Indian J. pure appl. Math., 32(1) : 69-84, January 2001

SECOND MINIMUM OF INHOMOGENEOUS INDEFINITE QUADRATIC FORMS OF SIGNATURE ± 2

MADHU RAKA AND URMILA RANI*

Centre For Advanced Study in Mathematics, Panjab University, Chandigarh, India

(*Received* 30 *October* 1998; *after Revision* 12 *May* 2000; *accepted* 26 *June* 2000)

It is proved that the minimum $C_{r+2,r} = 1/3$ of inhomogeneous indefinite quadratic forms of signature ± 2 for $r \geq 1$ is isolated and that the second minimum is 1/4. All the critical forms are also determined.

Key Words : Inhomogeneous Indefinite Quadratic Forms; Signature +2; Real Numbers; Inequality; Watson Conjecture

1. INTRODUCTION

Let $Q(x_1, x_2, \ldots, x_n)$ be a real indefinite quadratic form in n variables of determinant $D \neq 0$ and of type $(r, n-r)$. Let $C_{r,n-r}$ denote the infimum of all numbers $C > 0$ such that for any real numbers $c_1, c_2, \ldots, c_n$ there exist $(x_1, x_2, \ldots, x_n) \equiv (c_1, c_2, \ldots, c_n) \pmod 1$ satisfying

$$| Q(x_1, x_2, \ldots, x_n) | \leq (C \mid D \mid)^{1/n}. \qquad \ldots (1.1)$$

The values of $C_{r,n-r}$ are known for all n, r; for reference see Raka[15] and Dumir, Hans-Gill, Woods[9]. Davenport[5] showed that $C_{1,1}$ is not isolated. For $n \geq 3$, Vulakh[18] has shown that $C_{r,n-r}$ is isolated for rational forms; whereas for irrational forms it follows from results of Watson[19] and Margulis[13] that (1.1) is solvable for arbitrarily small values of C.

Let $n \geq 3$ and $C^{(k)}_{r,n-r}$ denote the kth successive inhomogeneous minimum. Clearly $C^{(k)}_{r,n-r} = C^{(k)}_{n-r,r}$. In this notation, Barnes[2] obtained $C^{(2)}_{2,1} = 4/15$ and $C^{(3)}_{2,1} = 1/4$. Sehmi and Dumir[17] obtained $C^{(2)}_{3,2}, C^{(3)}_{3,2}, C^{(2)}_{4,1}$. In a previous paper the authors[16] showed that $C^{(2)}_{3,1} = C^{(2)}_{1,3} = 1/4$. In this paper, we use this result and prove that $C^{(2)}_{r+2,r} = C^{(2)}_{r,r+2} = 1/4$ for all $r \geq 1$.

More precisely we prove that :

Theorem — *Let $Q(x_1, \ldots, x_n)$, $n \geq 4$ be a real indefinite quadratic form of signature ± 2 and determinant $D \neq 0$. Then given ann real numbers $c_1, \ldots, c_n$ there exist $(x_1, \ldots, x_n) \equiv (c_1, \ldots, c_n)$ (mod 1) such that*

*This paper forms a part of the Ph.D. thesis, Panjab University, written under the supervision of Dr. Madhu Raka, Research supported by C.S.I.R is gratefully acknowledged.

$$| Q(x_1, \dots, x_n) | \leq (|D|/4)^{1/n} \quad \dots (1.2)$$

except when Q is equivalent to $\rho Q_1, \rho \neq 0$ and $(c_1, \dots, c_n)$ is equivalent to $P_1 = (0, \dots, 0, 1/2, 1/2)$ (mod 1) where

$$Q_1 = x_1 x_2 + \dots + x_{n-3} x_{n-2} + x_{n-1}^2/2 + 3x_n^2/2.$$

Further equality holds in (1.2) if and only if Q ~ $\rho Q_i, \rho \neq 0$, i = 2, 3, 4 and $(c_1, \dots, c_n)$ is equivalent to P_i (mod 1) where

$$Q_2 = x_1 x_2 = \dots + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2, P_2 = (0, \dots, 0, 1/2, 1/2)$$

