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ORTHOGONAL SERIES AND SLLN IN NON-COMMUTATIVE L_2-SPACES[@]

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The notion of bundle convergence for single (ordinary) sequences in non Neumann algebras and their L_2-spaces was introduced by Henz, Jajte and Paszkiewicz in 1996. Bundle convergence is stronger than almost sure convergence,a nd it enjoys the property of additivity. We adopt this notion for double sequences. We prove a two-parameter extension of the classical Rademacher-Menshov theorem, a two-parameter strong law of large numbers for an orthogonal double sequence of vectors in a noncommuntative L_2-space, and a two-parameter extension of another strong law, even the one-parameter version of which seems to be new. Our basic tools are the Gelfand-Naimark-Segal representation theorem and the extension of the classical Rademacher-menshov inequality to the non-commutative case.

Key Words : **von Neumann Algebra $\mathcal{U}$; Faithful and Normal State ϕ; Completion $L_2 = L_2(\mathcal{U}, \phi)$; Gelfand-Naimark-Segal Representation Theorem; Bundle Convergence; Almost Sure Convergence; Orthogonal Sequence of Vectors in L_2; Rademacher-Menshov Inequality; Strong Law of Large Numbers**

1. VON NEUMANM ALGEBRAS : BASIC NOTIONS AND RESULTS

As a background, we give a brief account, without proofs, of the basic notions and results in von Neumann algebras. The reader interested in details may consult the books by Dixmier[3], Jajte[7,8] and Pedersen[17].

Let $\hbar$ be a Hilbert space over the field $\mathbb{C}$ of complex numbers, and $\mathcal{L}(\hbar)$ the family of all bounded linear operators acting on $\hbar$. Denote by $\mathcal{U}$ a von Neumann algebra in $\mathcal{L}(\hbar)$. To be more specific, $\mathcal{U}$ is a subalgera of $\mathcal{L}(\hbar)$, which is selfadjoint (that is, $A \in \mathcal{U}$ implies $A^* \in \mathcal{U}$), the identity operator $1 \in \mathcal{U}$, and $\mathcal{U}$ is closed in the weak operator topology. Denote by $\| \cdot \|_\infty$ the operator norm in $\mathcal{U}$:

$$\| A \|_\infty := \sup \{ \| Ah \|_\hbar : \| h \|_\hbar \leq 1 \}$$

wheıe $\| \cdot \|_\hbar$ means the norm in h.

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Denote by Proj $\mathcal{U}$ in the class of all projections P in $\mathcal{U}$:

$$P^2 = P \text{ and } P^* = P.$$

We say that $\mathcal{U}$ is σ-finite if any collection $\{P_\alpha\}$ in Proj $\mathcal{U}$ whose members are mutually orthogonal in the sense that

$$P_\alpha P_{\alpha'} = O \text{ whenever } \alpha \neq \alpha',$$

where O is the zero operator, has at most a countable cardinality. This is the case if $\hbar$ is separable. More exactly, $\mathcal{L}(\hbar)$ is α-finite if and only if the Hilbert space $\hbar$ is separable.

A linear functional ϕ acting on $\mathcal{U}$ is said to be positive if

$$\phi(A^* A) \geq 0 \text{ for all } A \in \mathcal{U}.$$

For such a functional ϕ, we necessarily have

$$\phi(A^*) = \overline{\phi(A)},$$

the Cauchy-Schwarz inequality holds true :

$$|\phi(B^* A)|^2 \leq \phi(B^* B)\, \phi(A^* A) \text{ for all } A, B \in \mathcal{U}, \qquad \text{... (1.1)}$$

and ϕ is bounded, more precisely:

$$|\phi(A)| \leq \phi(1) \|A\|_\infty \text{ for all } A \in \mathcal{U},$$

whence it follows that $\|\phi\| = \phi(1)$.

A positive linear functional ϕ acting on $\mathcal{U}$ is called

(i) a state if $\phi(1) = 1$;

(ii) faithful if for all $A \in \mathcal{U}_+$

$$\phi(A) = 0 \text{ implies } A = O,$$

where by $\mathcal{U}_+$ we denote the cone of the positive operators in $\mathcal{U}$.

It is well known that an operator $B \in \mathcal{U}$ belongs to $\mathcal{U}_+$ if and only if

$$B = A^* A \text{ for some } A \in \mathcal{U}.$$

Any increasing net (A_β) in $\mathcal{U}_+$ with an upper bound in $\mathcal{U}_+$ has a least upper bound (in abbreviation: l.u.b.), say $A \in \mathcal{U}_+$, and the net (A_β) converges strongly to A. We say that a positive linear functional ϕ acting on $\mathcal{U}$ is normal (or in other words: ϕ enjoys the Beppo Levi Property) if for all increasing nets (A_β) in $\mathcal{U}_+$ with an upper bound in $\mathcal{U}_+$ we have

$$\phi(\underset{\beta}{\text{l.u.b.}} A_\beta) = \underset{\beta}{\text{l.u.b.}}\ \phi(A_\beta).$$

Given a faithful and normal state ϕ acting on $\mathcal{U}$, we may introduce a scalar product on $\mathcal{U}$, by setting

$$\langle A, B \rangle := \phi(B^* A) \text{ for all } A,\ B \in \mathcal{U}. \qquad \text{... (1.2)}$$

By (1.1), it is easy to see that $(\mathcal{U}, \langle \cdot, \cdot \rangle)$ is a prehilbert space over C. We denote by $L_2 = L_2(\mathcal{U}, \phi)$ its completion, by $(\cdot, \cdot)$ the scalar product, and by $\|\cdot\|$ the norm in L_2.

