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COMMUTATIVITY THEOREMS FOR CERTAIN RINGS WITH POLYNOMIAL CONSTRAINTS

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We discuss the commutativity of certain rings with unity 1 and one sided s-unital rings under each of the following ring properties (see Introduction for properties). Also commutativity of rings is found under different sets of constraints on integral exponents.

Key Words : Commutativity; Commutator ideal; Polynomial Constraint; s-Unital Ring

1. INTRODUCTION

Throughout this paper, R will denote an associative ring, with $N(R)$, $Z(R)$, $C(R)$ and $(R, +)$ denoting respectively its set of nilpotent elements, its center, its commutator ideal and its additive group. For any $x, y \in R$, $[x, y] = xy - yx$. By $GF(q)$ we mean the Galois field (finite field) with q elements, and by $(GF)(q))_2$ the ring of all 2×2 matrices over $GF(q)$. We denote $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $(GF(p))_2$, for a prime p. As usual $\mathbf{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbf{Z}$, the ring of integers. Following [7], a ring R is said to be left (resp. right) s-unital ring if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further R is called s-unital if $x \in Rx \cap xR$ (see [1, 2]).

Now, we consider the following ring properties:

(c) For each $y \in R$, there exist polynomials $f(\lambda), g(\lambda), h(\lambda) \in \mathbf{Z}[\lambda]$ such that

$$g(y)\,[x, f(y)]\,h(y) = \pm x^p\,[x^r, y]\,y^q$$

for all $x \in R$ and fixed integers $r > 1$, $p \geq 0, q \geq 0$.

(c_1) For each $y \in R$, there exist polynomials $f(\lambda), g(\lambda), h(\lambda) \in \mathbf{Z}[\lambda]$ such that

$$g(y)\,[x, f(y)]\,h(y) = \pm y^q\,[x^r, y]\,x^p$$

for all $x \in R$ and fixed integers $r > 1$, $p \geq 0$, $q \geq 0$.

(c_2) For each $y \in R$, there exist polynomials $f(\lambda), g(\lambda), h(\lambda), \tilde{f}(\lambda), \tilde{g}(\lambda), \tilde{h}(\lambda) \in \mathbf{Z}[\lambda]$ such that

$$g(y)\,[x, y^t f(y)]\,h(y) = \pm x^p\,[x^r, y]$$

and $$\tilde{g}(y)\,[x, y^t \tilde{f}(y)]\,\tilde{h}(y) = \pm x^q\,[x^s, y]$$

where $p \geq 0, q \geq 0, t \geq 2, r > 1, s > 1$ are fixed integers with $(r, s) = 1$.

(c_3) For each $y \in R$, there exist polynomials $f(\lambda), g(\lambda), h(\lambda), \tilde{f}(\lambda), \tilde{g}(\lambda), \tilde{h}(\lambda), \in \mathbf{Z}[\lambda]$ such that

$$g(y)\,[x, y^t f(y)]\,h(y) = \pm [x^r, y]\,x^p$$

and $$\tilde{g}(y)\,[x, y^t \tilde{f}(y)]\,\tilde{h}(y) = \pm [x^s, y]\,x^q$$

for all $x \in R$ where $p \geq 0, q \geq 0, t \geq 2, r > 1, s > 1$ are fixed integers with $(r, s) = 1$.

(c_4) For each $x, y \in R$, there exist polynomials $f(\lambda), g(\lambda), h(\lambda), \tilde{f}(\lambda), \tilde{g}(\lambda), \tilde{h}(\lambda) \in \mathbf{Z}[\lambda]$ and $r = r(x, y) > 1$, $s = s(x, y) > 1, p = p(x, y) \geq 0, q = q(x, y) \geq 0, t \geq 2$ are integers such that r and s are relatively prime and R satisfies

$$g(y)\,[x, y^t f(y)]\,h(y) = \pm x^p\,[x^r, y]$$

and $$\tilde{g}(y)\,[x, y^t \tilde{f}(y)]\,\tilde{h}(y) = \pm x^q\,[x^s, y].$$

(c_5) For each $x, y \in R$, there exist polynomials $f(\lambda), g(\lambda), h(\lambda), \tilde{f}(\lambda), \tilde{g}(\lambda), \tilde{h}(\lambda) \in \mathbf{Z}[\lambda]$ and $r = r(x, y) > 1, s = s(x, y) > 1, p = p(x, y) \geq 0$, $q = q(x, y) \geq 0, t \geq 2$ are integers such that r and s are relatively prime and R satisfies

$$g(y)\,[x, y^t f(y)]\,h(y) = \pm [x^r, y]\,x^p$$

and $$\tilde{g}(y)\,[x, y^t \tilde{f}(y)]\,\tilde{h}(y) = \pm [x^s, y]\,x^q.$$

(CH) For each $x, y \in R$, there exist polynomials $f(\lambda), g(\lambda) \in \lambda^2 \mathbf{Z}[\lambda]$ such that

$$[x - f(x), y - g(y)] = 0.$$

Q(b) For all $x, y \in R$, $b\,[x, y] = 0$ implies that $[x, y] = 0$, where b is some positive integer.

