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REMARKS ON NONCONVEX MINIMIZATION THEOREMS AND FIXED POINT THEOREMS IN COMPLETE *D*-METRIC SPACES

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In this paper, we prove nonconvex minimization theorems in complete *D*-metric spaces which was defined by Dhage. These results improve the nonconvex minimization theorems and fixed point theorems of Kada-Suzuki-Takahashi, Caristi's fixed point theorem, Ekelands's ε-variational principle, Ume's fixed point theorems and Ciric's fixed point theorem. We also give a direct simple proof of the equivalence between Caristi's fixed point theorem and Ekelands's ε-variational principle in complete *D*-metric spaces.

Key Words : Minimization Theorem; Fixed Point Theorem; *D*-metric, *w*-Distance; Lower Semicontinuous

1. INTRODUCTION

In 1992, Dhage[3] introduced a *D*-metric space and obtain fixed point theorems on this space. Recently, Kada *et al.*[5] introduced the concept of *w*-distance (weak distance) in metric spaces and improved the Caristi's fixed point theorem[1], Ekelands's ε-variational principle[4] and Takahashi's nonconvex minimization theorems[7] in complete metric spaces by using the *w*-distance.

In this paper, we prove nonconvex minimization theorems in *D*-metric space which was defined by Dhage[3]. These results improve the nonconvex minimization theorems and fixed point theorems of Kada *et al.*[5 & 7], Caristi's fixed point theorem[1], Ekelands's ε-variational principle[4], Ume's fixed point theorems[8 & 9] and Ciric's fixed point theorem[2]. We also give a direct simple proof of the equivalence between Caristi's fixed point theorem[1] and Ekeland's ε-variational principle[4] in complete *D*-metric spaces.

2. PRELIMINARIES

Throughout this paper we denote by $I\!N$ the set of all positive integers and by $I\!R$ the set of all real numbers.

Definition 2.1 [3]— Let X be any set. A *D*-metric for X is a function $D : X \times X \times X \to I\!R$ such that

(A1) $D(x, y, z) \geq 0$ for all $x, y, z \in X$, and equality holds if and only if $x = y = z$,

(A2) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$ for all $x, y, z \in X$,

(A3) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$.

If D is a D-metric for X, then the ordered pair (X, D) is called a D-metric space or the set X together with D-metric is called a D-metric space.

Definition 2.2[3] — A sequence $\{x_n\}$ of points of a D-metric space X converges to a point $x \in X$ if for an arbitrary $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n > m > n_0$, $D(x_m, x_n, x) < \varepsilon$.

Definition 2.3[3] — A sequence $\{x_n\}$ of points of a D-metric space X is Cauchy sequence if for an arbitray $\varepsilon > 0$ there exists a positive integer n_0 such that for all $p > n > m \geq n_0$, $D(x_m, x_n, x_p) < \varepsilon$.

Definition 2.4[3]— A D-metric space X is a complete D-metric space if every Cauchy sequence $\{x_n\}$ in X converges in X.

Definition 2.5[3] — Let $x_0 \in X$ and $\varepsilon > 0$ be given. Then we define the open ball $B(x_0, \varepsilon)$ in X centered at x_0 of radius of ε by

$$B(x_0, \varepsilon) = \{y \in X \mid D(x_0, y, y) < \varepsilon \text{ if } y = x_0 \text{ and } \sup_{z \in X} D(x_0, y, z) < \varepsilon \text{ if } y \neq x_0\}.$$

The collection of all open balls $\{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ define the topology on X denoted by τ. Throughout this paper we assume that the D-metric space X is equipped with the topology τ.

Definition 2.6[5] — Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous; and

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

From the Definition 2.1 and Definition 2.5, we have the following Lemma :

Lemma 2.7 — The D-metric for X is a continuous function on $X \times X \times X$ in the topology τ on X.

Lemma 2.8[5] — Let X be a metric space with metric d and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:

(i) If $p(x_0, y) \leq \alpha_n$ and $p(x_0, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;

(ii) if $p(x_0, y_n) \leq \alpha_n$ and $p(x_0, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;

(iii) if $p(x_0, x_m) \leq \alpha_n$ for $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence; and

(iv) if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$ is a Cauchy sequence.