The celebrated Gelfand-Naimark-Segal representation theorem states that given a σ-finite von Neumann algebra $\mathcal{U}$ with a normal state ϕ, there exists a *-homomorphism π of $\mathcal{U}$ into the algebra of all bounded linear operators acting on L_2 and a cyclic vector $\omega \in L_2$ for

$$\pi(\mathcal{U}) := \{\pi(A) : A \in \mathcal{U}\}$$

such that $\phi(A)\,(\pi(A)\,\omega, \omega)$ for all $A \in \mathcal{U}$. ... (1.3)

If, in addition, ϕ is faithful, then ω is a separating vector for $\pi(\mathcal{U})$ in the sense that for all $A \in \mathcal{U}$

$$\pi(A)\,\omega = 0 \text{ implies } A = O,$$

where o means the zero vector in L_2. Consequently, in this case π is necessarily a one-to-one *-homomorphism. (However, "onto" is not true in general.)

We recall that π is called a *-homomorphism of $\mathcal{U}$ into the algebra of all bounded linear operators acting on L_2 if for all $A,\ B \in \mathcal{U}$ and $c \in \mathbb{C}$, the following identities hold true :

$$\pi(A+B) = \pi(A) + \pi(B),\ \pi(cA) = c\pi(A),$$

$$\pi(AB) = \pi(A)\,\pi(B), \text{ and } \pi(A^*) = \pi(A)^*.$$

Furthermore, a vector $\omega \in L_2$ is called cyclic for $\pi(\mathcal{U})$ if

$$[\pi(\mathcal{U})\omega] := [\pi(A)\omega : A \in \mathcal{U}] = L_2,$$

where by $[\cdot]$ we denote the closed linear subspace of L_2 spanned by the vectors indicated between the brackets. We may also that $\{\pi,\ L_2\}$ is a cyclic*-representation of $\mathcal{U}$ in L_2 with the cyclic vector $\omega \in L_2$.

Relations (1.2) and (1.3) can be combined into the following:

$$\langle A, B \rangle = (\pi(A)\,\omega, \pi(B)\,\omega) \text{ for all } A, B \in \mathcal{U};$$

in particular,

$$\langle A, B \rangle = \|\pi(A)\,\omega\|^2,$$

whence it follows easily that

$$\| \pi(A)\, \omega \| = \{\phi(A^* A)\}^{1/2} \leq \| A^* A \|_\infty^{1/2} = \| A \|_\infty \text{ for all } A \in \mathcal{U}. \qquad \text{... (1.4)}$$

Therefore, $\mathcal{U}$ (endowed with the scalar product $\langle \cdot, \cdot \rangle$ defined in (1.2)) can be identified with the subset $\pi(\mathcal{U})\omega$ of L_2, and whose closure coincides with L_2 itself. In other words, for each vector ξ in L_2 there exists a sequence $(A_k : k = 1, 2, ...)$ in $\mathcal{U}$ such that

$$\sum_{k=1}^{\infty} \pi(A_k)\, \omega = \xi, \qquad \text{... (1.5)}$$

the series being convergent in the norm of L_2.

2. Almost Sure Convergence and Bundle Convergence in $L_2(\mathcal{U}, \phi)$

The following definitions are motivated by (1.5). Given any $\xi \in L_2$ and $P \in \text{Proj } \mathcal{U}$, set

$$S_{\xi, P} = \left((A_k) \subset \mathcal{U} : \sum_{k=1}^{\infty} \pi(A_k)\, \omega = \xi \text{ in the norm of } L_2 \text{ while } \sum_{k=1}^{\infty} A_k P \text{ converges in the norm of } \mathcal{U} \right)$$

and

$$\| \xi \|_P := \inf \left\{ \left\| \sum_{k=1}^{\infty} A_k P \right\|_\infty : (A_k) \in S_{\xi, P} \right\}$$

with the usual convection that the infimum is meant to be ∞ in case $S_{\xi, P}$ is empty.

Obviously,

$$\| \xi + \eta \|_p \leq \| \xi \|_p + \| \eta \|_p \text{ for all } \xi, \eta \in L_2$$

and

$$\| \pi(A)\, \omega \|_p \leq \| AP \|_p \text{ for all } A \in \mathcal{U}.$$

The following definition of the almost sure convergence in L_2 is due to Hensz and Jajte[5] (see alsp [8, p. 2]). A sequence $(\xi_n : n = 1, 2 ...)$ of vectors in L_2 is said to converge almost surely (in abbreviation : a.s.) to some ξ in L_2 if for every $\varepsilon > 0$, there exists a projection $P \in \text{Proj } \mathcal{U}$ such that

$$\phi(1 - P) < \varepsilon \qquad \text{... (2.1)}$$

and

$$\| \xi_n - \xi \|_p \to 0 \text{ as } n \to \infty.$$

In particular, $\xi_n \to 0$ a.s. if for every $\varepsilon > 0$, there exist a projection $P \in \text{Proj } \mathcal{U}$ satisfying (2.1) and a matrix $(A_{nk} : n, k = 1, 2 ...)$ with entries in $\mathcal{U}$ such that

$$\sum_{k=1}^{\infty} \pi(A_{nk})\,\omega = \xi_n \text{ in the norm of } L_2 \ (n = 1, 2, \ldots)$$

and
$$\left\| \sum_{k=1}^{\infty} A_{nk} P \right\|_{\infty} \to 0 \text{ as } n \to \infty.$$

Unfortunately, the notion of a.s. convergence in a noncommutative von Neumann algebra lacks the property of additivity. This is a consequence of the fact that in the above definition of a.s. convergence we only require the existence of a family of projections with the identity operator 1 as a cluster point, such that the convergence of the associated sequence of operators is uniform on each of the subspaces corresponding to these projections.