A beautiful theorem due to Herstein[6] asserts that a ring satisfying the polynomial identity $(x + y)^m = x^m + y^m$ for some $m > 1$ must have nil commutator ideal. Among other classes of rings in which $C(R)$ is known to be nil is the class of rings satisfying the polynomial identity $[x^m, y] = [x, y^m]$ (see [5] for references). But the class of rings satisfying the latter identity includes the rings satisfying $(x+y)^m = x^m + y^m$. Motivated by this observation, Bell[4] established that a ring R with unity 1 satisfying the polynomial identity $[x^m, y] = [x, y^m]$ is commutative if $(R, +)$ is n-torsion free. In attempts to extend this result, several authors have considered various special cases of the properties (c) and (c_1) (cf. [1, 2, 4, 6, 9, 11, 12, 16]). In most of the cases the underlying polynomials are assumed to be monomials.

The object of this paper is to prove commutativity of rings with unity 1 satisfying either of the properties (c) or (c_1) together with the property $Q(r)$. Further if R satisfies (c_2) or (c_3), then $Q(r)$ is replaced by some other suitable constraints on the exponent r. Next, we establish commutativity of one-sided s-unital rings satisfying either of the properties (c_4) or (c_5), and hence, generalize many well-known commutativity results for rings.

2. Commutativity of Rings with Unity 1

***Theorem* 2.1** — *Let R be a ring with unity 1 satisfying (c). Moreover, if R satisfies $Q(r)$, then R is commutative (and conversely).*

***Theorem* 2.2** — *Let R be a ring with unity 1 satisfying (c_1). If R satisfies $Q(r)$, then R is commutative (and conversely).*

We begin with

Lemma 2.3 ([8] p. 221) — Let x, y be elements in a ring R such that $[x, [x, y]] = 0$. Then for any positive integer n, $[x^n, y] = nx^{n-1}[x, y]$.

Lemma 2.4 [10] Theorem) — Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, x_3, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:

(i) For any ring R satisfying the polynomial identity $f = 0$, $C(R)$ is nil.

(ii) For every prime p, $(G(F(p)))_2$ fails to satisfy $f = 0$.

Lemma 2.5 [3] Theorem 1 — Let R be a ring and suppose that, for any $x, y \in R$, there exists a polynomial $f(\lambda) \in \lambda \mathbf{Z}[\lambda]$ depending on x and y such that $[x, y] = [x, y]f(x)$. Then R is commutative.

Now, we establish the following results called Steps.

Step 2.1 — Let R be a ring with unity 1 and let $f: R \to R$ be any polynomial function of two variables with the property that $f(1 + x, y) = f(x, y)$, for all pairs of elements x and y.

(i) If for all $x, y \in R$ there exists an integer $m = m(x, y) \geq 1$ such that $x^m f(x, y) = 0$, then necessarily $f(x, y) = 0$.

(ii) If for all $x, y \in R$ there exists an integer $m = m(x, y) \geq 1$ such that $f(x, y)x^m = 0$, then necessarily $f(x, y) = 0$.

Proof (i) : Set an integer $m_1 = m(1 + x, y) \geq 1$ such that $(1+x)^{m_1} f(x, y) = 0$. If $n = \max\{m, m_1\}$, then $x^n f(x, y) = 0$ and $(1 = x)^n f(x, y) = 0$. Select $f(x, y) = \{(1+x) - x\}^{2n+1} f(x, y)$. On expanding the expression on the right-hand side by binomial theorem, this yield that $f(x, y) = 0$.

Similar arguments may be used if R satisfies (ii).

Remark 2.1 : Step 2.1 is proved in [15, Lemma 4] for a fixed exponent m, but with a slight modification in the proof we have established this for variable exponent m.

Step 2.2 — Let R be a ring with unity 1 satisfying either of the properties (c) or (c_1). Then $C(R) \subseteq N(R)$.

