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ON THE DIOPHANTINE EQUATION $X(X+1)(X+2)(X+3) = 6Y(Y+1)(Y+2)(Y+3)$

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In this paper we have proved that the only solution in positive integers of the equation in the title is $(X, Y) = (7, 4)$.

Key Words : Diophantine equation

1. INTRODUCTION AND RESULT

Several authors have studied the Diophantine equation

$$X(X+1)(X+2)(X+3) = DY(Y+1)(Y+2)(Y+3), \quad \dots (1)$$

where D is a positive nonsquare integer. For $D = 2, 3, 5$ and 7, all positive integer solutions of (1) have been obtained by Cohn[1], Ponnudural[2] Xuan[3] and Luo[4] respectively, which are listed as follows :-

If $D = 2$, then $(X, Y) = (5, 4)$

If $D = 3$, then $(X, Y) = (3, 2), (7, 5)$

If $D = 5$, then $(X, Y) = (2, 1)$

If $D = 7$, then $(X, Y) = (4, 2)$.

In 1982, Cao[5] considered the Diophantine equation

$$3X(X+1)(X+2)(X+3) = 2Y(Y+1)(Y+2)(Y+3)$$

and found that its only positive integer solution is $(X, Y) = (8, 9)$.

However, for the case $D = 6$ eq. (1) still remains open. In this paper we solve this case and obtain

Theorem — *The only solution in positive integers of the equation*

$$X(X+1)(X+2)(X+3) = 6Y(Y+1)(Y+2)(Y+3) \quad \dots (2)$$

is $(X, Y) = (7, 4)$.

2. PROOF

First we reduce (2) to

$$(X^2+3X+1)^2-6(Y^2+3Y+1)^2=-5.$$

It is easy to show that the Pellian equation $x^2-6y^2=-5$ has two classes of associated solutions with the fundamental solutions $\pm 1+\sqrt{6}$ (refer. [6]), then we have

$$(X^2+3X+1)+(Y^2+3Y+1)\sqrt{6}=\pm(x_n+y_n\sqrt{6})=\pm(1+\sqrt{6})(u_n+v_n\sqrt{6})$$

$$=+(1+\sqrt{6})(5+2\sqrt{6})^n,$$

or

$$(X^2+3X+1)+(Y^2+3Y+1)\sqrt{6}=\pm(\overline{x}_n+\overline{y}_n\sqrt{6})=\pm(-1+\sqrt{6})(u_n+v_n\sqrt{6})$$

$$=\pm(-1+\sqrt{6})(5+2\sqrt{6})^n,$$

where n is an integer and $\pm(u_n+v_n\sqrt{6})=\pm(5+2\sqrt{6})^n$ is the general solution of the Pellian equation $u^2-6v^2=1$. Since the two classes of solutions $\pm(x_n+y_n\sqrt{6})$ and $\pm(\overline{x}_n+\overline{y}_n\sqrt{6})$ are conjugate, we easily see $\overline{y}_n=y_{-n}$, so that if Y gives a positive integer solution of (2), then it must satisfy $Y^2+3Y+1=\pm y_n$, i.e.,

$$(2Y+3)^2=\pm 4y_n+5, \qquad \text{... (3)}$$

where n ranges over all integers.

Next we list the following relations which may be derived easily from the general solution of the above Pellian equation.

$$u_{n+2}=10u_{n+1}-u_n,\ \ u_0=1,\ u_1=5, \qquad \text{... (4)}$$

$$v_{n+2}=10v_{n+1}-v_n,\ \ v_0=0,\ v_1=2, \qquad \text{... (5)}$$

$$y_{n+2}=10y_{n+1}-y_n,\ \ y_0=1,\ y_1=7, \qquad \text{... (6)}$$

$$y_n=u_n+v_n, \qquad \text{... (7)}$$

$$u_{n+2km}\equiv(-1)^k u_n \pmod{u_m}, \qquad \text{... (8)}$$

$$v_{n+2km}\equiv(-1)^k v_n \pmod{u_m}, \qquad \text{... (9)}$$

$$y_{n+2km}\equiv(-1)^k y_n \pmod{u_m}, \qquad \text{... (10)}$$

$$u_{2n}=u_n^2+6v_n^2,\ \ v_{2n}=2u_nv_n.$$

Using them we can prove our conclusion.

Lemma 1 — $-4y_n+5$ is a square only for $n = 0$.