$$Q_3 = \left(x_1 + \frac{1}{2} x_2 \right) x_2 + x_3 x_4 + \dots + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2,$$

$$P_3 = (1/2, 0, \dots, 0, 1/2, 1/2)$$

$$Q_4 = \left(x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_{n-1} \right) x_2 + x_3 x_4 + \dots + x_{n-3} x_{n-2} + x_{n-1}^2/2 + 2x_n^2,$$

$$P_4 = (1/2), 0, \dots, 0, 1/2).$$

The symmetric inhomogeneous minimum for Q_1 can be easily seen to be 1/3 which is attained at the point P_1.

It follows from [9] that the second minimum for symmetric inhomogeneous indefinite quadratic forms in n variables of signature $8k \pm 2, k \in \mathbb{Z}$, is also 1/4.

2. PROOF OF THEOREM — We prove the result by induction on n. For n = 4, the result has been proved in [16]. Therefore, we can suppose that $n \geq 6$ and the result true for n – 2 variables. If Q is an incommensurable form then the result follows from Margulis[13] and Watson[19]. Therefore, we need to consider only rational forms. Rational forms in $n \geq 5$ variables, by Meyer's theorem, are necessarily zero forms.

Let $Q(x_1, \dots, x_n)$, $n \geq 6$, be a rational indefinite quadratic form of signature 2 and determinant $D \neq 0$. (We can assume without loss of generality that signature of Q is +2)

Let $d = (|D|/4)^{1/n}$.

If for any real number $c_1, \dots, c_n$ there exist $(x_1, \dots, x_n) \equiv (c_1, \dots, c_n)$ (mod 1) satisfying

$$\alpha < Q(x_1, \dots, x_n) < \beta, \quad \dots (2.1)$$

then we say that (2.1) is solvable.

Lemma 1 — Let α, β, γ be real numbers, $\gamma > 1/4$. Let n be an integer such that $n < 2\gamma \leq n + 1$. Then for any x_0, there exists $x \equiv x_0$ (mod 1) such that

$$| (x + \alpha)^2 + \beta | < \gamma \quad \dots (2.2)$$

provided

$$-(n^2/4+\gamma)<\beta<\gamma-1/4. \qquad \text{... (2.3)}$$

This is a result of Davenport[6].

Lemma 2 — Let $f(x_1, \ldots, x_n), n \geq 3$ be a zero form of determinant $\Delta \neq 0$. Let α, β be real numbers satisfying

$$\beta-\alpha>2\,|\,\Delta\,|^{1/n}. \qquad \text{... (2.4)}$$

Then

$$\alpha<f(x_1, \ldots, x_n)<\beta. \qquad \text{... (2.5)}$$

This is a result of Jackson[11].

Lemma 3 — Let Q represent a number $\mu \neq 0$ with $|\mu|<2d/3$ if $n \geq 8$ and $|\mu|<d/2$ if $n = 6$, then the inequality (1.2) is solvable with strict inequality.

PROOF : Without loss of generality we can suppose that Q represents μ primitively. On replacing Q by an equivalent form we can suppose that Q (1, 0, ..., 0) = μ and write

$$Q\,(x_1, \ldots, x_n)=\mu\,(x_1+\ldots.)^2+Q_{n-1}\,(x_2, \ldots, x_n),$$

where Q_{n-1} is a rational indefinite form in $(n-1) \geq 5$ variables and so by Meyor's Theorem, is a zero form. By homogeneity we can suppose that $\mu = \pm 1$, so that $d > 3/2$ if $n \geq 8$ and $d > 2$ if $n = 6$. Let m be an integer such that $m<2d \leq m+1$, so that $m \geq 3$ if $n \geq 8$ and $m \geq 4$ if $n = 6$.

Suppose first that $\mu = 1$.

