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STRING COSMOLOGIES WITH A SCALAR FIELD

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Bianchi-type III, VIII and IX cosmologies with a self-interacting scalar field are studied in the context of cosmic strings. Physical features of these cosmologies are briefly discussed.

Key Words : String Cosmology; Scalar Field; Bianchi Models

1. INTRODUCTION

The astronomical consideration reveals that the present day universe is of the Fredmann-Robertson-Walker (FRW) type. There is no definite evidence that the early universe was of the same type. Cosmologists are of the view that the early universe might have been rather of a different type and at some stage changed over to the present day FRW Universe.

Cosmic strings have attracted a lot of consideration recently to have a plausible description of the early stage of the universe. A number of authors[1-3,19] studied the problems. The presence of strings in the early universe is a by product of Grand Unified Theories (GUT)[4&5].

In spontaneously broken Gauge theories and the spontaneous broken symmetry in elementary particle physics have given rise to an intensive study of cosmic strings. In standard bigbang cosmology, it appears that the universe may have experienced a number of phase transitions[6]. These phase transitions can produce vacuum domain structures such as domain walls, strings and monopoles.[1] These vacuum structures may act as gravitational lenses[7] and may lead to density perturbation leading to the formation of galaxies[8&9].

Later, Letelier and Verdaguer[10] studied a new model of cloud formed by massive strings in the context of general relativity. They have considered the Bianchi type I and 'Kantowski-sachs' type models as they are supposed to be reasonable representation of the early universe. They observed that during the evolution of the universe the strings disappear and the particles become important and finally end up with galaxies.

Krori *et al.*[1] studied the problem of cosmic strings taking Bianchi types II, VI, VIII and IX. They[12] also studied the strings cosmology with the help of Bianchi type I, III and V. They observed that the universe was dominated by massive strings.

Self-interacting scalar fields play a central role in studies of the inflationary cosmology. Many of the authors[13-18] studied different aspects of scalar field in the evolution of the universe and FRW models.

So, it might be of considerable interest to study the cosmic strings with a self-interacting scalar field. For the purpose we have studied the problem taking Bianchi type III, VIII and IX.

The paper has been organised as follows. In § II the Einstein field equations for massive strings are given. In § III the field equations and solutions are derived. In § IV, we conclude with a brief physical interpretations of solutions.

2. EINSTEIN EQUATIONS COUPLED TO A CLOUD OF STRINGS AND A SCALAR FIELD

The Einstein equations for a cloud of strings with scalar field are :

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-[\rho u_\mu u_\nu-\lambda x_\mu x_\nu+\partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left\{\frac{1}{2}g^{\alpha\beta}\phi_\alpha\phi_\beta+V(\phi)\right\}\Bigg], \quad \text{... (2.1)}$$

where ρ is the energy density of the cloud of strings with particles attached to them. $\rho\equiv\rho_p+\lambda$, ρ_p being the rest energy density of the particles and λ is the cloud strings' tension density. The vector u^μ describes the cloud four velocity and x^μ represents a direction of anisotropy. 'ϕ' is a scalar field and $V(\phi)$ a potential function.

We have here

$$u^\mu u_\mu=-x^\mu x_\mu=1 \text{ and } u^\mu x_\mu=0. \quad \text{... (2.2)}$$

The contracted Bianchi identity for (2.1) is equivalent to

$$\nabla_\mu(\rho u^\mu)-\lambda x'^\nu u_\nu=0, \quad \text{... (2.3a)}$$

$$\nabla_\mu(\lambda x^\mu)-\rho\dot{u}^\nu x_\nu=0 \quad \text{... (2.3b)}$$

and

$$H^\nu_\nu(\rho\dot{u}^\nu-\lambda x'^\nu)=0 \quad \text{... (2.3c)}$$

and the scalar field ϕ satisfies the equation

$$\left[\partial_\mu\phi\,\partial_\nu\phi-g_{\mu\nu}\left\{\frac{1}{2}g^{\alpha\beta}\phi_\alpha\phi_\beta+V(\phi)\right\}\right];\nu=0, \quad \text{... (2.3d)}$$

where

$$x'^\nu=x^\mu\nabla_\mu(x^\nu),\ \dot{u}^\nu=u^\mu\nabla_\mu(u^\nu) \quad \text{... (2.4)}$$

and H^μ_ν is the 'projection operator' that projects in the directions that are perpendicular to both x^μ and u^μ, i.e., in the directions that are perpendicular to both x^μ and u^μ.