A stronger notion of convergence in L_2 (as well as in the von Neumann algebra $\mathcal{U}$) was recently introduced by Hensz, Jajte, and Paszkiewicz[6]. To this end, we consider a sequence (D_k) of operators in $\mathcal{U}_+$ such that

$$\sum_{k=1}^{\infty} \phi(D_k) < \infty. \qquad \ldots (2.2)$$

We associate with this sequence (D_k) a so-called bundle $\mathcal{P} = \mathcal{P}(D_k)$ as follows:

$$\mathcal{P} := \left\{ P \in \text{Proj } \mathcal{U} : \sup \left\{ \left\| P \left(\sum_{k=1}^{n} D_k \right) P \right\|_{\infty} : n \geq 1 \right\} < \infty \qquad \ldots (2.3)$$

and
$$\| PD_n P \|_{\infty} \to 0 \text{ as } \{n \to \infty\}.$$

It is a crucial fact that for any sequence (D_k) in $\mathcal{U}_+$ with property (2.2), the bundle defined in (2.3) is "abundant" enough in the sense that it contains projections P arbitrarily close to the identity operator $1 \in \mathcal{U}$.

Now, a sequence (ξ_n) of vectors in L_2 is said to be bundle convergent to some ξ in L_2, in sign:

$$\xi_n \xrightarrow{b} \xi \text{ as } n \to \infty,$$

if there exists a sequence (A_n) in $\mathcal{U}$ such that

$$\sum_{n=1}^{\infty} \| \xi_n - \xi - \pi(A_n)\,\omega \|^2 < \infty \qquad \ldots (2.4)$$

and there exists a bundle $\mathcal{P}$ such that for each P in $\mathcal{P}$, we have

$$\| A_n P \|_\infty \to 0 \text{ as } n \to \infty. \qquad \text{... (2.5)}$$

Since the intersection of two (or even a countable number of) bundles is a bundle again, bundle convergence enjoys the property of additivity. Furthermore, bundle convergence implies a.s. convergence, but not coversely, and the limit of a bundle convergent sequence is unique in the selfadjoint part of L_2. As for details, we refer the reader to [6].

In the classical commutative case where $\mathcal{h}$ is identified with $L_2(X, \mathcal{F}, \mu)$ *over a probability space,* $\mathcal{U}$ *with* $L_\infty(X, \mathcal{F}, \mu)$, *and any* A in $\mathcal{U}$ acts only by pointwise multiplication, both bundle convergence and a.s. convergence coincide with the usual almost everywhere convergence, due to Egorov's theorem. Now, a normal state ϕ acting on $\mathcal{U}$ is necessarily of the following form :

$$\phi = \phi_f, \text{ where } \phi_f(A) := \int_X A(t) f(t) d\mu(t) \text{ for all } A(t) \in L_\infty(X, \mathcal{F}, \mu),$$

with some $f \in L_1(X, \mathcal{F}, \mu)$ such that

$$f(t) \geq 0 \text{ a.e. and } \int_X f(t)\, d\mu(t) = 1;$$

while ϕ is faithful if and only if $f(t) > 0$ almost everywhere. These statements are immediate consequences of [3, Theorem 1 on p. 51]. Furthermore, this time $L_2(\mathcal{U}, \phi)$ can be identified with $L_2(X, \mathcal{F}, \mu)$, in the particular case where $\omega := 1_X$, the characteristic function of the whole space X, *is the cyclic vector.*

3. Main Results : One-Parameter Case

In the sequel, let $\mathcal{U}$ be a fixed σ-finite von Neumann algebra with a faithful and normal state ϕ and let $L_2 = L_2(\mathcal{U}, \phi)$ be the completion of $\mathcal{U}$ under the norm $A \to \phi(A^* A)^{1/2}$. *We remind the reader of (1.3), where* ω *is a cyclic and separatng vector in* $L_2(\mathcal{U}, \phi)$.

We recall that a sequence $(\xi_k : k = 1, 2 ...)$ of vectors in L_2 is called orthogonal if

$$(\xi_j, \xi_k) = 0 \text{ for all } 1 \leq j < k < \infty$$

Hence, it follows that

$$\left\| \sum_{k=m}^{n} \xi_k \right\|^2 = \sum_{k=m}^{n} \| \xi_k \|^2 \text{ for all } 1 \leq m \leq n < \infty.$$

The following noncommutative version of the classical Rademacher-Menshov theorem (see [18] and [10]) was proved by Hensz, Jajte and Paskiewicz[6].

Theorem 1 — *If* (ξ_k) *is an orthogonal sequence of vectors in* $L_2(\mathcal{U}, \phi)$ *such that*

$$\sum_{k=1}^{\infty} \| \xi_k \|^2 (\log 2k)^2 < \infty,$$

then
$$\sigma_n := \sum_{k=1}^{n} \xi_k \xrightarrow{b} \sigma \text{ as } n \to \infty$$

where s is the sum of the series $\sum_{k=1}^{\infty} \xi_k$ *in the norm of* $L_2(\mathcal{U}, \phi)$.

The logarithms are to the base 2 in this paper.

The following noncommutative version of the strong law of large numbers was also provided in [6].