By Lemma 1, the inequality

$$|\,(x_1+\ldots)^2+Q_{n-1}\,(x_2, \ldots, x_n)\,|<d$$

is solvable if we can solve

$$-(m^2/4+d)<Q_{n-1}\,(x_2, \ldots, x_n)<d-1/4. \qquad \text{... (2.6)}$$

By Lemma 2, (2.6) is solvable if

$$2d+(m^2-1)/4>2\,(|\,D\,|)^{1/(n-1)}=2(4d^n)^{1/(n-1)} \qquad \text{... (2.7)}$$

or if
$$f(d)=[2d+(m^2-1)/4]\,d^{-n/(n-1)}>2^{(n+1)/(n-1)} \qquad \text{... (2.8)}$$

$f(d)$ is a decreasing function of d and $d \leq (m+1)/2$, therefore, $f(d) \geq f((m+1)/2) = (m+3)\,2^{-(n-2)/(n-1)}\,(m+1)^{-1/(n+1)} = g(m)$. (say)

One can easily see that for $n=6, g(m) \geq g(4)>2^{7/5}$ and for $n \geq 8, g(m) \geq g(3) > 2^{(n+1)/(n-1)}$. This proves (2.8) and hence the Lemma for $\mu = 1$.

Now suppose that $\mu = -1$. By Lemma 1,

$$|-(x_1+\ldots.)^2+Q_{n-1}\,(x_2, \ldots, x_n)\,|<d$$

is solvable if we can solve

$$-(d-1/4) < Q_{n-1}(x_2, \ldots, x_n) < d + m^2/4. \qquad \ldots (2.9)$$

Applying Lemma 2, we find that (2.9) is solvable if we have inequality (2.7), which has been verified above. This completes the proof of the Lemma.

Lemma 4 — Let $\alpha, \beta, \gamma, a_2, c_1, c_2$ be any real numbers with $\gamma > 0$. Then there exist $(x_1, x_2) \equiv (c_1, c_2) \pmod 1$ such that

$$0 < (x_1 + a_2 x_2 + \alpha) x_2 + \beta < \gamma \qquad \ldots (2.10)$$

provided $c_2 \not\equiv 0 \pmod 1$ and $\gamma > 1/2$

or $\qquad c_2 \equiv 0 \pmod 1$ and $\gamma > 1$.

Proof is trivial.

Following Birch's reduction [3, Lemma 12], we can suppose, that either

$$Q(x_1, \ldots, x_n) = (x_1 + a_2 x_2 + \ldots) x_2 + m_2 (x_3 + b_4 x_4 + \ldots) x_4 + \ldots$$

$$+ m_k (x_{2k-1} + \ldots) x_{2k} + \phi (x_{2k+1}, \ldots, x_{2k+4}) \qquad \ldots (2.11)$$

where $n = 2k + 4, k \geq 1, m_i$ are positive integers and ϕ is a non zero form of type (3, 1) with determinant $\Delta, |\Delta| = 2^{2k}, D/m_2^2 m_3^2 \ldots m_k^2$ or

$$Q(x_1, \ldots, x_n) = (x_1 + a_2 x_2 + \ldots) x_2 + m_2 (x_3 + b_4 x_4 + \ldots) x_4 + \ldots) x_4 + \ldots$$

$$+ m_{k+1} (x_{2k+1} + \ldots) x_{2k+2} + \psi (x_{2k+2}, x_{2k+4}) \qquad \ldots (2.12)$$

where $n = 2k + 4$, $k \geq 1$ and ψ is a positive definite binary quadratic form with determinant $4^{k+2} d^n / m_2^2 \ldots m_{k+1}^2$; and

$$-1/2 < a_i \leq 1/2 \text{ for each } i. \qquad \ldots (2.13)$$

Remark 1 : Following Watson[20] [Page 21], we can suppose, for the form (2.12), that

(*i*) If $a_2 = 0$ then $a_i = 0$ for each i,

(*ii*) $a_{2r-1} = 0$ for $2 \leq r \leq k+1$, and

(*iii*) If $b_4 = 0$ then $a_4 = 0$.

Now (1.2) is solvable if

$$0 \leq Q(x_1, \ldots, x_n) + d \leq 2d, \qquad \ldots (2.14)$$

is solvable.

We consider the choices (2.11) and (2.12) separately.

