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MEROMORPHIC FUNCTIONS SHARING FOUR VALUES*

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In this paper, we proved three results on meromorphic functions that share four values and entire functions that share three values in a completely different method from the usual one previously used on this topic.

Key Words : Meromorphic Function; Shared Value; Nevanlinna Theory

1. INTRODUCTION AND RESULTS

In this paper, the term "meromorphic" will mean meromorphic in the whole complex plane. We say that two nonconstant meromorphic functions f and g share the value $c \in \hat{C}$ provided that $f(z) = c$ if and only if $g(z) = c$. We will state whether a shared value is by CM (counting multiplicities) or by IM (ignoring multiplicities). It is assumed that the reader is familiar with the usual notations and fundamental results of Nevanlinna's theory of meromorphic functions (see [8]). In particular, $s(r, f)$ will denote any quantity that satisfies $s(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of r of finite linear measure,

Nevenlinna proved the following two well-known theorems :

Theorem A [1] — *If f and g are nonconstant meromorphic functions that share five values IM, then $f \equiv g$.*

Theorem B [1] — *If f and g are distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3, a_4 CM, then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2, are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In [2], Gundersen gave a clever example which shows that one can not simply replace all the *CM*-shared values by *IM*-shared values in Theorem B. However, he has also pointed out that the hypothesis of Theorem B can be relaxed to a certain extent. He proved the following results.

Theorem C[3] — *If two nonconstant meromorphic functions f and g share two values CM and share two other values IM, then f and g share all four values CM.*

Remark : From the proof of Theorem C, it is not difficult to see that Theorem C remains valid if *CM* is replaced by "*CM*". We will state the definition of "*CM*" later in this section.

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The main open question in the theory of meromorphic functions that share four values is: If two nonconstant meromorphic functions share three values *IM* and share a fourth value *CM*, then do the functions necessarily share all four values *CM*? Mues, Gundersen and Wang proved some partial results on this open question.

For the statement of their results, we need two definitions. Let f and g be nonconstant meromorphic functions that share the value a *IM*. We denote by $\overline{N}_E\left(r, \frac{1}{f-a}\right)$ the counting function of those a-points of f and g where a is taken by f and g with the same multiplicity, counted only once without regard to multiplicity.

Definition 1 — Supposing two nonconstant meromorphic functions f and g share the value a *IM*, we define

$$\tau(a,f) = \liminf_{r \to \infty} \frac{\overline{N}_E\left(r, \frac{1}{f-a}\right)}{\overline{N}\left(r, \frac{1}{f-a}\right)},$$

if $\overline{N}\left(r, \frac{1}{f-a}\right) \not\equiv 0$, $\tau(a,f) = 1$ otherwise.

Obviously, $\tau(a,f) = \tau(a,g)$. For simplicity, we denote it by $\tau(a)$. Furthermore, it can be seen that $\tau(a) = 1$ for a *CM* shared value a.

Definition 2 — We say that f and g share the value a "*CM*", if a is shared by f and g and if

$$\overline{N}\left(r, \frac{1}{f-a}\right) - \overline{N}_E\left(r, \frac{1}{f-a}\right) = s(r,f),$$

and

$$\overline{N}\left(r, \frac{1}{g-a}\right) - \overline{N}_E\left(r, \frac{1}{g-a}\right) = s(r, g).$$

In 1989, Mues proved the following theorem.

Theorem D[4] — *Let f and g be two nonconstant meromorphic functions that share for values a_1, a_2, a_3, a_4. If a_1 is shared CM and $\tau(a_2) > \frac{2}{3}$, then f and g share all four values CM.*

Remark : It is easy to see from the proof of Theorem D that Theorem D remains to be valid if we replace sharing a_1 *CM* by sharing a_1 "*CM*".

In 1993, Wang proved the following result.

Theorem E[5] — *Let f and g be two distinct non constant meromorphic functions sharing four values. Suppose that two of these values, say a_1 and a_2 satisfy min $\{\tau(a_1), \tau(a_2)\} > \frac{4}{5}$, then f and g share all four values CM.*

Recently, Song and Chang proved the following result.