Theorem 2 — *If* (ξ_k) *is an orthogonal sequence in* $L_2(\mathcal{U}, \phi)$ *such that*

$$\sum_{k=1}^{\infty} \frac{\| \xi_k \|}{k^2} (\log 2k)^2 < \infty, \qquad \text{... (3.1)}$$

then the arithmetic means

$$\zeta_n := \frac{1}{n} \sum_{k=1}^{n} \xi_k \xrightarrow{b} 0 \text{ as } n \to \infty.$$

The proofs of both theorems are based on the following noncommutative version of the famous Rademacher-Menshov inequality. (See [6] and also [8, Lemma 5.5.2 on p. 65].)

Theorem 3 — *Given a finite orthogonal sequence* $(\xi_k : 1 \le k \le n)$ *in* $L_2(\mathcal{U}, \phi)$ *and a number* $\delta > 0$ *there exists a sequence* $(A_k \, 1 \le k < n)$ *of operators in* $\mathcal{U}$ *and an operator B in* $\mathcal{U}_+$ *such that*

$$\| \xi_k - \pi(A_k)\, \omega \| < \delta \text{ for all } 1 \le k \le n,$$

$$\left| \sum_{k=1}^{l} A_k \right|^2 \le B \text{ for all } 1 \le + \le n,$$

and
$$\phi(B) \le (\log 2n)^2 \sum_{k=1}^{n} \| \xi_k \|^2 + \delta.$$

Here the symbol $| \cdot |$ is defined by

$$| A | := (A^* A)^{1/2} \text{ for all } A \in \mathcal{U}. \qquad \text{... (3.2)}$$

The square root makes sense, since $A^* A \in \mathcal{U}_+$.

Unfortunately, the traditional triangle inequality

$$|A_1 + A_2| \leq |A_1| + |A_2| \text{ for all } A_1, A_2 \in \mathcal{U}, \qquad \text{... (3.3)}$$

does not hold in general. As an illustration, we present the phenomenon in the simplest pattern of 2×2 matrices. For example, if

$$A_1 := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } A_2 := \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix},$$

then,

$$|A_1|, = A_1, |A_2| = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } |A_1 + A_2| = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}.$$

Clearly, inequality (3.3) does not hold in this case.

Fortunately, the following weaker substitute for the triangle inequality is available in any von Neumann algebra $\mathcal{U}$ (see, e.g. [8, p.4]).

Lemma 1 — If $c_k \in \mathbb{C}$ and $A_k \in \mathcal{U}$ for all $1 \leq k \leq n$, then

$$\left| \sum_{k=1}^{n} c_k A_k \right|^2 \leq \sum_{k=1}^{n} |c_k|^2 \sum_{k=1}^{n} |A_k|^2. \qquad \text{... (3.4)}$$

It is plain that (3.4) would follow from (3.3), via the familiar Cauchy inequality. But the converse implication is not true in general. In spite of the lack of the triangle inequality in von Neumann algebras, inequality (3.4) is enough to prove Theorems 1-3 above.

We cite an important sufficient condition for bundle convergence (see [6, pp. 30-31]).

Lemma 2 — If $(\xi_n) \subset L_2(\mathcal{U}, \phi)$ and

$$\sum_{n=1}^{\infty} \| \xi_n \|^2 < \infty$$

then

$$\xi_n \xrightarrow{b} o \text{ as } n \to \infty.$$

We point out two more problems in the noncommutative setting.

(i) If $A, B, \in \mathcal{U}$, then $A \vee B \in \mathcal{U}_+$ in general. This means that forming a maximum makes no sense in a von Neumann algebra $\mathcal{U}$. Instead, one has to construct appropriate majorants in $\mathcal{U}_+$ in the proofs.

(ii) If P and Q are projections in $\mathcal{U}$, such that

$$\phi(1 - P) < \varepsilon \text{ and } \phi(1 - Q) < \varepsilon$$

for some $\varepsilon > 0$, where f is a faithful and normal state, then we cannot state that $\phi(1 - (P \wedge Q))$ is also "small". In particular, it may even happen that $P \wedge Q$ contains only the zero operator O; and consequently, $\phi(1-(P \wedge Q)) = 1$.

If we consider the second order mean

$$\tau_n := \frac{1}{n} \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) \xi_k \text{ for } n = 1, 2, \ldots$$

instead of the first order arithmetic mean ζ_n, the bundle convergence of the sequence (τ_n) can be guaranteed by a weaker condition than (3.1). This is stated in the following

***Theorem* 4** — *If (ξ_k) is an orthogonal sequence in $L_2(\mathcal{U}, \phi)$ such that*

$$\sum_{k=1}^{\infty} \frac{\|\xi_k\|^2}{k^2} < \infty, \qquad \ldots (3.5)$$

then

$$\tau_n \xrightarrow{b} 0 \text{ as } n \to \infty. \qquad \ldots (3.6)$$

In the book [8, theorem 5.2.13 on p. 78], Jajte proved a.s. convergence instead of (3.6) under condition (3.5). In the classical commutative case, Theorem 4 was proved by the first named author [13].

In order to present the proving technique when bundle convergence is involved, we give the detailed proof of Theorem 4, which is new in the literature.