$$H^\mu_\alpha=\delta^\mu_\alpha-u^\mu u_\alpha+x^\mu x_\alpha \quad \text{... (2.5)}$$

Eq. (2.3) are the evolutions for the cloud of strings and scalar field; these equations are also the integrability conditions for (2.1).

The deceleration parameter q is given by

$$q = -3\theta^2\left[\dot{\theta} + \frac{1}{3}\theta^2\right], \text{ where } \theta = u^{\alpha};\ \alpha \qquad \text{... (2.6)}$$

and ';' means the covariant derivative.

3. Field Equations and Their Integrals

A. *Bianchi Type* - III *Cosmological Model*

The line element for Bianchi type III universe is

$$ds^2 = dt^2 - A^2\,dx^2 - B^2\,e^{2x}\,dy^2 - C^2\,dz^2, \qquad \text{... (3.1)}$$

where A, B, C are functions of t only.

From eqs. (2.1), (2.2) and (3.1), we may write

$$u^{\mu} = u_{\mu} = (1, 0, 0, 0) \qquad \text{... (3.2)}$$

and x^{μ} must be taken along any of the directions $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

Without losing generality we choose x^{μ} parallel to $\frac{\partial}{\partial z}$, so that

$$x^{\mu} = (0,\ 0,\ 0,\ c^{-1}). \qquad \text{... (3.3)}$$

Applying eqs. (3.1), (3.2) and (3.3) in the Bianchi identities (2.3) one gets

$$\dot{\rho} + (\rho - \lambda)\frac{\dot{C}}{C} + \rho\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0 \qquad \text{... (3.4)}$$

and

$$\dot{\phi}\,\dot{\phi}^{\cdot} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{\phi}^2 - \dot{V}(\phi) = 0, \qquad \text{... (3.5)}$$

where a dot $(\cdot)$ denotes derivative with respect to t. Also it is to be noted that as a consequence of Einstein equations ρ and λ are functions of t alone.

The Einstein field eqs. (2.1) for the metric (3.1) are equivalent to

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{1}{A^2} = \rho + \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right), \qquad \text{... (3.6)}$$

$$\frac{\dot{B}^{\cdot}}{B} + \frac{\dot{C}^{\cdot}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \qquad \text{... (3.7)}$$

$$\frac{\dot{A}^{\cdot}}{A} + \frac{\dot{C}^{\cdot}}{C} + \frac{\dot{A}\dot{C}}{AC} = -\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \qquad \text{... (3.8)}$$

$$\frac{\dot{A}^{\cdot}}{A}+\frac{\dot{B}^{\cdot}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{1}{A^2}=\lambda-\left(\frac{1}{2}\dot{\phi}^2+V(\phi)\right) \qquad \text{... (3.9)}$$

and

$$\frac{\dot{A}}{A}-\frac{\dot{B}}{B}=0. \qquad \text{... (3.10)}$$

We have seven (7) variables viz., $A, B, C, \lambda, \rho, \phi$ and V and have six eqs. (3.5)-(3.10), so we choose $A = t^m$. A solution for (3.5)-(3.10) is given by

$$A = B = t^m,\ m > 0, \qquad \text{... (3.11)}$$

$$C = t^{2m-1}, \qquad \text{... (3.12)}$$

$$\lambda = \frac{2(1-m)(2m-1)}{t^2}-\frac{1}{t^{2m}}, \qquad \text{... (3.13)}$$

$$\rho = -\frac{2(2m-1)^2(m-1)}{(4m-3)}\frac{1}{t^2}-\frac{1}{t^{2m}}, \qquad \text{... (3.14)}$$

$$\phi = \frac{\sqrt{2(7m^2-8m+2)}}{4m-3}\log t, \qquad \text{... (3.15)}$$

$$V(\phi) = -\frac{2(2m-1)(7m^2-8m+2)}{4m-3}\frac{1}{t^2} \qquad \text{... (3.16)}$$

and

$$\rho_p = \rho - \lambda = \frac{4(m-1)^2(2m-1)}{4m-3}\frac{1}{t^2}. \qquad \text{... (3.17)}$$

