)\,(\phi_1 + \psi_1)\,y\}]$$

$$- \; \frac{3(1-\mu)}{2r_1^5}\,y\,[2A_1 + 2A_2 - A_3$$

$$- \; \frac{3}{r_1^2}\,\{A_2\,(x-\mu)^2 + A_1\,y^2$$

$$+ \; 2\;\,(A_2 - A_1)\,(x-\mu)\,(\phi_1 + \psi_1)\,y\}]$$

$$+ \frac{3\mu}{r_2^5} [(A'_1 - A'_2)(\phi_2 + \psi_2)(x + 1 - \mu) - A'_1 y$$

$$+ \frac{1}{r_2^2} y \; \{A'_2 (x + 1 - \mu)^2 + A'_1 y^2$$

$$+ 2(A'_2 - A'_1)(x + 1 - \mu)(\phi_2 + \psi_2) y\}]$$

$$- \frac{3\mu}{2r_2^5} y \, [2A'_1 + 2A'_2 - A'_3$$

$$- \frac{3}{r_2^2} \{A'_2 (x + 1 - \mu)^2 + A'_1 y^2$$

$$+ 2 (A'_2 - A'_1)(x + 1 - \mu)(\phi_2 + \psi_2) y\}] = 0. \quad \dots (9)$$

If the primaries are

(a) spheres, (b) bigger primary an oblate body and the smaller one a sphere and (c) spheroids (oblate bodies), then by solving $\Omega_x = 0 = \Omega_y$, in each case, we get the same libration points as given in corresponding cases of section 1.

It may be noted that though the equatorial plane is not coincident with the plane of motion in the above three cases but for small values of v_i, ϕ_i, ψ_i (i = 1, 2), we are getting the same libration points in each of the above cases.

SECTION 3

We consider eqs. (9)

Let the triaxial rigid body of mass m_1 be nearly a sphere of radius R_0, then

$$a \simeq R_0 + \sigma_1,$$

$$b \simeq R_0 + \sigma_2$$

and $$c \simeq R_0 + \sigma_3,$$

where $$\sigma_1, \sigma_2, \sigma_3 << 1.$$

Therefore,

$$A_1 = \lambda_1 + \mu_1 \sigma_1 \text{ (Using eq. 5).}$$

where $$\lambda_1 = \frac{R_0^2}{5R^2} \text{ and } \mu_1 = \frac{2R_0}{5R^2}.$$

Similarly,

$$A_2 = \lambda_1 + \mu_1 \sigma_2,$$

and $$A_3 = \lambda_1 + \mu_1 \sigma_3.$$

Again, let the triaxial rigid body of mass m_2 be also nearly a sphere of radius R'_0, then

$$A'_1 = \lambda'_1 + \mu'_1 \sigma'_1 \text{ (using eq. 5)}$$

where $$\lambda'_1 = \frac{R_0'^2}{5R^2} \text{ and } \mu'_1 = \frac{2R'_0}{5R^2}.$$

Similarly,

$$A'_2 = \lambda'_1 + \mu'_1 \sigma'_2,$$

and $$A'_3 = \lambda'_1 + \mu'_1 \sigma'_3$$

where $$\sigma'_1, \sigma'_2, \sigma'_3 << 1.$$

Therefore, eqs. (9) become

$$\Omega_x = n^2 x - \frac{1-\mu}{r_1^3}(x-\mu) - \frac{\mu}{r_2^3}(x+1-\mu)$$

$$+ \frac{3(1-\mu)}{r_1^5}(x-\mu)\left[-\mu_1 \sigma_2 + \frac{1}{r_1^2}\left\{\mu_1 \sigma_2 (x-\mu)^2 + \mu_1 \sigma_1 y^2\right\}\right]$$

$$- \frac{3(1-\mu)}{2r_1^5}(x-\mu)\,[\mu_1 (2\sigma_1 + 2\sigma_2 - \sigma_3)$$

$$\left. - \frac{3}{r_1^2}\left\{\mu_1 \sigma_2 (x-\mu)^2 + \mu_1 \sigma_2 y^2\right\}\right]$$

$$+ \frac{3\mu}{r_2^5}(x+1-\mu)\left[-\mu'_1 \sigma'_2 + \frac{1}{r_2^2}\left\{\mu'_1 \sigma'_2 (x+1-\mu)^2 + \mu'_1 \sigma'_1 y^2\right\}\right]$$