2.1 When Q is of the form (2.11)

We need the following result which follows from Oppenheim[14]

Lemma 5 — If $\phi(x, y, z, t)$ is a non zero indefinite quaternary quadratic form of type (3,1) and determinant Δ, then ϕ represents α number a such that

$$0 < |a| \leq (4|\Delta|/7)^{1/4}. \qquad \ldots (2.15)$$

Further if $\phi \sim \rho\phi' = \rho(x^2 + y^2 + z^2 - t^2 - xt - yt - zt)$ then ϕ represents a satisfying

$$0 < |a| \leq (4|\Delta|/15)^{1/4}. \qquad \ldots (2.16)$$

Lemma 6 — Let α, β, γ be real numbers with $\gamma > 1$. Let n be an integer such that $n < \gamma \leq n + 1$. Then for any real number x_0

(i) there exists $x \equiv x_0 \pmod 1$ satisfying

$$0 < (x + \alpha)^2 + \beta < \gamma$$

provided

$$-n^2/4 < \beta < \gamma - 1/4 \text{ and} \qquad \ldots (2.17)$$

(ii) there exists $x \equiv x_0 \pmod 1$ satisfying

$$0 < -(x + \alpha)^2 + \beta < \gamma$$

provided

$$1/4 < \beta < n^2/4 + \gamma. \qquad \ldots (2.18)$$

This follows from Dumir[7 & 8].

Lemma 7 — If Q is of the form (2.11) and $c_2 \equiv 0 \pmod 1$ then (1.2) is solvable with strict inequality.

PROOF : By Lemma 4, (2.14) and hence (1.2) is solvable with strict inequality if $2d > 1$ i.e., $d > 1/2$.

Let therefore $d \leq 1/2$.

Here $Q(x_1, \ldots, x_n) = (x_1 + \ldots)x_2 + Q_{n-2}$,

where $Q_{n-2} = m_2(x_3 + \ldots.)x_4 + \ldots + m_k(x_{2k-1} + \ldots.)x_{2k} + \phi$ is a form of signature 2 in $(n - 2)$ variables with determinant $4|D|$ and ϕ is a non zero quaternary form; therefore by induction hypothesis there exist $(x_3, \ldots, x_n) \equiv (c_3, \ldots, c_n) \pmod 1$ satisfying

$$|Q_{n-2}(x_3, \ldots, x_n)| < (4|D|/4)^{1/(n-2)} = (4d^n)^{1/(n-2)}$$

Now take $x_2 = 0$ and choose $x_1 \equiv c_1 \pmod 1$ arbitrarily so that

$$|Q_n| = |Q_{n-2}| < (4d^n)^{1/(n-2)} \leq d.$$

Thus (1.2) follows with strict inequality.

Lemma 8 — If $c_2 \not\equiv 0 \pmod 1$ and $n \geq 8$, then (2.14) and hence (1.2) is solvale with strict inequality.

PROOF : By Lemma 5, ϕ and hence Q represents a such that

$$0 < |a| \leq (4\,\Delta|/7)^{1/4}.$$

If $|a| < 2d/3$, then the result follows from Lemma 3 and hence we can suppose that

$$2d/3 \leq |a| \leq (4\,|\,\Delta\,|/7)^{1/4} = (4^{k+2}\, d^{\,n}/7m_2^2 \,....\, m_k^2)^{1/4}. \qquad ...\ (2.19)$$

From (2.19) we get

$$2d \geq 2\,(7m_2^2 \ldots m_k^2/3^4\, 4^k)^{1/(n-4)} \geq 2(7/3^4\, 4^{(n-4)/2})^{1/(n-4)}$$

$$= (7/3^4)^{1/(n-4)} \geq (7/3^4)^{1/4} > 1/2$$

for $n \geq 8$. But then (2.14) is solvable with strict inequality by Lemma 4. Hence the Lemma.

Lemma 9 — If $n = 6$, $c_2 \not\equiv 0 \pmod 1$ and

$$\phi \sim \rho\,\phi' = \rho\,(x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_3 x_6 - x_4 x_6 - x_5 x_6)$$

then (1.2) is solvable with strict inequality.