3. Main Results

The following theorem is nonconvex minimization theorem in a complete D-metric space.

Theorem 3.1 — *Let X be a complete D-metric space and let $g : X \times X \to X$ be a continuous function such that*

$$\max \{D(x, z, g(x, z)), D(x, y, g(x, z)), D(y, z, g(x, z))\}$$
$$\leq D(x, y, g(x, y)) + D(y, z, g(y, z)) \text{ for all } x, y, z \in X.$$

for each $x \in X$, $D(x, y, g(x, y)$ is a lower semicontinuous at y in X.

Assume that $f: X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Suppose that for each $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + D(u, v, g(u, v)) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

PROOF : Suppose $\inf_{x \in X} f(x) < f(y)$ for every $y \in X$. Since f is a proper, let $u_1 \in X$ and $f(u_1) < \infty$. Then, by hypothesis, we obtained sequence $\{u_n\}$ in X satisfying the following conditions:

$$u_{n+1} \in S_n = \{x \in X : f(x) + D(u_n, x, g(u_n, x)) \leq f(u_n)\} \quad \text{... (3.1)}$$

$$\alpha_n = \inf_{x \in S_n} f(x)$$

and
$$f(u_{n+1}) < \alpha_n + \frac{1}{n}.$$

Since $f(u_{n+1}) + D(u_n, u_{n+1}, g(u_n, u_{n+1})) \leq f(u_n)$, $\{f(u_n)\}$ is nonincreasing. So, $\lim_{n \to \infty} f(u_n)$ exists. Let $\beta = \lim_{n \to \infty} f(u_n)$. We claim that $\{u_n\}$ is a D-Cauchy. From (3, 1), we have

$$D(u_n, u_{n+1}, (u_n, u_{n+1})) \leq f(u_n) - f(u_{n+1}). \quad \text{... (3.2)}$$

Let n, p, t and $r \in I\!N$. Then by hypothesis of Theorems 3.1 and (3.2),

$$D(u_n, u_{n+p}, g(u_n, u_{n+p+t})) \leq f(u_n) - f(u_{n+p+t}), \quad \text{... (3.3)}$$

$$D(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})) \leq f(u_n) - f(u_{n+p+t}) \quad \text{... (3.4)}$$

and
$$D(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t})) \leq f(u_n) - f(u_{n+p+t}). \quad \text{... (3.5)}$$

From (3.3), (3.4) and (3.5), we have

$$\begin{aligned} D(u_n, u_{n+p}, u_{n+p+t}) &\leq D(u_n, u_{n+p}, g(u_n, u_{n+p+t})) \quad \text{... (3.6)} \\ &+ D(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})) \\ &+ D(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t})) \\ &\leq 3\ \{f(u_n) - f(u_{n+p+t})\}. \end{aligned}$$

By (3.6), $\{u_n\}$ is a D-Cauchy. Let $u_n \to v_0$. By hypothesis of Theorem 3.1, Lemma 2.7 and (3.4), we have

$$D(u_n, v_0, g(u_n, v_0)) \leq f(u_n) - \beta \leq f(u_n) - f(v_0).$$

By hypothesis, there exists $v_1 \in X$ such that $v_1 \neq v_0$ and $f(v_1) + D(v_0, v_1, g(v_0, v_1)) \leq f(v_0)$. Hence, we obtain

$$f(v_1) + D(u_n, v_1, g(u_n, v_1)) \leq f(v_1) + D(v_0, v_1, g(v_0, v_1)) \quad \text{... (3.7)}$$

$$+ D(u_n, v_0, g(u_n, v_0))$$

$$\leq f(v_0) + D(u_n, v_0, g(u_n, v_0))$$

$$\leq f(u_n)$$

and hence $v_1 \in S_n$. Since

$$f(v_0) \leq f(u_{n+1}) < \alpha_n + \frac{1}{n} \leq f(v_1) + \frac{1}{n}$$

for every $n \in I\!N$, we have $f(v_0) \leq f(v_1)$. Then, $f(v_0) = f(v_1)$ and we have $D(v_0, v_1, g(v_0, v_1)) = 0$. Thus $v_0 = v_1$. This is a contradiction.