$$- \frac{3\mu}{2r_2^5}(x+1-\mu)\,[\mu'_1 (2\sigma'_1 + 2\sigma'_2 - \sigma'_3)$$

$$- \frac{3}{r_2^2}\{\mu'_1 \sigma'_2 (x+1-\mu)^2 + \mu'_1 \sigma'_1 y^2\}] = 0$$

and
$$\Omega_y = n^2 y - \frac{1-\mu}{r_1^3} y - \frac{\mu}{r_2^3} y$$

$$+ \frac{3(1-\mu)}{r_1^5} y \left[\frac{1}{r_1^2} \left\{ \mu_1 \sigma_2 (x-\mu)^2 + \mu_1 \sigma_1 y^2 \right\} - \mu_1 \sigma_1 \right]$$

$$- \frac{3(1-\mu)}{2r_1^5} y \left[\mu_1 (2\sigma_1 + 2\sigma_2 - \sigma_3) - \frac{3}{r_1^2} \left\{ \mu_1 \sigma_2 (x-\mu)^2 + \mu_1 \sigma_1 y^2 \right\} \right]$$

$$+ \frac{3\mu}{r_2^5} y \left[\frac{1}{r_2^2} \left\{ \mu'_1 \sigma'_2 (x+1-\mu)^2 + \mu'_1 \sigma'_1 y^2 \right\} - \mu'_1 \sigma'_1 \right]$$

$$- \frac{3\mu}{2r_2^5} y \, [\mu'_1 + 2\sigma'_1 + 2\sigma'_2 - \sigma'_3)$$

$$- \frac{3}{r_2^2} \left\{ \mu'_1 \sigma'_2 (x+1-\mu)^2 + \mu'_1 \sigma'_1 y^2 \right\} \Bigg] = 0. \qquad \text{... (10)}$$

It may be noted that upto the first order of v_i, ϕ_i, ψ_i (i = 1, 2) Ω_x, Ω_y are independent of these Euler's angles.

The mean motion, n, given in equation (6), becomes

$$n^2 = 1 + \frac{3}{2} \mu_1 (2\sigma_1 - \sigma_2 - \sigma_3) + \frac{3}{2} \mu'_1 (2\sigma'_1 - \sigma'_2 - \sigma'_3) \qquad \text{... (11)}$$

Two cases arise

Case (a) — Triangular Libration Points ($y \neq 0$),

The triangular libration points are the solutions of the equations

$$n^2 x - \frac{1-\mu}{r_1^3} (x-\mu) - \frac{\mu}{r_2^3} (x+1-\mu)$$

$$+ \frac{3(1-\mu)}{r_1^5} (x-\mu) \left[-\mu_1 \sigma_2 + \frac{1}{r_1^2} \left\{ \mu_1 \sigma_2 (x-\mu)^2 + \mu_1 \sigma_1 y^2 \right\} \right]$$

$$- \frac{3(1-\mu)}{2r_1^5} (x-\mu) \, [\mu_1 (2\sigma_1 + 2\sigma_2 - \sigma_3)$$

$$- \frac{3}{r_1^2} \left\{ \mu_1 \sigma_2 (x-\mu)^2 + \mu_1 \sigma_1 y^2 \right\} \Bigg]$$

$$+ \frac{3\mu}{r_2^5} (x+1-\mu) \left[-\mu'_1 \sigma'_2 + \frac{1}{r_2^2} \left\{ \mu'_1 \sigma'_2 (x+1-\mu)^2 + \mu'_1 \sigma'_1 y^2 \right\} \right]$$

$$- \frac{3\mu}{2r_2^5}(x+1-\mu)\,[\mu'_1\,(2\sigma'_1+2\sigma'_2-\sigma'_3)$$

$$\left. - \frac{3}{r_2^2}\left\{\mu'_1\,\sigma'_2\,(x+1-\mu)^2+\mu'_1\,\sigma'_1\,y^2\right\}\right] = 0 \text{ (Using eq. 10)}$$

and
$$n^2 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}$$

$$+ \frac{3(1-\mu)}{r_1^5}\left[\frac{1}{r_1^2}\left\{\mu_1\,\sigma_2\,(x-\mu)^2+\mu_1\,\sigma_1\,y^2\right\}-\mu_1\,\sigma_1\right]$$

$$- \frac{3(1-\mu)}{2r_1^5}\,[\mu_1\,(2\sigma_1+2\sigma_2-\sigma_3)$$

$$\left. - \frac{3}{r_1^2}\,\{\mu_1\,\sigma_2\,(x-\mu)^2+\mu_1\,\sigma_1\,y^2\}\right]$$