PROOF : (i) First, we shall approximate each ξ_k by some $\pi(A_k)\,\omega$, A_k is in $\mathcal{U}$, such that

$$\|\xi_k - \pi(A_k)\,\omega\| < \delta_k, \; k = 1, 2, \ldots, \qquad \ldots (3.7)$$

where (δ_k) is a sequence of positive numbers with

$$\sum_{k=1}^{\infty} k^2 \delta_k^2 < \infty. \qquad \ldots (3.8)$$

By the Cauchy inequality for real numbers, hence it follows that

$$\sum_{k=1}^{\infty} \delta < \infty. \qquad \ldots (3.9)$$

We introduce the following operators in $\mathcal{U}$:

$$T_n := \frac{1}{n} \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) A_k, \; n = 1, 2, \ldots,$$

and represent τ_n in the following form :

$$\tau_n = \tau_{2^p} - \pi(T_{2^p})\,\omega) + (\pi(T_n)\,\omega - \pi(T_{2p})\,\omega) + (\tau_n - \pi(T_n)\,\omega), \qquad \ldots (3.10)$$

where $n \in I_p := \{2^p, 2^p + 1, \ldots, 2^{p+1} - 1\}$

for some p^3 0. If $p = 0$, the right-hand side in (3.10) collapses into one term : τ_{2^0}.

In order to prove (3.6), it is sufficient to prove that each one of the four terms on the right-hand side of (3.10) is bundle convergent to zero, by using the additivity property of bundle convergence. Observe that the bundle convergence of the second term on the right-hand side of (3.10) follows immediately from that of the fourth term. So, we have to prove the bundle convergence of the first, third, and fourth terms on the right-hand side of (3.10).

(*ii*) By orthogonality and (3.5)

$$\sum_{p=0}^{\infty} \| \tau_{2^p} \|^2 = \sum_{p=0}^{\infty} \frac{1}{2p} \sum_{k=1}^{2^p} \left(1 - \frac{k\Gamma}{2^p}\right)^2 \| \xi_k \|^2$$

$$\leq \sum_{p=0}^{\infty} 2^{-2p} \sum_{p=0}^{2^p} \| \xi_k \|$$

$$= \sum_{k=1}^{\infty} \| \xi_k \|^2 \sum_{p: 2^p \geq k} 2^{-2p}$$

$$\leq \frac{4}{3} \sum_{k=1}^{\infty} \| \xi_k \|^2 \frac{1}{k^2} < \infty.$$

By Lemma 2, hence it follows that

$$\tau_{2^p} \xrightarrow{b} 0 \text{ as } p \to \infty. \qquad \text{... (3.11)}$$

(*iii*) It is plain that

$$\tau_n = \pi(T_n)\,\omega = \frac{1}{n} \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) \eta_k, \text{ where } \eta_{k:} = \xi_k - \pi(A_k)\,\omega.$$

Again by the Cauchy inequality and (3.7), hence it follows that

$$\| \tau_n = \pi(T_n)\,\omega \|^2 = \frac{1}{n^2} \left\| \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) \eta_k \right\|^2$$

$$\leq \frac{1}{n^2} \left(\sum_{k=1}^{n} \frac{1}{k} k \| \eta_k \| \right)^2$$

$$\leq \frac{1}{n^2}\left(\sum_{k=1}^{n} \frac{1}{k^2}\right)\left(\sum_{k=1}^{n} k^2 \delta_k^2\right)$$

By (3.8),

$$\sum_{n=1}^{\infty} \| \tau_n - \pi(T_n)\,\omega \|^2 \leq \frac{\pi^2}{6}\left(\sum_{k=1}^{\infty} k^2 \delta_k^2\right) \sum_{k=1}^{\infty} \frac{1}{n^2} < \infty$$

and Lemma 2 implies that

$$\tau_n = \pi(T_n)\,\omega \xrightarrow{b} 0 \text{ as } n \to \infty. \qquad \text{... (3.12)}$$

(iv) Finally, we have to prove that

$$\pi(T_n)\,\omega - \pi(T_{2^p})\,\omega \xrightarrow{b} 0 \text{ as } n \to \infty,\; n \in I_p. \qquad \text{... (3.13)}$$

To this effect, we have to construct an appropriate bundle. Our reasoning hinges on the following estimate: Given any $n \in I_p^0 := \{2^p + 1, 2^p + 2, \ldots, 2^{p+1} + 1\}$ for some $p \geq 1$, by Lemma 1, we have

$$| T_n - T_{2^p} |^2 = \left| \sum_{k=2^p+1}^{n} (T_k - T_{k-1}) \right|^2$$

$$\leq (n - 2^p) \sum_{k=2^p+1}^{n} | T_k - T_{k-1} |^2.$$

This motivates the definition

$$D_p := 2^p \sum_{k \in I_p^0} | T_k - T_{k-1} |^2.$$

It is plain that $D_p \in \mathcal{U}_+$ and

$$| T_n - T_{2^p} |^2 \leq D_p \text{ for all } n \in I_p. \qquad \text{... (3.14)}$$

We claim that

$$\sum_{p=1}^{\infty} \phi(D_p) < \infty, \qquad \text{... (3.15)}$$

so the sequence $(D_p : p = 1, 2, \ldots)$ determines a bundle, say $\mathcal{P}$. By definition,

$$\phi(D_p) = 2^p \sum_{k \in I_p^0} \phi(|T_k - T_{k-1}|^2)$$

$$= 2^p \sum_{k \in I_p^0} \| \pi(T_k - T_{k-1})\, \omega \|^2$$

$$\le 2^{p+1} \sum_{k \in I_p^0} \left(\| \pi(\tau_k)\, \omega - \tau_k - \pi(\tau_{k-1})\, \omega + \tau_{k-1} \|^2 + \| \tau_k - \tau_{k-1} \|^2 \right) \quad \ldots (3.16)$$