PROOF : Replacing x by $x + 1$ in (c), we get $g(y)\,[x, f(y)]\,h(y) = \pm (x+1)^p\,[(x+1)^r\,y]\,y^q$. An application of (c) yields that $(x+1)^p\,[(x+1)^r, y]\,y^q = x^p\,[x^r, y]\,y^q$ that is $\{(x+1)^p\,[(x+1)^r, y] - x^p\,[x^r\,y]\}\ y^q = 0$, for all $x, y \in R$. Now Step 2.1 gives that $(x+1)^p\,[(x+1)^r, y] = x^p\,[x^r, y]$, for all $x, y \in R$.. This is a polynomial identity and we observe that $x = -e_{11} + e_{12}$ and $y = e_{11}$ fail to satisfy this equality. Hence by Lemma 2.4, $C(R)$ is a nil ideal.

On the other hand, if R satisfies (c_1), then using the same arguments with $x = e_{11} + e_{21}, y = e_{11}$, we get the required result.

PROOF OF THEOREM 2.1 : Suppose that R satisfies the property (c). Let $b \in N(R)$. Then there exists an integer l such that

$$b^k \in Z(R) \text{ for all } k \geq l,\, l \text{ minimal.} \qquad \dots (2.1)$$

If $l = 1$, then for each such $b \in Z(R)$. Let $l > 1$. Replacing x by b^{l-1} in (c), we get $g(y)\,[b^{l-1}, f(y)]\,h(y) = \pm\, b^{p(l-1)}\,[b^{r(l-1)}, y]\,y^q$, for all $y \in R$. By using (2.1) and the fact that $r(l-1) \geq l$ for $r > 1$, we obtain

$$g(y)\,[b^{l-1}, f(y)]\,h(y) = 0. \qquad \dots (2.2)$$

Replacing x by $1 + b^{l-1}$ in (c), we get

$$g(y)\,[1 + b^{l-1}, f(y)]\,h(y) = \pm\,(1 + b^{(l-1)})^p\,[(1 + b^{(l-1)})^r, y]\,y^q.$$

Now (2.2), yields that $(1 + b^{(l-1)})^p\,[(1 + b^{[(l} - 1)/)^r, y]\,y^q = 0$. Since $1 + b^{l-1}$ is invertible, the last equation implies that $[(1 + b^{(l-1)})^r, y]\,y^q = 0$, for all $y \in R$. Now application of Step 2.1 yields that

$$[(1 + b^{(l-1)})^r, y] = 0. \qquad \dots (2.3)$$

Combining (2.3) with (2.1), we get

$$0 = [(1 + b^{(l-1)})^r, y] = [1 + b^{l-1}, y].$$

It follows that $r\,[b^{l-1}, y] = 0$, for all $y \in R$ and the property $Q(r)$ gives that $b^{l-1} \in Z(R)$. This contradicts the minimality of l in (2.1). Hence, $l = 1$ and (2.1) implies that $N(R) \in Z(R)$. Combining this fact with Step (2.2), we get

$$C(R) \subseteq N(R) \subseteq Z(R). \qquad \dots (2.4)$$

Replacing x by $x + 1$ in (c), we get

$$\{(x+1)^p\,[(x+1)^r, y] - x^p\,[x^r, y]\}\ y^q = 0$$

for all $x, y \in R$. Again an application of Step 2.1, we get

$$(x+1)^p\,[(x+1)^r, y] = x^p\,[x^r, y]$$

for all $x, y \in R$. Thus, in view of (2.4) and lemma 2.3, we have

$$r[x, y]\ \{(x+1)^{p+r-1} - x^{p+r-1}\} = 0.$$

That is,

$$r[x, y\ \{(x+1)^{p+r-1} - x^{p+r-1}\}] = 0, \text{ for all } x, y \in R.$$

An application of property $Q(r)$, we get

$$[x, y\ \{(x+1)^{p+r-1} - x^{p+r-1}\}] = 0. \qquad \dots (2.5)$$

Equation (2.5) is a polynomial identity and can be rewritten in the form $[x, y] = [x, y]\, xq(x)$ for some $q(\lambda) \in \mathbf{Z}[\lambda]$. Hence, by Lemma 2.5, R is commutative.

Similar arguments can be used if R satisfies the hypotheses of Theorem 2.2. We omit the details to avoid repetition.

The following results are the immediate consequences of the above theorems.

Corollary 2.6 (1, Theorem 2) — Let $n > 1$, m, r, s and t be fixed non negative integers. Let R be a ring with unity 1 satisfying $y^r\,[x, y^m]\,y^s = \pm\,[x^n, y]\,x^t$. Moreover, if R has the property $Q(n)$, then R is commutative.