$$+ \frac{3\mu}{r_2^5}\left[\frac{1}{r_2^2}\left\{\mu'_1\,\sigma'_2\,(x+1-\mu)^2+\mu'_1\,\sigma'_1\,y^2\right\}-\mu'_1\,\sigma'_1\right]$$

$$- \frac{3\mu}{2r_2^5}\,[\mu'_1\,(2\sigma'_1+2\sigma'_2-\sigma'_3)$$

$$\left. - \frac{3}{r_2^2}\,\{\mu'_1\,\sigma'_2\,(x+1-\mu)^2+\mu'_1-\sigma'_1\,y^2\}\right] = 0. \qquad \text{... (12)}$$

If we take $\sigma_1=\sigma_2=\sigma_3=0$ and $\sigma'_1=\sigma'_2=\sigma'_3=0$, the solution of eq. (12) is given by $r_1=r_2=1$ and from eq. (11), $n = 1$.

Now, we suppose that the solution for eq. (12) when σ_i, σ'_i (i = 1, 2, 3) are not equal to zero as

$$r_1 = 1+\alpha$$

and
$$r_2 = 1+\beta, \text{ where } \alpha, \beta << 1. \qquad \text{... (13)}$$

Putting the values of r_1 and r_2 from eq. (13) in eq. (3), we get

$$x = \mu - \frac{1}{2} + \beta - \alpha$$

and
$$y = \pm\frac{3^{1/2}}{2}\left\{1+\frac{2}{3}(\alpha+\beta)\right\}. \qquad \text{... (14)}$$

Putting the values of r_1, r_2 from eq. (13) and x, y from eq. (14) in eq. (12), rejecting higher order terms, we get

$$\alpha=\frac{\mu_1}{\mu-1}\left(\frac{11}{8}\sigma_1-\frac{11}{8}\sigma_2\right)+\frac{\mu_1'}{\mu-1}\left(\sigma'_1-\frac{1}{2}\sigma'_2-\frac{1}{2}\sigma'_3\right)+\frac{\mu\mu_1}{\mu-1}\left(-\frac{11}{8}\sigma_1+\frac{11}{8}\sigma_2\right)$$
$$+\frac{\mu\mu'_1}{\mu-1}\left(-\frac{3}{2}\sigma'_1+\sigma'_2+\frac{1}{2}\sigma'_3\right)$$

and

$$\beta=\frac{\mu_1}{\mu}\left(\frac{1}{2}\sigma_1-\frac{1}{2}\sigma_2\right)+\mu_1\left(-\frac{3}{2}\sigma_1+\sigma_2+\frac{1}{2}\sigma_3\right)$$
$$+\mu_1'\left(-\frac{11}{8}\sigma'_1+\frac{11}{8}\sigma'_2\right)$$

Then we get the co-ordinates (x, y) of the two triangular librations points $L_{4,5}$ as

$$x=\mu-\frac{1}{2}+\frac{\mu_1}{\mu}\left(\frac{1}{2}\sigma_1-\frac{1}{2}\sigma_2\right)+\mu_1\left(-\frac{3}{2}\sigma_1+\sigma_2+\frac{1}{2}\sigma_3\right)$$
$$+\mu'_1\left(-\frac{11}{8}\sigma'_1+\frac{11}{8}\sigma'_2\right)$$
$$-\frac{\mu_1}{\mu-1}\left(\frac{11}{8}\sigma_1-\frac{11}{8}\sigma_2\right)-\frac{\mu_1'}{\mu-1}\left(\sigma'_1-\frac{1}{2}\sigma'_2-\frac{1}{2}\sigma'_3\right)$$
$$-\frac{\mu\mu_1}{\mu-1}\left(-\frac{11}{8}\sigma_1+\frac{11}{8}\sigma_2\right)$$
$$-\frac{\mu\mu_1'}{\mu-1}\left(-\frac{3}{2}\sigma'_1+\sigma'_2+\frac{1}{2}\sigma'_3\right)$$