From the representation

$$\tau_k - \tau_{k-1} = \sum_{j=1}^{k} \left(\frac{(j-1)+(2k-1)}{k^2 (k-1)^2} - \frac{1}{k(k-1)} \right)^2 \xi_j \, k \ge 2, \quad \ldots (3.17)$$

it follows, via orthogonality, that

$$\| \tau_k - \tau_{k-1} \|^2 = \sum_{j=1}^{k} \left(\frac{(j-1)+(2k-1)}{k^2 (k-1)^2} - \frac{1}{k(K-1)} \right)^2 \| \xi_j \|^2$$

$$\le \frac{1}{k^2 (k-1)^2} \sum_{j=1}^{k} \| \xi_j \|^2. \quad \ldots (3.18)$$

In fact, the value of the difference in parentheses in (3.17) increases from $- 1/k\ (k - 1)$ (corresponding to $j = k$) to $1/k^2$ (corresponding to $j = k$). By (3.5) and (3.18),

$$\sum_{p=1}^{\infty} 2^p \sum_{k \in I_p^0} \| \tau_k - \tau_{k-1} \|^2 \le \sum_{p=1}^{\infty} 2^p \sum_{k \in I_p^0} \frac{1}{k^2 (k-1)^2} \sum_{j=1}^{k} \| \xi_j \|^2 \quad \ldots (3.19)$$

$$\le \sum_{p=1}^{\infty} 2^p \sum_{j=1}^{2^{p+1}-1} \| \xi_j \|^2 \sum_{k \in I_p^0} \frac{1}{k^2 (k-1)^2}$$

$$\le \sum_{p=1}^{\infty} \frac{1}{(2^p+1)^2} \sum_{j=1}^{2^{p+1}-1} \| \xi_j \|^2$$

$$= \sum_{j=1}^{\infty} \| \xi_j \|^2 \sum_{p \,:\, 2^{p+1} > j} \frac{1}{(2p+1)^2}$$

$$\leq \frac{16}{3} \sum_{j=1}^{\infty} \| \xi_j \|^2 \frac{1}{j^2} < \infty.$$

Similarly to (3.17), the representation

$$t_k - p(T_k)w - t_{k-1} + p(T_k - 1)w$$

$$= \sum_{j=1}^{k} \left(\frac{(j-1)(2k-1)}{k^2 (k-1)^2} - \frac{1}{k(k-1)} \right) \eta_j, \text{ where } h_j := x_j - p(A_j)w,$$

holds true for $k \geq 2$. By the Cauchy inequality in L_2 and (3.8), we have

$$\| \tau_k - \pi(T_k)\,\omega - \tau_{k-1} + \pi(T_{k-1})\,\omega \|^2 \leq \frac{1}{k^2 (k-1)^2} \sum_{j,h=1}^{k} |(\eta_j, \eta_{j_1})|$$

$$\leq \frac{1}{k^2 (k-1)^2} \sum_{j,h=1}^{k} \delta_j \delta_{j_1} = \frac{1}{k^2 (k-1)^2} \left(\sum_{j,j_1=1}^{k} \delta_j \right)^2.$$

Hence by (3.9), it follows that

$$\sum_{p=1}^{\infty} 2^p \sum_{k \in I_p^0} \| \tau_k - \pi(T_k)\,\omega - \tau_{k-1} + \pi(T_{k-1})\,\omega \|^2 \qquad \text{... (3.20)}$$

$$\leq \sum_{p=1}^{\infty} 2^p \sum_{k \in I_p^0} \frac{1}{k^2 (k-1)^2} \left(\sum_{j=1}^{k} \delta_j \right)^2$$

$$\leq \left(\sum_{j=1}^{\infty} \delta_j \right)^2 \sum_{k=2}^{\infty} \frac{1}{k^2 (k-1)^2} < \infty.$$

Combining (3.16), (3.19), and (3.20) gives (3.15).

(v) Coming back to the proof of (3.13), let $P \in \mathcal{P}$ and $n \in I_p$ for some $p \geq 1$. Since $T_n - T_2{}^p \in \mathcal{U}$, this time the fulfillment of (2.4) is trivial. It remains to check the fulfillment of (2.5). By (1.4), (3.14), and the second property of a bundle expressed in definition (2.3), we conclude that

$$\| (T_n - T_2^p)\, P \|_\infty^2 = \| P \,|\, T_n - T_2^p \,|^2\, P \|_\infty$$

$$\leq PD_p P \|_\infty \to 0 \text{ as } p \to \infty.$$

This proves (3.13).

Combining (3.10)-(3.13) yields (3.6). □

4. MAIN RESULTS

We shall consider a double sequence $(\zeta_{mn} : m, n = 1, 2, ...)$ of vectors in $L_2 := L_2(\mathcal{U}, \phi)$, where m and n run over the positive integers, independently of one another. The notions of a bundle and bundle convergence can be easily adopted to this two-parameter case (to distinguish from the one-parameter case described in Section 2) as follows :

To a given double sequence $(D_{kl} : k, l, 1, 2, ..)$ of operators in $\mathcal{U}_+$ for which

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \phi(D_{kl}) < \infty, \qquad \text{... (4.1)}$$

we can associate a double bundle $\mathcal{P} = \mathcal{P}(D_{kl})$ as follows.