The following theorem is generalizations of the results of [5], [6], [7] and [9].

***Theorem* 3.2** — *Let (X, d) be a complete metric space with a w-distance p, let $g : X \times X \to X$ be a function, let $L : X \times X \times X \to [0, \infty)$ be a mapping and let $\varphi : X \to [0, \infty)$ be a mapping such that*

$$L(x, y, z) = \max \{p(x, y), p(x, z), \varphi(x)\}, \text{ for all } x, y, z \in X, \quad \text{... (3.8)}$$

$$\max \{L(x, z, g(x, z)), L(x, y, g(x, z)), L(y, z, g(x, z))\}$$

$$\leq L(x, y, g(x, y)) + L(y, z, g(y, z)) \text{ for all } x, y, z \in X, \quad \text{... (3.9)}$$

for each $x \in X$, $L(x, y, g(x, y))$ is a lower semicontinuous at y in X, and ... (3.10)

$$\sup \{\varphi(x) : x \in X\} < \infty. \quad \text{... (3.11)}$$

Assume that $f : X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Suppose that for each $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + L(u, v, g(u, v)) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

PROOF : Suppose $\inf_{x \in X} f(x) < f(y)$ for every $y \in X$. Since f is a proper, let $u_1 \in X$ and $f(u_1) < \infty$. Then, by hypothesis, we obtain sequence $\{u_n\}$ in X satisfying the following conditions:

$$u_{n+1} \in S_n = \{x \in X : f(x) + L(u_n, x, g(u_n, x)) \leq f(u_n)\}, \quad \text{... (3.12)}$$

$$\alpha_n = \inf_{x \in S_n} f(x)$$

and $$f(u_{n+1}) < \alpha_n + \frac{1}{n}.$$

Since $f(u_{n+1}) + L(u_n, u_{n+1}, g(u_n, u_{n+1})) \le f(u_n)$, $\{f(u_n)\}$ is nonincreasing. So, $\lim_{n \to \infty} f(u_n)$ exists. Let $\beta = \lim_{n \to \infty} f(u_n)$. We claim that $\{u_n\}$ is a Cauchy sequence. From (3.12), we have

$$L(u_n, u_{n+1}, g(u_n, u_{n+1})) \le f(u_n) - f(u_{n+1}). \quad \text{... (3.13)}$$

Let n, p, t and $r \in I\!N$. Then by hypothesis of Theorem 3.2 and (3.13),

$$L(u_n, u_{n+p}, g(u_n, u_{n+p+t})) \le f(u_n) - f(u_{n+p+t}). \quad \text{... (3.14)}$$

$$L(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})) \le f(u_n) - f(u_{n+p+t}) \quad \text{... (3.15)}$$

and $$L(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t})) \le f(u_n) - f(u_{n+p+t}). \quad \text{... (3.16)}$$

From (3.14), (3.15) and (3.16), we have

$$\begin{aligned} L(u_n, u_{n+p}, u_{n+p+t}) &\le L(u_n, u_{n+p}, g(u_n, u_{n+p+t})) \quad \text{... (3.17)} \\ &+ L(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})) \\ &+ L(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t}) \\ &\le 3\ \{f(u_n) - f(u_{n+p+t})\}. \end{aligned}$$

By (3.17) and Lemma 2.8, $\{u_n\}$ is a Cauchy sequence. Let $u_n \to v_0$. By hypothesis of Theorem 3.2 and (3.15), we have

$$L(u_n, v_0, g(u_n, v_0)) \le f(u_n) - \beta \le f(u_n) - f(v_0).$$

By hypothesis, there exists $v_1 \in X$ such that $v_1 \ne v_0$ and $f(v_1) + L(v_0, v_1, g(v_0, v_1)) \le f(v_0)$. Hence, we obtain