Corollary 2.7 — Suppose that $r > 1$, p, q are fixed non-negative integers and let R be a ring with unity 1 in which for all $x \in R$ there exist integers $l = l(y) \geq 0, m = m(y) \geq 0, n = n(y) \geq 0$ depending on y, such that $y^l\,[x, y^m]\,y^n = \pm\,x^p\,[x^r, y]\,x^q$ for all $x \in R$. Moreover, if R has the property $Q(r)$, then R is commutative (and conversely).

Remark 2.2 : If we drop the restriction of unity 1 in the hypothesis of Theorems 2.1 and 2.2, R may be badly non-commutative.

Example 2.8 — Let B_s be the ring of $s \times s$ matrices over a division ring B and $A_s = \{a_j \in B_s \mid a_{ij} = 0 \text{ if } i \geq j\}$. A_s is necessarily non-commutative and for any positive integer $s >$ 2. But A_3 satisfies (c) or (c_1).

The above example shows that Theorems 2.1 and 2.2 are not valid for arbitrary rings.

Remark 2.3 : The following ring shows that the property $Q(r)$ in the hypotheses of the Theorems 2.1 and 2.2, cannot be deleted.

Example 2.9 — Consider the ring $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in GF(2) \right\}$.

The non-commutative ring R with unit 1 satisfies $[x^4, y] = [x, y^4]$, for all $x, y \in R$.

3. Commutativity through a Streb Theorem

In view of the Example 2.9, one might ask a natural question: What additional conditions are needed to force the commutativity of R if the property $Q(r)$ is dropped from conditions of the Theorems 2.1 and 2.2?

The following theorems give answer to the above question.

***Theorem* 3.1** — *Let R be a left s-unital ring satisfying the property* (c_2). *Then R is commutative (and conversely).*

***Theorem* 3.2** — *Let R be a right s-unital ring satisfying the property* (c_3). *Then R is commutative (and conversely).*

In order to develop the proof of the above theorems, we begin with the following types of rings:

$(1)_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(1)_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(1) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(2) $M_\sigma(F) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \mid \alpha, \beta \in F \right\}$, where F is a finite field and σ is a non-trivial automorphism of F.

(3) A non-commutative ring with no non-zero divisors of zero.

(4) $\mathcal{A} = \langle 1 \rangle + T$, T is a non-commutative subring of $\mathcal{A}$ such that $T[T, T] = [T, T]\, T = 0$.

In 1989, Streb[17] gave a classification of non-commutative rings which has been effectively used as a tool to get several commutativity results (cf. [9, 11, 13, 14]). From the proof of [14, Corollary 1], it is trivial to observe that if R is a non-commutative s-unital ring, then there exists a factorsurbing $\mathcal{A}$ of R which is of type $(1)_l$, (2), (3) or (4). This yields the following result which plays a vital role in our subsequent discussion (cf. [14, Meta Theorem]).

Proposition 3.3 — Let P be a ring property which is inherited by factor subrings. If no ring of type $(1)_l$, (2), (3) or (4) satisfies P, then every left s-unital ring satisfying P is commutative.

Remark 3.1 : Obviously, one can remark that the dual of Proposition 3.3 holds, that is, if P is a ring property which is inherited by factorsurbing and if no ring of type $(1)_r$, (2), (3) or (4) satisfies P then every right s-unital ring satisfying P is commutative.

Lemma 3.4[13] — Let R be a left (resp. right) s-unital ring and not right (resp. left) s-unital. Then R has a factorsurbing of R which is of type $(1)_l$ (resp. $(1)_r$).

Lemma 3.5[14]— Let R be a ring with unity 1 satisfying the property (CH). If R is non-commutative, then there exists a factorsurbing of R which is of type (1) or (2).

Now we prove

Step 3.1 — Let a ring $\mathcal{A}$ be a factorsuring of R of type $(1)_l$ or (2). Then $\mathcal{A}$ does not satisfy (c_4).

PROOF : Let $\mathcal{A}$ be of type $(1)_l$. Taking $x = e_{11} + e_{12}$ and $y = e_{12}$ in (c_4), we get $g(e_{12})\,[e_{11} + e_{12}, e_{12}^t f(e_{12})]\, h(e_{12}) = \pm (e_{11} + e_{12})^p\, [(e_{11} + e_{12})^r, e_{12}] = e_{12} \neq 0$ for some integers γ 1, $t \geq 2, p \geq 0$ and polynomials $f(\lambda, g(\lambda), h(\lambda) \in \mathbf{Z}[\lambda]$. This is a contradiction because $g(e_{12})\,[e_{11} + e_{12}, e_{12}^t f(e_{12})]\, h(e_{12}) = 0$.