$$\mathcal{P} := \left\{ P \in \text{Proj} : \sup_{m, n \geq 1} \left\| P \left(\sum_{k=1}^{m} \sum_{l=1}^{n} D_{kl} \right) P \right\|_\infty \text{ is finite and} \right. \qquad \text{... (4.2)}$$

$$\left. \| PD_{kl} P \|_\infty \to 0 \text{ as } k + l \to \infty \right\}.$$

Now, a double sequence (ζ_{mn}) of vectors in L_2 is said to be bundle convergent to some in ζ in L_2, in sign:

$$\zeta_{mn} \xrightarrow{b} \zeta \text{ as } m + n \to \infty,$$

if there exists a double sequence (Z_{mn}) in $\mathcal{U}$ such that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \| \zeta_{mn} - \zeta - \pi(Z_{mn}) \omega \|^2 < \infty \qquad \text{... (4.3)}$$

and there exists a bundle $P = P(D_{kl})$ $(D_{kl} \in \mathcal{U}_+$ and the fulfillment of (4.1) are always assumed) such that for each $P \in \mathcal{P}$,

$$\| Z_{mn} P \|_\infty \to 0 \text{ as } m + n \to \infty. \qquad \text{... (4.4)}$$

In fact, these definitions (4.1)-(4.4) are simple reformulations of (2.2)-(2.5) from single to double sequences, while the members of the double sequence (ζ_{kl}) are reindexed into members of a single sequence (ζ^0_n), for example by making use of the familiar diagonalization process due to Cantor. To be more precise, we set

$$\zeta_1^0 := \zeta_{11}, \zeta_2^0 := \zeta_{12}, \zeta_3^0 := \zeta_{21}, \zeta_4^0 := \zeta_{13}, \zeta_5^0 := \zeta_{22}, \zeta_6^0 := \zeta_{31}, \zeta_7^0 := \zeta_{14} \text{ etc.}$$

It is only condition (4.2) which requires a slight explanation. In fact, according to the above diagonalization process, one should have required that

$$\sup \left\{ \left\| P\left(\sum_{(k,l):k+l\leq m} D_{kl} + \sum_{t=1}^{n} D_{m+1-l,l} \right) P \right\|_\infty \quad \text{for } n = 1, 2, ..., m; m = 1, 2, \ldots \right\}$$

is finite, instead of the first requirement in (4.2). Due to the fact that the D_{kl} are positive (selfadjoint) operators in $\mathcal{U}$, these two requirements are equivalent. Even it is indifferent how the double sequence (ζ_{kl}) is rearranged into a single sequence.

After these preliminaries, we recall that a double sequence $(\xi_{kl}$ (: $k, l = 1, 2$ ....) of vectors in L_2 is called orthogonal if

$$(\xi_{kl}, \xi_{k_1, l_1}) = 0 \text{ whenever } (k, l) \neq (k_1, l_1).$$

In particular, hence it follows that

$$\left\| \sum_{(k,l)\in \mathcal{N}} \xi_{kl} \right\|^2 = \sum_{(k,l)\in \mathcal{N}} \| \xi_{kl} \|^2$$

for every finite set $\mathcal{N}$ of lattice points in R^2 with positive integer coordinates.

The extension of the two-parameter strong law of large number proved in[9] reads as follows :

Theorem 5 — *If (ξ_{kl}) is an orthogonal double sequence in $L_2(\mathcal{U}, \phi)$ such that*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \| \xi_{kl} \|^2 \frac{(\log 2k)^2}{k^2} - \frac{(\log 2l)^2}{l^2} < \infty, \qquad \text{... (4.5)}$$

then $$\zeta_{mn} := \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} \xi_{kl} \xrightarrow{b} 0 \text{ as } m+n \to \infty.$$

In the classical commutative case, Theorem 1 was proved by the second named author [11].

If we consider the second order mean τ_{mn} (see definition (4.6) below) instead of the first arithmetic mean ζ_{mn}, then boundle convergence of the double sequence (τ_{mn} : $m, n = 1, 2, \ldots$) can be guaranteed by a weaker condition than (4.5). This is stated in the following theorem, which was also proved in [9].

Theorem 6 — *If (ξ_{mn}) is a double orthogonal sequence in $L_2(\mathcal{U}, \phi)$ such that*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\| \xi_{kl} \|^2}{k^2 l^2} < \infty,$$

then $$\tau_{mn} := \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} \left(1 - \frac{k-1}{m}\right)\left(1 - \frac{l-1}{n}\right) \xi_{kl} \xrightarrow{b} 0 \text{ as } m+n \to \infty. \qquad \dots (4.6)$$

In the classical commutative case, Theorem 6 was proved by the second named author [15].

The proofs of Theorems 5 and 8 below are based on two noncommutative versions of the famous Rademacher-Menshov inequality. The first of them relates to the one-parameter case (see Theorem 3 in Section 3), while the second one relates to the two-parameter case.

Theorem 7 — *Given a finite double orthogonal sequence* $(\xi_{kl} : 1 \le k \le M, 1 \le l \le N)$ *in* $L_2(\mathcal{U}, \phi)$ *and a number* $\delta > 0$, *there exist a double sequence* $(A_{kl} : 1 \le k \le m, 1 \le l \le N)$ *of operators in* $\mathcal{U}$ *and an operator* B *in* $\mathcal{U}_+$ *such that*

$$\| \xi_{kl} - \pi(A_{kl})\, \omega \| \; \delta \text{ for all } 1 \le k \le M, 1 \le * \le N;$$

$$\left| \sum_{k=1}^{k_1} \sum_{l=1}^{l_1} A_{kl} \right| \le B \text{ for all } 1 \le k_1 \le M, 1 \le l_1 \le N;$$

and $$\phi(B) \le (\log 2M)^2 (\log 2N)^2 \sum_{k=1}^{M} \sum_{l=1}^{N} \| \xi_{kl} \|^2 + \delta$$

The commutative version of Theorem 7 was first proved by Agnew [1].