$$\begin{aligned} f(v_1) + L(u_n, v_1, g(u_n, v_1)) &\le f(v_1) + L(v_0, v_1, g(v_0, v_1)) \quad \text{... (3.18)} \\ &+ L(u_n, v_0, g(u_n, v_0)) \\ &\le f(v_0) + L(u_n, v_0, g(u_n, v_0)) \\ &\le f(u_n) \end{aligned}$$

and hence $v_1 \in S_n$. Since

$$f(v_0) \le f(u_{n+1}) < \alpha_n + \frac{1}{n} \le f(v_1) + \frac{1}{n}$$

for every $n \in IN$, we have $f(v_0) \leq f(v_1)$. Then, $f(v_0) = f(v_1)$ and we have $L(v_0, v_1, g(v_0, v_1)) = 0$. By hypothesis, there exists $v_2 \in X$ such that $v_2 \neq v_1$ and $f(v_2) + L(v_1, v_2, g(v_1, v_2)) \leq f(v_1)$. As in (3.18), we have $f(v_2) + L(u_n, v_2, g(u_n, v_2)) \leq f(u_n)$ and hence $v_2 \in S_n$. Therefore, we have $f(v_1) = f(v_0) \leq f(v_2) \leq f(v_1)$. This implies $L(v_0, v_1, g(v_0, v_1)) = L(v_1, v_2, g(v_1, v_2)) = 0$. By (3.8) and Lemma 2.8, we have $v_1 = v_2$. This is a contradiction.

In Theorem 3.2, putting $g(x, y) = y$ for all $x, y \in X$ and $\varphi(x) = 0$ for all $x \in X$, then we obtain the following corollary.

Corollary 3.3 [5, Theorem 1] — Let (X, d) be a complete metric space, and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-distance p on X such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v0 + p(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

The following theorem is a Caristi type fixed point theorem in a complete D-metric space.

***Theorem* 3.4** — *Let X and g be as in Theorem 3.1. Assume that $f : X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Let T be a mapping from X into itself. Suppose that*

$$f(Tx) + D(x, Tx, g(x, Tx)) \leq f(x)$$

for every $x \in X$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $D(x_0, Tx_0, g(x_0, Tx_0)) = 0$.

PROOF : Since f is proper, there exists $u \in X$ such that $f(u) < \infty$. Let

$$Y = \{x \in X : f(x) \leq f(u)\}.$$

Then, since f is lower semicontinuous, Y is closed. Hence, Y is complete D-metric space. Let $x \in Y$. Then, since

$$f(Tx) + D(x, Tx, g(x, Tx)) \leq f(x) \leq f(u),$$

we have $Tx \in Y$. So Y is invariant under T. Assume that $Tx \neq x$ for every $x \in Y$. Then, by Theorem 3.1, there exists $v_0 \in Y$ such that $f(v_0) = \inf_{x \in Y} f(x)$. Since $f(Tv_0) + D(v_0, Tv_0, g(v_0, Tv_0)) \leq f(v_0)$ and $f(v_0) = \inf_{x \in Y} f(x)$, we have $f(Tv_0) = f(v_0) = \inf_{x \in Y} f(x)$ and $D(v_0, Tv_0, f(v_0, Tv_0)) = 0$. Thus $v_0 = Tv_0$. This is a contradiction. Therefore T has a fixed point x_0 in Y. Since $f(x_0) < \infty$ and

$$f(x_0) + D(x_0, x_0, g(x_0, x_0)) = f(Tx_0) + D(x_0, Tx_0, g(x_0, Tx_0)) \leq f(x_0),$$

we have $D(x_0, x_0, g(x_0, x_0)) = 0$.

The following theorem is generalizations of the fixed point theorem of Kada, Suzuki and Thakashi[5] and Caristi's fixed point theorem [1]

***Theorem* 3.5** — *Let (X, d) be a complete metric space with a w-distance p, let $g: X \times X \to X$ be a function, let $L: X \times X \times X \to [0, \infty)$ be amapping and let $\varphi: X \to [0, \infty)$ be a mapping satisfying (3.8) ~ (3.11). Assume that $f: X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Let T be a selfmapping of X such that*

$$f(Tx) + L(x, Tx, g(x, Tx)) \leq f(x) \text{ for every } x \in X.$$

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $L(x_0, Tx_0, g(x_0, Tx_0)) = 0$.