The deceleration parameter q is given by

$$q = \frac{-4(4m-1)^3(m-1)}{t^4}. \qquad \text{... (3.18)}$$

The solutions (3.11) to (3.14) identicaly satisfy the equation (3.4).

B. Bianchi Type VIII and IX Cosmological Models

The line element for Bianchi type VIII ($\delta = -1$) and
Bianchi type IX ($\delta = +1$) is

$$ds^2 = dt^2-(sdx-shdz)^2-(Rdy)^2-(Rfdz)^2, \qquad \text{... (3.19)}$$

where S and R are functions of t only

and

$$f(y)=\begin{pmatrix}\sin y\\ \sin hy\end{pmatrix}, h(y)=\begin{pmatrix}\cos y\\ \cos hy\end{pmatrix} \text{ and } \delta=\begin{pmatrix}+1\\ -1\end{pmatrix}$$

From (3.19), (2.1) and (2.2) we may write

$$u^{\mu} = u_{\mu} = (1, 0, 0, 0) \quad \text{... (3.20)}$$

and choose x^{μ} parallel to $\frac{\partial}{\partial x}$, so that

$$x^{\mu} = (0, s^{-1}, 0, 0). \quad \text{... (3.21)}$$

The restriction (2.3) for the metric (3.19) are

$$\dot{\rho} + (\rho - \lambda)\frac{\dot{S}}{S} + 2\rho\frac{\dot{R}}{R} = 0 \quad \text{... (3.22)}$$

and

$$\dot{\phi}\,\ddot{\phi} + \left(\frac{\dot{S}}{S} + \frac{2\dot{R}}{R}\right)\dot{\phi}^2 - \dot{V}(\phi) = 0, \quad \text{... (3.23)}$$

where ρ and λ are functions of t only.

The field eqs. (2.1) for the metric (3.19) are equivalent to

$$2\frac{\dot{S}\dot{R}}{SR} + \frac{1}{R^2}(\dot{R}^2 + \delta) - \frac{1}{4}\frac{S^2}{R^4} = \rho + \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \quad \text{... (3.24)}$$

$$2\frac{\ddot{R}}{R} + \frac{1}{R^2}(\dot{R}^2 + \delta) - \frac{3}{4}\frac{S^2}{R^4} = \lambda + \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \quad \text{... (3.25)}$$

and

$$\frac{\ddot{S}}{S} + \frac{\ddot{R}}{R} + \frac{\dot{S}\dot{R}}{SR} + \frac{1}{4}\frac{S^2}{R^4} = -\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \quad \text{... (3.26)}$$

The solutions for the eqs. (3.23) to (3.26) are

$$R = t^m, m > 0, \quad \text{... (3.27)}$$

$$S = t^{2m-1}, \quad \text{... (3.28)}$$

$$\lambda = (-4m^2 = 6m - 3)\frac{1}{t^2} + \delta\frac{1}{t^{2m}}, \quad \text{... (3.29)}$$

$$\rho = \frac{(2m-1)(4m^2 - 6m + 3)}{3 - 4m}\frac{1}{t^2} + \delta\frac{1}{t^{2m}}, \quad \text{... (3.30)}$$

$$\phi = \sqrt{\frac{(9-14m)(2m-1)}{2(3-4m)}}\log t, \quad \text{... (3.31)}$$

$$V = -\frac{(2m-1)^2(9-14m)}{2(3-4m)}\frac{1}{t^2} \quad \text{... (3.32)}$$

and
$$\rho_p = \frac{2(1-m)(4m^2-6m+3)}{3-4m}\frac{1}{t^2}. \quad \text{... (3.33)}$$

The deceleration parameter q is given by

$$q = -\frac{4(4m-1)^3(m-1)}{t^4}. \quad \text{... (3.34)}$$

Eq. (3.27) to (3.30) identically satisfy eq. (3.22).