In the proofs of Theorems 5 and 8 below we also use the following :

Lemma 3 — (cf. [4, pp. 30-31]) — If $(\zeta_{mn}) \subset L_2(\mathcal{U}, \phi)$ and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \| \zeta_{mn} \|^2 < \infty,$$

then $$\zeta_{mn} \xrightarrow{b} 0 \text{ as } m+n \to \infty$$

We recall that if a double sequence $(\xi_{kl} : k, l = 2, 3, \dots)$ of vectors in L_2 is orthogonal and

$$\sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \| \xi_{kl} \|^2 < \infty,$$

then, by the completeness of L_2, the sum

$$\sigma := \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \xi_{kl} \qquad \dots (4.7)$$

as well as the remainder sums

$$\rho_{mn} := \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} \xi_{kl}$$

exist in the norm of L_2 for all $m, n \geq 1$. Denote by

$$\sigma_{mn} := \sum_{k=2}^{m} \sum_{l=2}^{\infty} \xi_{kl}, \; m, n \geq 2,$$

the rectangular partial sums of the double series in (4.7). It is plain that

$$\sigma - \sigma_{mn} = \rho_{m1} + \rho_{1n} - \rho_{mn}, \; m, n \geq 1; \qquad \text{... (4.8)}$$

with the agreement that $\sigma_{mn} := o$ in case min $(m, n) = 1$, where o is the zero vector in L_2.

The two-parameter noncommutative Rademacher-Menshov theorem proved in [16] reads as follows:

Theorem 8 — *If* $(\xi_{kl} : k, l = 2, 3, ...)$ *is an orthogonal double sequence in* $L_2(\mathcal{U}, \phi)$ *such that*

$$\sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \| x_{kl} \|^2 (\log k)^2 (\log l)^2 < \infty. \qquad \text{... (4.9)}$$

then $\quad \rho_{mn} \xrightarrow{b} 0$ as $m + n \to \infty$. ... (4.10)

We raise the following problem. Theorem 8 can be viewed as the two-parameter Rademacher-Menshov theorem in noncommutative L_2-space. By virtue of (4.8), we may interpret the conclusion of Theorem 8 in the form that

$$\sigma_{mn} - \sigma \to o \text{ as } m, n \to \infty,$$

where σ is defined in (4.7). This kind of interpretation resembles us to the notion of regular convergence of double complex series, introduced by Hardy[4] and rediscovered by the first named author[12&14] in an equivalent form.

However, the problem of how to attribute a precise meaning to the limit relation just above in the sense that the rectangular partial sum σ_{mn} is arbitrarily "close" to its sum σ as both indices m and n are large enough, is still open. In this context, limit relation (4.10) says that the remainder sum ρ_{mn} of the series in (4.7) is arbitrarily small if at least one of the indices m and n is large enough.

On the other hand, the definition of the limit relation

$$\rho_{mn} \xrightarrow{b} 0 \text{ as } m, n \to \infty \qquad \text{... (4.11)}$$

which would be the noncommutative counterpart of convergence in Pringsheim's sense, has not yet been clear. We emphasize that max $(m, n) \to \infty$ in (4.10), while min $(m, n) \to \infty$ in (4.11).

By definition, (4.10) guarantees the existence of a sequence (R_{mn}) of operators in $\mathcal{U}$ satisfying

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \| \rho_{mn} - \pi(R_{mn})\, \omega \|^2 < \infty, \qquad \text{... (4.12)}$$

(cf. (4.3)). and the existence of a bundle P such that for each $P \in \mathcal{P}$,

$$\| R_{mn} P \|_{\infty} \to 0 \text{ as } m + n \to \infty. \qquad \text{... (4.13)}$$

Motivated by (4.8), set

$$\tilde{R}_{mn} := R_{m1} + R_{1n} - R_{mn},\ m, n = 2, 3, \ \dots\ .$$

It is plain that $\tilde{R}_{mn} \in u$ and for each $P \in P$,

$$\| \tilde{R}_{mn} P \|_{\infty} \to 0, \text{ as } m, n \to \infty,$$

but we are in trouble as to the fulfillment of an analogue of (4.12). Since

$$\sigma - \sigma_{mn} - \pi(\tilde{R}_{mn})\, \omega \ = (\rho_{m1} - \pi(R_{m1})\, \omega) + (\rho_{1n} - \pi(R_{1n})\, \omega) - (\rho_{mn} - (R_{mn})\, \omega),$$

we can only state, via the triangle inequality in L_2, that

$$\| \sigma - \sigma_{mn} - \pi(\tilde{R}_{mn})\, \omega \| \to 0 \text{ as } m, n \to \infty,$$

without involving any summation with respect of m and/or n.

To sum up, in any reasonable definition of the limit relation (4.11) one has to require the fulfillment of (4.13) with $m, n \to \infty$ instead of $m + n \to \infty$. However, it is not clear, what kind of substitute for (4.12) expressing a certain rate of approximation makes sense.

Finally, we make a historical remark. In the classical commutative case, the almost sure (almost everywhere) convergence of double orthogonal series in Pringsheim's sense, under condition (4.9), was first proved by Agnew[1], while the regular convergence of the same double series was proved by the first named author[12]. The reader interested in details in the classical theory of orthogonal series may consult with the monography by Alexits[2].

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