Theorem F^6 — *Let f and g be distinct nonconstant meromorphic functions that share four values* a_1, a_2, a_3, a_4. *If three values, say* a_1, a_2 *and* a_3, *satisfy* $\min_{1 \le j \le 3} \{\tau(a_j)\} > \frac{3}{4}$, *then f and g share all four values CM.*

In this paper, we shall prove the following partial results on the open question in a completely different method from the usual one previously used.[2-7]

Theorem 1 — *Let f and g be distinct nonconstant meromorphic functions that share four values* a_1, a_2, a_3, a_4. *If* $\min_{1 \le i \le 4} \{\tau(a_i)\} > \frac{2}{3}$, *then f and g share all four values CM.*

Theorem 2 — *Let f and g be distinct nonconstant meromorphic functions that share four values* a_1, a_2, a_3, a_4. *Suppose that one of these values, say* a_4, *is shared CM and that* $\min_{1 \le i \le 3} \{\tau(a_i)\} > \frac{1}{2}$, *then f and g share all your values CM.*

A natural sub-question of the above open question is: If two nonconstant entire functions share three finite values *IM*, then do the functions necessarily share all three finite values *CM*? Mues proved the following partial result on this question.

Theorem G^7 — *There does not exist two nonconstant entire functions f and g that share three finite values* a_1, a_2, a_3 *with the property that f and g have different multiplicities at any* a_i*-point* ($i = 1, 2, 3$).

Form our Theorem 2, we can immediately obtain the following result on the sub-question.

Theorem 3 — *Let f and g be distinct nonconstant entire functions that share three finite values* a_1, a_2, a_3. *If* $\min_{1 \le i \le 3} \{\tau(a_i)\} > \frac{1}{2}$, *then f and g share all three finite values CM.*

2. LEMMAS

For the proof of our theorems, we need three lemmas that are all well known.

Lemma 1[1] — Let f and g be distinct nonconstant meromorphic functions that share four values *IM*. Then we have:

$$T(r,f) = T(r,g) + s(r,f),\ T(r,g) = T(r,f) + s(r,g).$$

From it we can see that $s(r,f) = s(r,g)$. For convenience, we will denote them by $s(r)$ in the following discussion.

Lemma 2[3] — Let f and g be distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3, a_4 *IM*. Then the following statements hold :

(I) $N_1\left(r, \frac{1}{f'}\right) = s(r)$ and $N_1\left(r, \frac{1}{g'}\right) = s(r)$ where $N_1\left(r, \frac{1}{f'}\right)$ and $N_1\left(r, \frac{1}{g'}\right)$ "count" respectively only those points in $N\left(r, \frac{1}{f'}\right)$ and $N\left(r, \frac{1}{g'}\right)$ which do not occur when $f(z) = g(z) = a_i$ for some $i = 1, 2, 3, 4$.

(II) for $i = 1, 2, 3, 4$, denote by $\overline{N}_2\left(r, \frac{1}{f-a_i}\right)$ the counting function of those a_i-points where a_i is taken by f and g simultaneously with multiplicity greater than one, each point in the counting function is counted only once. Then

$$\sum_{i=1}^{4} \overline{N}_2\left(r, \frac{1}{f-a_i}\right) = s(r).$$

Lemma 3[4] — Let f and g be distinct nonconstant meromorphic functions that share four values 0, 1, ∞ and c *IM*. Then the function

$$\psi \equiv \frac{f'\, g'\, (f-g)^2}{f(f-1)\,(f-c)\, g(g-1)\,(g-c)} \qquad \text{... (2.1)}$$

is an entire function and satisfies

$$T(r, \psi) = s(r). \qquad \text{... (2.2)}$$

3. Proof of Theorem 1

Without loss of generality we may assume that f and g share 0, 1, ∞ and c *IM*. Let β_0 be the function defined by

$$\beta_0 \equiv \frac{f'}{f} - \frac{g'}{g}. \qquad \text{... (3.1)}$$

From the fundamental estimate of the logarithmic derivative, Lemma 1 and Lemma 2, we have

$$T(r, \beta_0) \le \overline{N}\left(r, \frac{1}{f}\right) - \overline{N}_E\left(r, \frac{1}{f}\right) + \overline{N}(r, f) - \overline{N}_E(r, f) + s(r). \qquad \text{... (3.2)}$$

Rewrite (2.1) as

$$\psi \frac{(f-1)\,(f-c)\,(g-1)\,(g-c)}{f'\, g'} \equiv \frac{f}{g} + \frac{g}{f} - 2. \qquad \text{... (3.3)}$$

By taking the derivative in both sides of the above identify and using it again, we deduce that

$$\left(\frac{f}{g} - \frac{g}{f}\right)\beta_0 \equiv \psi' \frac{(f-1)\,(f-c)\,(g-1)\,(g-c)}{f'\, g'}$$

$$-\ \psi \frac{(f''\, g' + f' + g'')\,(f-1)\,(f-c)\,(g-1)\,(g-c)}{(f'\, g')^2}$$

$$+ \ \psi\left[\frac{(f-c)(g-1)(g-c)}{g'} + \frac{(f-1)(g-1)(g-c)}{g'}\right]$$

$$+ \ \psi\left[\frac{(f-1)(f-c)(g-c)}{f'} + \frac{(f-1)(f-c)(g-1)}{f'}\right]. \qquad \text{... (3.4)}$$