PROOF : Using Theorem 3.2, the proof follows by the method similar to Theorem 3.4.

In Theorem 3.5, putting $g(x, y) = y$ for all $x, y \in X$ and $\varphi(x) = 0$ for all $x \in X$, then we obtain the following corollary.

Corollary 3.6 [5, Theorem 2] — Let (X, d) be a complete metric space, and let $f: X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Let T be a mapping from X into itself. Assume that there exists a w-distance p on X such that $f(Tx) + p(x, Tx) \leq f(x)$ for every $x \in X$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.

The following theorem is Ekeland type ε-variational principle in a complete D-metric space.

***Theorem* 3.7** — *Let X and g be as in Theorem 3.1. Let $f: X \to (-\infty, \infty]$ be a proper lower semicontinuous function bounded from below. Then*

(1) For each $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \leq f(u)$ and $f(z) > f(v) - D(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$;

(2) for each $\varepsilon > 0$ and $u \in X$ with $D(u, u, g(u, u)) = 0$ and $f(u) \leq \inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that $f(v) \leq f(u), D(u, v, g(u, v)) \leq 1$ and $f(z) > f(v) - \varepsilon D(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$.

PROOF : (1) Let $u \in X$ and $f(u) < \infty$ and let

$$Y = \{x \in X : f(x) \leq f(u)\}.$$

Then Y is nonempty and complete D-metric space. Hence, we may prove that there exists an element $v \in Y$ such that

$$f(z) > f(v) - D(v, z, g(v, z))$$

for every $z \in X$ with $z \neq v$. Suppose not. Then, for every $x \in Y$, there exists $z \in X$ such that $z \neq x$ and $f(z) + D(x, z, g(x, z)) \leq f(x)$. Since $f(z) \leq f(x) \leq f(u)$, $z \in X$ is an element of Y. By Theorem 3.1, there exists $x_0 \in Y$ such that $f(x_0) = \inf_{x \in Y} f(x)$. Then, there exists $x_1 \in Y$ such that $x_1 \neq x_0$ and $f(x_1) + D(x_0, x_1, g(x_0, x_1)) \leq f(x_0)$. Hence we have $f(x_1) = f(x_0)$ and $D(x_0, x_1, g(x_0, x_1)) = 0$. Thus $x_0 = x_1$. This is a contradiction. Therefore, there exists $v \in Y$ such that $f(z) > f(v) - D(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$. Let

$$Z = \{x \in X : f(x) \leq f(u) - \varepsilon D(u, x, g(u, x))\}. \quad \text{... (2)}$$

Then Z is nonempty and complete D-metric space. As in the proof of (1), we have that there exists $v \in Z$ such that $f(z) > f(v) - \varepsilon D(v, z, g(v, z))$ for every $z \in X$ with $z \neq v$. On the other hand, since $v \in Z$, we have

$$f(v) \leq f(u) - \varepsilon D(u, v, g(u, v)) \leq f(u)$$

and $$D(u, v, g(u, v)) \leq \frac{1}{\varepsilon}[f(u) - f(v)] \leq \frac{1}{\varepsilon}\left[f(u) - \inf_{x \in X} f(x)\right] \leq \frac{1}{\varepsilon}\varepsilon = 1.$$

The proof is complete.

The following theorem is generalizations of Ekeland's ε-variational principle[4] and Ekeland type theorem of Kada, Suzuki and Thakahshi[5].

***Theorem* 3.8** — *Let (X, d) be a complete metric space with a w-distance p, let $g : X \times X \to X$ be a function, let $L : X \times X \times X \to [0, \infty)$ be a mapping and let $\varphi : X \to [0, \infty)$ be a mapping satisfying (3.8) ~ (3.11). Assume that $f : X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Then the following (1) and (2) hold:*

(1) For any $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \leq f(u)$ and $f(w) > f(v) - L(u, w, g(u, w))$ for every $x \in X\, w \neq v$;

(2) for any $\varepsilon > 0$ and $u \in X$ with $L(u, u, g(u, u)) = 0$ and $f(u) \leq \inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that $f(v) \leq f(u)$, $L(u, v, g(u, v)) \leq 1$ and $f(w) > f(v) - \varepsilon L(u, w, g(v, w))$ for every $w \in X$ with $w \neq v$.