and

$$y=\pm\frac{3^{1/2}}{2}\left[1+\frac{2}{3}\left\{\frac{\mu_1}{\mu-1}\left(\frac{11}{8}\sigma_1-\frac{11}{8}\sigma_2\right)\right.\right.$$
$$+\frac{\mu_1'}{\mu-1}\left(\sigma'_1-\frac{1}{2}\sigma'_2-\frac{1}{2}\sigma'_3\right)+\frac{\mu\mu_1}{\mu-1}\left(-\frac{11}{8}\sigma_1+\frac{11}{8}\sigma_2\right)$$
$$+\frac{\mu\mu_1'}{\mu-1}\left(-\frac{3}{2}\sigma'_1+\sigma'_2+\frac{1}{2}\sigma'_3\right)$$
$$+\frac{\mu_1}{\mu}\left(\frac{1}{2}\sigma_1-\frac{1}{2}\sigma_2\right)+\mu_1\left(-\frac{3}{2}\sigma_1+\sigma_2+\frac{1}{2}\sigma_3\right)$$
$$\left.\left.+\mu_1'\left(-\frac{11}{8}\sigma'_1+\frac{11}{8}\sigma'_2\right)\right\}\right].$$

Case (b) — Collinear Libration Points, $y = 0$.

The collinear libration points are the solutions of the equations $y = 0$

and

$$f(x) = n^2 x - \frac{1-\mu}{r_1^3}(x-\mu) - \frac{\mu}{r_2^3}(x+1-\mu)$$

$$+ \frac{3(1-\mu)}{r_1^5}(x-\mu)\left[-\mu_1\sigma_2 + \frac{1}{r_1^2}\left\{\mu_1\sigma_2(x-\mu)^2 + \mu_1\sigma_1 y^2\right\}\right]$$

$$- \frac{3(1-\mu)}{2r_1^5}(x-\mu)\Big[\mu_1(2\sigma_1 + 2\sigma_2 - \sigma_3)$$

$$\left. - \frac{3}{r_1^2}\{\mu_1\sigma_2(x-\mu)^2 + \mu_1\sigma_1 y^2\}\right]$$

$$+ \frac{3\mu}{r_2^5}(x+1-\mu)\left[-\mu_1'\sigma_2' + \frac{1}{r_2^2}\left\{\mu_1'\sigma_2'(x+1-\mu)^2 + \mu_1'\sigma_1' y^2\right\}\right]$$

$$- \frac{3\mu}{2r_2^5}(x+1-\mu)\Big[\mu_1'(2\sigma_1' + 2\sigma_2' - \sigma_3')$$

$$\left. - \frac{3}{r_2^2}\{\mu_1'\sigma_2'(x+1-\mu)^2 + \mu_1'\sigma_1' y^2\}\right] = 0, \quad \ldots (15)$$

where $r_1 = |x-\mu|$ and $r_2 = |x+1-\mu|$.

Obviously they lie on the x-axis nd their abscissae are the roots of the eq. (15). Since $f(x) > 0$ in each of the open intervals $(-\infty, \mu-1)$, $(\mu-1, \mu)$ and (μ, ∞), the function f is strictly increasing in each of them.

Also,

$$f(x) \to -\infty \text{ as } x \to -\infty, (\mu-1)+0 \text{ or } \mu+0$$

and

$$f(x) \to \infty \text{ as } x \to \infty, (\mu-1)-0 \text{ or } \mu-0.$$

Therefore, there exists one and only one value of x in each of the above intervals such that $f(x) = 0$.

Further, $f(\mu-2) < 0, f(0) \geq 0$ and $f(\mu+1) > 0$. Therefore, there are only three real roots of eq. (15), one lying in each of the intervals $(\mu-2, \mu-1)$, $(\mu-1, 0)$ and $(\mu, \mu+1)$. Thus there are three collinear libration points.

4. CONCLUSIONS

In the restricted three body problem when both the primaries are triaxial rigid bodies, there are in general five libration points, three collinear and two triangular.

In case the equatorial plane is coincident with the plane of motion we have five libration points which are given by Szebehely[6], Bhatnagar and Hallan[9], Khanna and Bhatnagar[4 & 5] and Sharma, Taqvi and Bhatnagar[10].

Upto first order in ν_i, ϕ_i, ψ_i $(i = 1, 2)$ the libration points are independent of the orientation of the principal axes to the synodic axes.

Further in the case when the bodies are nearly spherical the libration points are again independent of ν_i, ϕ_i, ψ_i $(i = 1, 2)$.

We will be studying the stability of these libration points in our subsequent paper.

ACKNOWLEDGEMENT

We are thankful to the Centre for Fundamental Research in Space Dynamics and Celestial Mechanics (CFRSC) for providing all facilities in the completion of this work.

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