Consider $\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \mid \alpha, \beta \in F \right\}$, where F is a finite field with a non-trivial automorphism σ. It is noticed that $N(\mathcal{A}) = Fe_{12}$. Let $\mathcal{A}$ satisfy (c_4). For any $c \in N(\mathcal{A})$ and arbitrary unit u there exist integers $r = r(u, c) > 1$, $s = s(u, c) > 1$, $p = p(u, c) \geq 0, q = q(u, c) \geq 0$ and $t \geq 2$ with the property that r and s are relatively prime, and polynomials $f(\lambda), g(\lambda), h(\lambda), \tilde{f}(\lambda)$ $\tilde{g}(\lambda)$, $\tilde{h}(\lambda) \in \mathbf{Z}[\lambda]$ such that

$$g(c)\,[u, c^t f(x)]\,h(c) = \pm u^p\,[u^r, c]$$

and

$$\tilde{g}(c)\,[u, c^t \tilde{f}(c)]\,\tilde{h}(c) = \pm u^q\,[u^s, c].$$

But $c^t = 0$ because $t \geq 2$, and u is a unit. Then the last two equations imply that $[u^r, c] = 0$ and $[u^s, c] = 0$. The relative primeness of r and s shows that $[u, c] = 0$. Since the non-central element $c = e_{12}$, we see that $[u, e_{12}] = 0$. This implies that $e_{12} \in Z(R)$, which leads to a contradiction. Thus $\mathcal{A}$ which is of type (2) cannot satisfy (c_4).

Remark 3.2 : If a ring $\mathcal{A}$ is of type $(1)_r$ or (2), then by using the similar arguments as Step 3.1, with the choice of $x = e_{12} + e_{22}$ and $y = e_{12}$ in (c_5), we can show that $\mathcal{A}$ does not satisfy (c_5).

PROOF OF THEOREM 3.1 : It is enough to prove that no ring of type $(1)_l$, (2), (3) or (4) satisfies (c_2). From the proof of Step 3.1, one can observe that no ring of type $(1)_l$ and (2) satisfies (c_2).

Let R be the ring of type (3). If R satisfies (c_2), then in view of the above, R can not be of type $(1)_l$. Hence in view of Lemma 3.4, R is also right s-unital and hence s-unital. Thus, by [7, Proposition 1] we can assume that R has unity 1.

Further, it suffices to show that R has nil commutator ideal (that is, $C(R) \subseteq N(R)$). Replace x by $1 + x$ in (c_2) and combine (c_2) to get $(x+1)^p\,[(x+1)^r, y] = x^p\,[x^r, y]$, for all $x, y \in R$.

Since the rest of the proof carries over almost verbatim as that of Step 2.2, we omit the details to avoid repetition. Since R has nil commutator ideal, Lemma 2.4 shows that no ring of type (3) can satisfy (c_2).

Next, let R be a ring of type (4). Assume that $a, b \in T$ such that $[a, b] \neq 0$. Then there exist polynomials $f(\lambda), g(\lambda), h(\lambda), \tilde{f}(\lambda), \tilde{g}(\lambda), \tilde{h}(\lambda) \in \mathbf{Z}[\lambda]$ such that

$$r[a, b] = \pm(1+a)^p\,[(1+a)^r, b] = g(b)\,[a, b^t f(b)]\,h(b) = 0$$

and

$$s[a, b] = \pm(1+a)^q\,[(1+a)^s, b] = g(b)\,[a, b^t f(b)]\,h(b) = 0.$$

This implies that $[a, b] = 0$, which gives a contradiction.

Hence no ring of type $(1)_l$, (2), (3) or (4) satisfies (c_2) and so, by Proposition 3.3, R is commutative.

PROOF OF THEOREM 3.2 : The proof of this theorem follows exactly as the proof of Theorem 3.1, except at one point when Step 3.1 is used. Keeping the Remarks 3.1 and 3.2 in mind, it can be trivially shown that no ring of type $(1)_r$, (2), (3) or (4) satisfies (c_3). We omit the details of the proof to avoid repetition.

In a recent paper Komatsu *et al.*[14] proved that if R is a non-commutative ring with unity 1 satisfying the property (CH), then there exists a factor subring of R which is of type (1) or (2) (see Lemma 3.5). Thus, in view of the proof of the Theorems 3.1 and 3.2, one can prove the following :

***Theorem* 3.6** — *Let R be a left (resp. right) s-unital ring satisfying (c_4) (resp. c_5)). Suppose that R satisfies (CH). Then R is commutative (and conversely).*

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