PROOF : Using Theorem 3.2, the proof follows by method similar to Theorem 3.7.

In Theorem 3.8, putting $g(x, y) = y$ for all $x, y \in X$ and $\varphi(x) = 0$ for all $x \in X$, then we obtain the following corollary.

Corollary 3.9 [5, Theorem 3] — Let (X, d) be a complete metric space, let p be a w-distance on X and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Then the following (1) and (2) hold :

(1) For any $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \leq f(u)$ and $f(w) > f(v) - p(v, w)$ for every $x \in X\, w \neq v$;

(2) for any $\varepsilon > 0$ and $u \in X$ with $p(u, u) = 0$ and $f(u) \leq \inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that $f(v) \leq f(u)$, $p(u, v) \leq 1$ and $f(w) > f(v) - \varepsilon p(v, w)$ for every $w \in X$ with $w \neq v$.

The following theorem is Kada-Suzuki-Takahashi type fixed point theorem in complete D-metric space.

***Theorem* 3.10** — *Let X, D and g be as in Theorem 3.1. Let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that*

$$D(Tx, T^2x, g(Tx, T^2x)) \leq rD(x, Tx, g(x, Tx))$$

for every $x \in X$ and that

$$\inf\{D(x, y, g(x, y)) + D(x, Tx, g(x, Tx)) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$. Then, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $D(v, v, g(v, v)) = 0$.

PROOF : let $u \in X$ and define the sequence $\{u_n\}_{n=0}^{\infty}$ satisfying the following: $u_0 = u$ and $u_n = T^n u$ for any $n \in \mathbb{N}$. Then we have, for any $n \in \mathbb{N}$,

$$D(u_n, u_{n+1}, g(u_n\, u_{n+1})) \leq rD(u_{n-1}, u_n, g(u_{n-1}, u_n))$$

$$\le r^2 D(u_{n-2}, u_{n-1}, g(u_{n-2}, u_{n-1}))$$

$$\vdots$$

$$\le r^n D(u, u_1, g(u, u_1)).$$

From hypothesis of Theorem 3.1, we have

$$D(u_n, u_{n+p}, g(u_n, u_{n+p})) \le \sum_{j=0}^{p-1} D(u_{n+j}, u_{n+j+1}, g(u_{n+j}, u_{n+j+1})) \qquad \text{... (3.19)}$$

$$\le \frac{r^n(1-r^p)}{1-r} D(u, u_1, g(u, u_1)$$

and

$$D(u_n, u_{n+p}, u_{n+p+t}) \le D(u_n, u_{n+p}, g(u_n, u_{n+p+t}))$$

$$+ D(u_n, u_{n+p+t}, g(u_n, u_{n+p+t}))$$

$$+ D(u_{n+p}, u_{n+p+t}, g(u_n, u_{n+p+t}))$$

$$\le 2\ \{D(u_n, u_{n+p}, g(u_n, u_{n+p}))$$

$$+ D(u_{n+p}, u_{n+p+t}, g(u_{n+p}, u_{n+p+t}))\}$$

$$+ D(u_n, u_{n+p+t}, g(u_n, u_{n+p+t})).$$

Thus

$$D(u_n, u_{n+p}, u_{n+p+t}) \le \frac{5r^n}{1-r} D(u, u_1, g(u, u_1)).$$

Since X is complete D-metric space, $\{u_n\}$ converges to some point $z \in X$. By hypothesis of Theorem 3.1 and (3.19),

$$D(u_n, z, g(u_n, z)) \le \frac{r^n}{1-r} D(u, u_1, g(u, u_1)).$$

Assume that $z \ne Tz$. Then, by hypothesis, we have

$$0 < \inf\ \{D(x, z, g(x, z)) + D(x, Tx, g(x, Tx)) : x \in X\}$$

$$\le \inf\ \{D(u_n, z, g(u_n, z)) + D(u_n, u_{n+1}, g(u_n, u_{n+1})) : n \in N\}$$

$$\le \inf \left\{ \frac{r^n}{1-r} D(u, u_1, g(u, u_1)) + r^n D(u, u_1, g(u, u_1)) : n \in N \right\}$$

$$= 0.$$

This is a contradiction. Therefore, we have $z = Tz$. If $v = Tv$, then

$$D(v, v, g(v, v)) = D(Tv, T^2v, g(Tv, T^2v))$$

$$\leq r\, D(v, Tv, g(v, Tv)) = rD(v, v, g(v, v)).$$

Hence $D(v, v, g(v, v)) = 0$.