4. Discussion

A : Bianchi Type III Solutions

At the early era ($t \to 0$) for m in the interval $\frac{4+\sqrt{2}}{7} \le m < 1$, we have from the solutions (3.11) to (3.18), $\lambda > 0, \rho > 0, \rho_p > 0$ and ϕ real. Also we have

$$\frac{\rho_p}{|\lambda|} = \frac{2(m-1)}{3-4m}. \quad \text{... (4.1)}$$

From (4.1) we have,

$$\frac{\rho_p}{|\lambda|} > 1 \text{ for } \frac{4+\sqrt{2}}{7} \le m < \frac{5}{6}, \quad \text{.. (4.2)}$$

$$\frac{\rho_p}{|\lambda|} < 1 \text{ for } \frac{5}{6} < m < 1 \quad \text{... (4.3)}$$

and
$$\frac{\rho_p}{|\lambda|} = 1 \text{ for } m = \frac{5}{6}. \quad \text{... (4.4)}$$

Thus at the early era strings and particles exist. Further (4.2) suggests that the particles dominate over the strings for $\frac{4+\sqrt{2}}{7} \le m < \frac{5}{6}$. (4.3) implies that the strings dominate over the particles for $\frac{5}{6} < m < 1$. Also from (4.4) we have the strings and particles equally dominant for $m = \frac{5}{6}$.

At the later stage (t is very large), we have

$$\frac{\rho_p}{|\lambda|} = \frac{(m-1)^2(2m-1)}{4m-3} t^{2(m-1)}; \frac{4+\sqrt{2}}{7} \le m < 1. \quad \text{... (4.5)}$$

Thus (4.5) suggests that at the later stage the strings dominate over the particles.

Since $q > 0$ for $\frac{4+\sqrt{2}}{7} \le m < 1$, there is no inflation at any stage. Further $A = B \ne C$, so the universe is anisotropic throughout the process of evolution. For $m = \frac{4+\sqrt{2}}{7}$, ϕ and V vanish and the expansion of the universe is guided by a particular value of m only.

B. Bianchi Type VIII and IX solutions

It is observed from solutions (3.27) to (3.34) that at the early era for $\frac{1}{2} < m \leq \frac{9}{14}, \lambda < 0, \rho > 0, \rho_p > 0, q > 0$ and ϕ real for Bianchi type VIII solutions and $\lambda > 0, \rho > 0, \rho_p > 0, q > 0$ and ϕ real for Bianchi type IX solutions. Thus at the early era strings exist with negative λ but the particles exist with positive ρ_p for Bianchi type VIII solutions, and the strings and particles both exist with positive values of λ and ρ_p for Bianchi type IX solutions.

At the early era $(t \to 0)$

$$\frac{\rho_p}{|\lambda|} = \frac{2(1-m)}{3-4m} > 1 \text{ for } \frac{1}{2} < m < \frac{9}{14} \quad \ldots (4.6)$$

and at the later stage (t very large)

$$\frac{\rho_p}{|\lambda|} = \frac{2(1-m)\,(4m^2 - 6m + 3)}{3-4,} t^{2(m-1)}, \frac{1}{2} < m < \frac{9}{14} \quad \ldots (4.7)$$

(4.6) and (4.7) state that at the early era the universe is dominated by massive strings and at a later phase, the strings dominate over the particles.

In both the cases $R \neq S$ and $q > 0$, so the universe is anisotropic and non-inflationary. Moreover, for $m = \frac{9}{14}$, ϕ and V vanish and one has the solutions obtained by Krori *et al.*[11]

We would like to highlight the role of the scalar field in the above solutions. It is observed that due to the presence of the scalar field the power index m of the metric co-efficients has a range of values, when ϕ vanishes m has a fixed value. Thus we may conclude that ϕ plays a significant role in the expansion of the universe during the period of its operation.

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