From (3.4)

$$\frac{f'g'}{(f-1)(f-c)(g-1)(g-c)}\left(\frac{f}{g}-\frac{g}{f}\right)\beta_0$$

$$\equiv \psi' + \psi\left(\frac{f'}{f-1}+\frac{f'}{f-c}+\frac{g'}{g-1}+\frac{g'}{g-c}-\frac{f''}{f'}-\frac{g''}{g'}\right). \qquad \text{... (3.5)}$$

If $\overline{N}_E\left(r,\frac{1}{f-1}\right)=s(r)$ or $\overline{N}_E\left(r,\frac{1}{f-c}\right)=s(r)$, it can be seen that f and g share 1 or c "CM" from the assumption $\tau(1)>0$ and $\tau(c)>0$ of Theorem 1. Noting the remark of Theorem C, we can deduce that the conclusion of Theorem 1 holds.

Now we may suppose that both $\overline{N}_E\left(r,\frac{1}{f-1}\right)\neq s(r)$ and $\overline{N}_E\left(r,\frac{1}{f-c}\right)\neq s(r)$.

Let z_1 be a common simple 1-point of f and g which satisfies $\beta_0(z_1)\neq 0$ and $\psi(z_1)\neq 0$, and let f and g have the following Laurent expansions at z_1:

$$f \equiv 1 + a_1(z-z_1) + a_2(z-z_1)^2 =+ \ldots; \qquad \text{... (3.6)}$$

and

$$g \equiv 1 + b_1(z-z_1) + b_2(z-z_1)^2 + \ldots. \qquad \text{... (3.7)}$$

where a_1 and b_1 are nonzero constants. By a careful calculation, we can see that z_1 is a simple pole of both sides of identity (3.5) respectively.

Set

$$A_1 \equiv \frac{f'g'}{(f-1)(f-c)(g-1)(g-c)}\left(\frac{f}{g}-\frac{g}{f}\right)\beta_0,$$

$$A_2 \equiv \psi' + \psi\left(\frac{f'}{f-1}+\frac{f'}{f-c}+\frac{g'}{g-1}+\frac{g'}{g-c}-\frac{f''}{f'}-\frac{g''}{g'}\right).$$

From a simple calculation, we can obtain the residues of A_1 and A_2 at z_1 :

$$\text{Res}\ (A_1, z_1) = \frac{2\beta_0^2(z_1)}{(1-c)^2} \text{ and } Res\ (A_2, z_1) = 2\psi(z_1)$$

where $\beta_0(z_1) = a_1 - b_1$. It follows from (3.5) that

$$\psi(z_1)=\frac{\beta_0^2(z_1)}{(1-c)^2}. \qquad \ldots (3.8)$$

Similarly, let z_c be a common simple c-point of f and g which satisfies $\beta_0(z_c)\neq 0$ and $\psi(z_c)\neq 0$, then z_c is also a simple pole of both sides of identity (3.5) respectively. A simple calculation gives

$$\psi(z_c)=\frac{\beta_0^2(z_c)}{(1-c)^2}. \qquad \ldots (3.9)$$

If $\psi\not\equiv\dfrac{\beta_0^2}{(1-c)^2}$, we can deduce from Lemma 2 and (2.2), (3.2) that

$$\overline{N}_E\left(r,\frac{1}{f-1}\right)+\overline{N}_E\left(r,\frac{1}{f-c}\right)-s(r)\leq N\left(r,\frac{1}{\psi-\dfrac{\beta_0^2}{(1-c)^2}}\right)+s(r)$$

$$\leq 2T(r,\beta_0)+s(r)\leq 2\overline{N}\left(r,\frac{1}{f}\right)-2\overline{N}_E\left(r,\frac{1}{f}\right)+2\overline{N}(r,f)-2\overline{N}_E(r,f)+s(r). \qquad \ldots (3.10)$$

If $\beta_0(z_1)=0$, it follows from (3.8) that $\psi(z_1)=0$. Similarly, if $\beta_0(z_c)=0$, we also obtain $\psi(z_c)=0$ from (3.9). Hence from (2.2) we can deduce that (3.10) is always valid whether $\beta_0(z^*)\,\psi(z^*)=0$ or not, where z^* is z_1 or z_c.