The following theorem is generalizations of Ciric's fixed point theorem[2], the fixed point theorem of Kada, Suzuki and Thakashi[5] and Ume's fixed point theorem[8].

***Theorem* 3.11** — *Let (X, d) be a complete metric space with a w-distance p, let $g: X \times X \to X$ be a function, let $L: X \times X \times X \to [0, \infty)$ be mapping and let $\varphi: X \to [0, \infty)$ be a mapping satisfying (3.8) ~ (3.11). Assume that $f: X \to (-\infty, \infty]$ is a proper lower semicontinuous function bounded from below. Let T be a selfmapping of X such that*

$$L(Tx, T^2x, g(Tx, T^2x)) \leq r \cdot L(x, Tx, g(x, Tx))$$

for all $x \in X$ and that

$$inf\ \{L(x, y, g(x, y)) + L(x, Tx, g(x, Tx)) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $L(v, v, g(v, v)) = 0$.

PROOF : Using Theorem 3.2, the proofs follows by method similar to Theorem 3.10.

In Theorem 3.11, putting $g(x, y) = y$ for all $x, y \in X$ and $\varphi(x) = 0$ for all $x \in X$, then we obtain the following corollary.

Corollary 3.12 [5, Theorem 5] — Let (X, d) be a complete metric space, let p be a w-distance on X and let T be a mapping from X into itself. Suppose tha' there exists $r \in [0, 1)$ such that

$$p(Tx, T^2x) \leq rp(x, Tx)$$

for every $x \in X$ and that

$$\inf\ \{p(x, y) + p(x, Tx) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Corollary 3.13[8] — Let (X, d) be a complete metric space with a w-distance p and let T be a self-mapping of X. Suppose that there exists $r \in [0, 1)$ such that

$$p(Tx, Ty) \leq r \cdot \max\ \{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\} \quad \text{... (3.20)}$$

for every $x, y \in X$ and that

$$\inf\ \{p(x, y) + p(x, Tx) : x \in X\} > 0 \quad \text{... (3.21)}$$

for every $y \in X$ with $y \neq Ty$. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

PROOF : By Lemma 2.6 in[8], for very $x \in X$

$$\alpha(x) := \sup\ [p\,(T^i x, T^j x) \mid i, j\ I\!N \bigcup \{0\}] < \infty.$$

Define $q : X \times X \to [0, \infty)$ by $q(x, y) = \max\ \{\alpha(x), p(x, y)\}$ for every $x, y \in X$. By Lemma 3 in[5], q is a w-distance on X. Let $x \in X$. Then from (3.20),

$$q(Tx, T^2x) = \max[\alpha(Tx), p(Tx, T^2x)\} = \alpha(Tx) \leq \tau \cdot \alpha(x) = \tau \cdot q(x, Tx).$$

Let $g : X \times X \to X$ be a function such that $g(x, y) = y$ for all $x, y \in X$ and let $\varphi : X \to [0, \infty)$ be a mapping such that $\varphi(x) = 0$ for all $x \in X$. Define $D : X \times X \times X \to [0, \infty)$ by

$$L(x, y, z) = \max\{\varphi(x), q(x, y), q(x, z)\}$$

for all $x, y, z \in X$. Then those conditions are satisfied all conditions of Theorem 3.11. Therefore, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Corollary 3.14[2] — Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that

$$d\,(Tx, Ty) \leq r \cdot \max\ \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \qquad \dots (3.22)$$

for all $x, y \in X$ and some $r \in [0, 1)$. Then T has a unique fixed point.