PROOF : From (6), obviously $y_n > 1$ for $n \neq 0$, then $-4y_n+5$ is negative and impossible to be a square number. When $n = 0$, $-4y_n+5 = 1^2$, as concluded.

Lemma 2 — If $n \not\equiv 0, -2 \pmod{60}$ then $4y_n+5$ is not a square.

PROOF : By (6), $2 \nmid n$ implies $y_n = \pm 2 \pmod 5$ and $4y_n+5 \equiv \pm 3 \pmod 5$, which are quadratic nonresidues modulo 5, so that $4y_n+5$ is not a square for odd n. Thus, in what follows we can restrict n to be even.

Using (6) we take modulo 89 to the sequence $\{4y_n+5\}$ and get its residue sequence with period 10 (to obtain expected period $2k$ we usually take a prime factor of υ_k as the modulo). When $n \equiv 2, 4, 6 \pmod{10}$, $4y_n+5 \equiv 14, 82, 66 \pmod{89}$ respectively, which are quadratic nonresidues modulo 89. Hence these values of n may be excluded and there remain $n \equiv 0, 8 \pmod{10}$, i.e., $n \equiv 0, 8, 10, 18, 20, 28 \pmod{30}$.

Take modulo 179 to $\{4y_n+5\}$, the residue sequence has period 30. Since $n \equiv 10, 18 \pmod{30}$ imply $4y_n+5 \equiv 104, 137 \pmod{179}$ respectively, both of which are quadratic nonresidues modulo 179, so may be excluded. Thus there remain $n \equiv 0, 8, 20, 28 \pmod{30}$, which are equivalent to $n \equiv 20, 28, 30, 38, 50, 58 \pmod{60}$.

Take modulo 97, the residue sequence of $\{4y_n+5\}$ has period 12. We can exclude $n \equiv 2, 4, 8 \pmod{12}$, since they imply $4y_n+5 \equiv 87, 83, 20 \pmod{97}$ respectively, which are quadratic nonresidues modulo 97. Hence $n \equiv 8, 20, 28, 38, 50 \pmod{60}$ may be excluded, and there remain $n \equiv 0, 30, 58 \pmod{60}$.

Finally we take modulo 92188801, then the residue sequence of $\{4y_n+5\}$ has period 40. When $n \equiv 10, 30 \pmod{40}$, then $4y_n+5 \equiv 1536325, 90652486 \pmod{92188801}$ respectively, both of which, shown by the calculations, are quadratic nonresidues modulo 92188801. Thus $n \equiv 10 \pmod{20}$, so also $n \equiv 30 \pmod{60}$, may be excluded and therefore there remain only $n \equiv 0, 58 \pmod{60}$, i.e., $n \equiv 0, -2 \pmod{60}$. The proof is complete.

Lemma 3 — If $n \equiv 0 \pmod{60}$, then $4y_n+5$ is a square only for $n = 0$.

PROOF : If $n \equiv 0 \pmod{60}$ and $n \neq 0$, then we can write $n = 2 \cdot 3 \cdot 5 \cdot 2^t \cdot k$, where $t \geq 1, 2 \nmid k$. Let

$$m = \begin{cases} 2^t & \text{if } t \equiv 0 \pmod 3 \\ 3 \cdot 2^t & \text{if } t \equiv 1 \pmod 3 \\ 5 \cdot 2^t & \text{if } t \equiv 2 \pmod 5 \end{cases},$$

then it is easy to check that $m \equiv \pm 1 \pmod 7$ and $2 | m$. Using (7) to (10) we obtain

$$y_n \equiv \pm y_{2m} \equiv \pm (u_{2m} + v_m) \equiv \pm v_{2m} \pmod{u_{2m}}$$

and $$4y_n + 5 \equiv \pm 4v_{2m} + 5 \pmod{u_{2m}}.$$

Since $2|m$ implies $u_m \equiv 1 \pmod 8$ and $v_m \equiv 0 \pmod 2$, by (11), we have

$$\left(\frac{4y_n+5}{u_{2m}}\right)=\left(\frac{\pm 4v_{2m}+5}{u_{2m}}\right)=\left(\frac{\pm 8u_m v_m+5u_m^2-30v_m^2}{u_m^2+6v_m^2}\right)=\left(\frac{10u_m^2 \pm 8u_m v_m}{u_m^2\, 6v_m^2}\right)$$

$$=\left(\frac{2u_m}{u_m^2+6v_m^2}\right)\left(\frac{5u_m \pm 4v_m}{u_m^2+6v_m^2}\right)=\left(\frac{6v_m^2}{u_m}\right)\left(\frac{u_m^2+6v_m^2}{5u_m \pm 4v_m}\right)$$

$$=\left(\frac{-1}{u_m}\right)\left(\frac{16u_m^2+6\cdot(4v_m)^2}{5u_m \pm 4v_m}\right) \text{ (since } 6v_m^2=u_m^2-1)$$

$$=\left(\frac{166}{5u_m \pm 4v_m}\right)=-\left(\frac{83}{5u_m \pm 4v_m}\right)=-\left(\frac{5u_m \pm 4v_m}{83}\right).$$