Now we can suppose that $\psi\equiv\dfrac{\beta_0^2}{(1-c)^2}$. Since ψ is an entire function, so β_0 is also an entire function. Hence from (3.1) we obtain that f and g share 0 and ∞ *CM*. Thus the conclusion of Theorem 1 follows from Theorem *C*.

Set

$$F\equiv\frac{f-1}{f-c} \qquad \ldots (3.11)$$

and

$$G\equiv\frac{g-1}{g-c}. \qquad \ldots (3.12)$$

From the assumption of Theorem 1 and (3.11), (3.12), we deduce that F and G share 0, 1, ∞ and $\frac{1}{c}$ *IM*, and

$$T(r,F)=1+o(1))\,T(r,f),\quad T(r,G)=(1+o(1))\,T(r,g).$$

Thus we have $s(r,F)=s(r,G)=s(r)$. Furthermore, it is easy to see that

$$\tau(1,f)=\tau(0,F),\quad \tau(c,f)=\tau(\infty,F),$$

$$\tau(0,f)=\tau\left(\frac{1}{c},F\right)\quad \tau(\infty,f)=\tau(1,F).$$

Hence F and G satisfy all the conditions of Theorem 1, too. From the above discussion of this section, we can see that either F and G share 0, 1, ∞ and $\frac{1}{c}$ CM, i.e., f and g share 0, 1, ∞ and c CM or the following inequality holds :

$$\overline{N}_E\left(r,\frac{1}{F-1}\right)+\overline{N}_E\left(r,\frac{1}{F-\frac{1}{c}}\right)-s(r)$$

$$\leq 2\,\overline{N}\left(r,\frac{1}{F}\right)-2\,\overline{N}_E\left(r,\frac{1}{F}\right)+2\,\overline{N}(r,F)-2\,\overline{N}_E(r,F)+s(r),$$

which gives

$$\overline{N}_E(r,f)+\overline{N}_E\left(r,\frac{1}{f}\right)-s(r)$$

$$\leq 2\,\overline{N}\left(r,\frac{1}{f-1}\right)-2\,\overline{N}_E\left(r,\frac{1}{f-1}\right)+2\,\overline{N}\left(r,\frac{1}{f-c}\right)-2\,\overline{N}_E\left(r,\frac{1}{f-c}\right)+s(r)\,. \quad \text{... (3.13)}$$

From (3.10) and (3.13),

$$3\,\overline{N}_E(r,f)+3\,\overline{N}_E\left(r,\frac{1}{f}\right)+3\,\overline{N}_E\left(r,\frac{1}{f-1}\right)+3\,\overline{N}\left(r,\frac{1}{f-c}\right)$$

$$\leq 2\,\overline{N}(r,f)+2\,\overline{N}\left(r,\frac{1}{f}\right)+2\,\overline{N}\left(r,\frac{1}{f-1}\right)+2\,\overline{N}\left(r,\frac{1}{f-c}\right)+s(r). \quad \text{... (3.14)}$$

It follows from the assumption of Theorem 1 and (3.14) that

$$\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)=s(r),$$

which is a contradiction. The proof of Theorem 1 is completed.

4. Proof of Theorem 2 and Theorem 3

It is easy to see that Theorem 3 is a special case of Theorem 2. So we only have to prove Theorem 2. Since f and g share ∞ CM, we can obtain from (3.10) that

$$\overline{N}_E\left(r,\frac{1}{f-1}\right)+\overline{N}_E\left(r,\frac{1}{f-c}\right)\leq 2\,\overline{N}\left(r,\frac{1}{f}\right)-2\,\overline{N}_E\left(r,\frac{1}{f}\right)+s(r). \quad \text{... (4.1)}$$

Let β_1 be the function defined by

$$\beta_1\equiv\frac{f'}{f-1}-\frac{g'}{g-1}. \quad \text{... (4.2)}$$

From the Lemma of logarithmic derivative, Lemma 1 and Lemma 2, we have

$$T(r,\beta_1)\le \overline{N}\left(r,\frac{1}{f-1}\right)-\overline{N}_E\left(r,\frac{1}{f-1}\right)+s(r). \quad \text{... (4.3)}$$

From (2.1),

$$\psi\frac{fg\,(f-c)\,(g-c)}{f'\,g'}\equiv\frac{f-1}{g-1}+\frac{g-1}{f-1}-2. \quad \text{... (4.4)}$$