PROOF : Since a metric d is a w-distance, (3.22) implies (3.20). By Lemma 2.5 in [8], (3.21) is satisfied. Therefore, by Corollary 3.13, the result follows.

The following theorem is a generalization of the results of [5], [6], [7] and [9].

***Theorem* 3.15** — *Let (X, d) be a complete metric space, let T be a continuous mapping from X into itself and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-distance p on X such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$., there exists $v \in X$ with $v \neq u$ and*

$$f(v) + \max\ \{p\,(Tu, v), p(Tu, Tv)\} \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

PROOF : Let $L(x, y, z) = \max\ \{p(Tx, Ty), p(Tx, Tz), p(Tx, y), p(Tx, z)\}$ for all $x, y, z \in X$, let $g(x, y) = y$ for all $x, y \in X$ and let $\varphi(x) = 0$ for all $x \in X$. Then Theorem 3.15 follows by the method similar to Theorem 3.2.

Using Theorem 3.15, we have the following corollary.

Corollary 3.16 [5, Corollary 1] — Let (X, d) be a complete metric space, let T be a continuous mapping from X into itself and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + \max\ \{d(Tu, v), d(Tu, Tv)\} \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

4. THE EQUIVALENCE RELATIONS

In this section, we study the relation between Caristi's fixed point theorem [1] and Ekeland's ε-variational principle [4] in complete D-metric spaces.

***Theorem* 4.1** — *Theorem 3.4 is equivalent to Theorem 3.7.*

PROOF : First we shall prove that Theorem 3.4 implies Theorem 3.7. Let $u \in X$ be such that $f(u) < \infty$ and let

$$Y = \{x \in X : f(x) \leq f(u)\}.$$

Then from hypothesis of Theorem 3.4 we obtain that Y is nonemtpy closed set. Hence Y is a complete D-metric space. Now we show that there exists an element $v \in Y$ such that

$$f(z) > f(v) - D(v, z, g(v, z)) \qquad \text{... (4.1)}$$

for every $z \in Y$ with $z \neq v$. Suppose not. Then, for every $x \in Y$, there exists $z \in Y$ such that $z \neq x$ and $f(z) + D(x, z, g(x, z)) \leq f(x)$. Let $T : Y \to Y$ be a mapping such that for every $x \in Y$, $Tx = z \neq x$ and $f(Tx) + D(x, Tx, g(x, Tx)) \leq f(x)$. By Theorem 3.4, there exists a $v \in Y$ such that $v = Tv$. This contradicts the definition of T. Therefore (4.1) holds. Also we show that there exists an element $v \in Y$ such that

$$f(z) > f(v) - D(v, z, g(v, z)) \qquad \text{... (4.2)}$$

for every $z \in X - Y$ with $z \neq v$. Suppose not. Then, for every $x \in Y$, there exists $z \in X - Y$ such that $z \neq x$ and $f(z) + D(x, z, g(x, z)) \leq f(x)$. This implies that $z \in Y$ and $z \notin Y$. From this contradiction we obtain that (4.2) holds. By (4.1) and (4.2) we have (1) of Theorem 3.7. Similarly we obtain (2) of Theorem 3.7.

Therefore, Theorem 3.4 implies Theorem 3.7. Next we shall prove that Theorem 3.7 implies Theorem 3.4. Suppose that Theorem 3.4 does not hold. Then we have

$$Tx \neq x \text{ and } f(Tx) + D(x, Tx, g(x, Tx)) \leq f(x). \qquad \text{... (4.3)}$$

for every $x \in X$. By (4.3) and Theorem 3.7, we obtain

$$f(Tv) + D(v, Tv, g(v, Tv)) \leq f(v) < f(Tv) + D(v, Tv, g(v, Tv)),$$

which is contradiction. Therefore Theorem 3.7 implies Theorem 3.4.

By method similar to Theorem 4.1, we have the following theorem,

***Theorem* 4.2** — *Theorem 3.5 is equivalent to Theorem 3.8.*

Using Theorem 4.2, we have the following corollary.

Corollary 4.3 — Corollary 3.6 is equivalent to Corollary 3.9.

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