Both residue sequences of $\{5u_n \pm 4v_n\}$ modulo 83 have period 7, and $m \equiv \pm 1 \pmod 7$ imply $5u_m \pm 4v_m \equiv 17$ or $33 \pmod{83}$. Since $\left(\frac{17}{83}\right)=\left(\frac{33}{83}\right)=1$, then $\left(\frac{4y_n+5}{u_{2m}}\right)=-1$, so that $4y_n+5$ cannot be a square.

If $n = 0$, then $4y_n+5=3^2$, which completes the proof. ■

Lemma 4 — If $n \equiv -2 \pmod{60}$, then $4y_n+5$ is a square only for $n=-2$.

PROOF : If $n \equiv -2 \pmod{60}$ and $n \neq -2$, then put $n=-2+2\cdot k\cdot 3\cdot 5\cdot 2^t$, where $t \geq 1$, $2 \nmid k$. Let $m=2^t$ or $3\cdot 2^t$ or $5\cdot 2^t$ or $15\cdot 2^t$ (to be determined), then, by (10),

$$4y_n+5 \equiv -4y_{-2}+5 \equiv -111 \pmod{u_m}.$$

Since $2|m$ implies $u_m \equiv 1 \pmod 3$, we get

$$\left(\frac{4y_n+5}{u_m}\right)=\left(\frac{-111}{u_m}\right)=\left(\frac{u_m}{111}\right)=\left(\frac{u_m}{37}\right).$$

The residue sequence of $\{u_m\}$ modulo 37 has period 19. When $m \equiv \pm 1, \pm 5, \pm 6, \pm 8, \pm 9 \pmod{19}$, $u_m \equiv 5, 17, 31, 13, 22 \pmod{37}$ respectively, all the which are quadratic nonresidues modulo 37. Thus it suffices to choose m such that m is congruent to one of $\pm 1, \pm 5, \pm 6, \pm 8, \pm 9$ modulo 19. Notice that 2 is a primitive root modulo 19, the residue sequence of $\{2^t\}$ modulo 19 has period 18 with respect to t. We let

$$m=\begin{cases} 2^t & \text{if } t \equiv 0, 3, 5, 7, 8, 9, 12, 14, 16, 17 \pmod{18} \\ 3\cdot 2^t & \text{if } t \equiv 1, 4, 10, 13 \pmod{18} \\ 5\cdot 2^t & \text{if } t \equiv 2, 11 \pmod{18} \\ 15\cdot 2^t & \text{if } t \equiv 6, 15 \pmod{18} \end{cases},$$

then it is easy to check that for each case of t, m satisfies our requirement. Hence, we have

$$\left(\frac{4y_n+5}{u_m}\right)=-1,$$

so that $4y_n+5$ cannot be a square.

If $n = -2$, then $4y_n+5=11^2$, which completes our proof. ■

PROOF OF THEOREM — Consider (3). If $(2Y+3)^2 = -4y_n+5$, then, by Lemma 1, $(2Y+3)^2=-4y_0+5=1$, so $Y = -1, -2$, which only give trivial solutions of (2).

If $(2Y+3)^2=4y_n+5$, then, by lemmas 2 to 4, $n = 0, -2$ and $(2Y+3)^2=3^2$ or 11^2. The former implies $Y = 0, -3$, only give trivial solutions of (2); the latter implies $Y = 4, -7$, but only $Y = 4$ gives a positive integer solution $(X, Y) = (7, 4)$, which is the unique solution in positive integers of (2). The proof is complete. ■

Remark : Because throughout the paper we have not used the equation $X^2+3X+1=\pm x_n$, so actually we have proved that the more general equation $X(X+2)=6Y(Y+1)(Y+2)(Y+3)$ has only one positive integer solution $(X, Y)=(70, 4)$.

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