By taking derivative in both sides of (4.4) and using it, we get

$$\frac{f'\,g'}{fg\,(f-c)\,(g-c)}\left(\frac{f-1}{g-1}-\frac{g-1}{f-1}\right)\beta_1$$

$$\equiv\psi'+\psi\left(\frac{f'}{f}+\frac{f'}{f-c}+\frac{g'}{g}+\frac{g'}{g-c}-\frac{f''}{f'}-\frac{g''}{g'}\right). \quad \text{... (4.5)}$$

If $\psi\not\equiv\frac{\beta_1^2}{c^2}$, noting that $\tau(0)>0$ and $\tau(c)>0$, with the similar method which has been used to obtain (3.10) in section 3, we can deduce from (4.3) and (4.5) that

$$\overline{N}_E\left(r,\frac{1}{f}\right)+\overline{N}_E\left(r,\frac{1}{f-c}\right)-s(r)\le N\left(r,\frac{1}{\psi-\frac{\beta_1^2}{c^2}}\right)+s(r)$$

$$\le 2T(r,\beta_1)+s(r)\le 2\,\overline{N}\left(r,\frac{1}{f-1}\right)-2\,\overline{N}_E\left(r,\frac{1}{f-1}\right)+s(r). \quad \text{... (4.6)}$$

If $\psi\equiv\frac{\beta_1^2}{c^2}$, since ψ is an entire function, so β_1 is an entire function, too. Thus from (4.2), we can see that f and g share 1 and ∞ *CM*. It follows from Theorem C that the conclusion of Theorem 2 holds.

Set

$$\beta_c\equiv\frac{f'}{f-c}-\frac{g'}{g-c}. \quad \text{...(4.7)}$$

From the Lemma of logarithmic derivative, Lemma 1 and Lemma 2, we have

$$T(r,\beta_c)\le \overline{N}\left(r,\frac{1}{f-c}\right)-\overline{N}_E\left(r,\frac{1}{f-c}\right)+s(r). \quad \text{... (4.8)}$$

Rewriting (2.1) as

$$\psi\frac{fg\,(f-1)\,(g-1)}{f'\,g'}\equiv\frac{f-c}{g-c}+\frac{g-c}{f-c}-2. \quad \text{... (4.9)}$$

By taking derivative in both sides of (4.9) and using it, we have

$$\frac{f'g'}{fg(f-1)(g-1)}\left(\frac{f-c}{g-c}-\frac{g-c}{f-c}\right)\beta_c$$

$$\equiv \psi' + \psi\left(\frac{f'}{f}+\frac{f'}{f-1}+\frac{g'}{g}+\frac{g'}{g-1}-\frac{f''}{f'}-\frac{g''}{g'}\right). \quad \text{... (4.10)}$$

If $\psi \not\equiv \beta_c^2$, noting that $\tau(0) > 0$, and $\tau(1) > 0$, with the similar method which has been used to obtain (3.10), we can deduce from (4.8) and (4.10) that

$$\overline{N}_E\left(r,\frac{1}{f}\right)+\overline{N}_E\left(r,\frac{1}{f-1}\right)-s(r) \le N\left(r,\frac{1}{\psi-\beta_c^2}\right)+s(r)$$

$$\le 2T(r,\beta_c)+s(r) \le 2\overline{N}\left(r,\frac{1}{f-c}\right)-2\,\overline{N}_E\left(r,\frac{1}{f-c}\right)+s(r). \quad \text{... (4.11)}$$

If $\psi \equiv \beta_c^2$, since ψ is an entire function, so β_c is also an entire function. Therefore, from (4.7), we can deduce that f and g share c and ∞ CM it follows from Theorem C that the conclusion of Theorem 2 is true.

Now we suppose that

$$(\psi-\beta_c^2)\left(\psi-\frac{\beta_1^2}{c^2}\right)\left[\psi-\frac{\beta_0^2}{(1-c)^2}\right] \not\equiv 0.$$

From (4.1), (4.6) and (4.11)

$$4\,\overline{N}_E\left(r,\frac{1}{f}\right)+4\,\overline{N}_E\left(r,\frac{1}{f-1}\right)+4\,\overline{N}_E\left(r,\frac{1}{f-c}\right)$$

$$\le 2\,\overline{N}\left(r,\frac{1}{f}\right)+2\,\overline{N}\left(r,\frac{1}{f-1}\right)+2\,\overline{N}\left(r,\frac{1}{f-c}\right)+s(r). \quad \text{... (4.12)}$$

From the condition $\min\{\tau(0), \tau(1), \tau(c)\} > \frac{1}{2}$ and (4.12), we have

$$\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{f-c}\right)=s(r),$$

which contradicts Nevanlinna's second fundamental theorem.This completes the proof of Theorem 2.

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