PROOF : Here

$$Q = (x_1 + a_2 x_2 + \ldots)\, x_2 + \rho\,(x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_3 x_6 - x_4 x_6 - x_5 x_6).$$

Since Q represents ρ, so by Lemma 3, if $0 < \rho < d/2$ then (1.2) is solvable with strict inequality. Let therefore

$$d/2 \leq \rho = (64\, d^6/7)^{1/4}. \qquad ...\ (2.20)$$

Also if $2d > 1/2$, then (2.14) is solvable by Lemma 4.

So we have from (2.20) that

$$\sqrt{7}/32 \leq d \leq 1/4. \qquad ...\ (2.21)$$

Q can be written as

$$Q = \rho\left(x_3 - \frac{x_6}{2} + \frac{a_3 x_2}{2\rho}\right)^2 + \rho\left(x_4 - \frac{x_6}{2} + \frac{a_4 x_2}{2\rho}\right)^2$$

$$+\ \rho\left(x_5 - \frac{x_6}{2} + \frac{a_5 x_2}{2\rho}\right)^2 - \frac{7}{4}\rho x_6^2 + (x_1 + a_2' x_2 + a_6' x_6)\, x_2$$

where a_2' and a_6' are some real numbers.

From (2.20) and (2.21) we have

$$2d/\rho = (7/4d^2)^{1/4} \geq (28)^{1/4} > 2.3. \quad \text{... (2.22)}$$

Let p_1 be an integer such that $p_1 < 2d/\rho \leq p_1 + 1$, so that $p_1 \geq 2$.

Now there exist $x_i \equiv c_i \pmod 1$, $1 \leq i \leq 6$, satisfying

$0 < Q/\rho + d/\rho < 2d/\rho$, if by Lemma 6, we can find $x_i \equiv c_i \pmod 1$, i = 1, 2, 4, 5 and 6 such that,

$$-p_1^2/4 < (x_4 - ....)^2 + (x_5 - ....)^2 + \frac{1}{\rho}(x_1 + ...)\, x_2 - 7x_6^2/4 + d/\rho < 2d/\rho - 1/4$$

i.e., if

$$0 < (x_4 - ...)^2 + (x_5 - ...)^2 + \frac{1}{\rho}(x_1 + ...)\, x_2 - 7x_6^2/4 + p_1^2/4 + d/\rho \; < \; 2d/\rho + (p_1^2 - 1)/4. \quad \text{... (2.23)}$$

Let p_2 be an integer such that

$$p_2 < 2d/\rho + (p_1^2 - 1)/4 \leq p_2 + 1.$$

From (2.22) and using $p_1 \geq 2$, we have $p_2 \geq 3$. Let p_3 be an integer such that

$$p_3 < 2d/\rho + (p_1^2 - 1)/4 + (p_2^2 - 1)/4 \leq p_3 + 1.$$

Then since $p_1 \geq 2, p_2 \geq 3$, we find from (2.22) that $p_3 \geq 5$. Applying the same Lemma 6 twice we find that (2.23) is solvable if there exist $(x_1, x_2, x_6) \equiv (c_1, c_2, c_6) \pmod 1$ such that

$$0 < (x_1 + a_2' x_2 + a_6' x_6)\, x_2/\rho - 7x_6^2/4 + p_1^2/4 + p_2^2/4 + p_3^2/4 + d/\rho$$

$$< \; 2d/\rho + (p_1^2 - 1)/4 + (p_2^2 - 1)/4 + (p_3^2 - 1)/4 = \delta \text{ (say)} \quad \text{.... (2.24)}$$

Choose $x_6 \equiv c_6 \pmod 1$ arbitrarily, $x_2 \equiv c_2 \pmod 1$ such that

$0 < | x_2 | \leq 1/2$ and then choose $x_1 \equiv c_1 \pmod 1$ to satisfy

$$0 < (x_1 + ...)\, x_2/\rho - 7x_6^2/4 + p_1^2/4 + p_2^2/4 + p_3^2/4 + d/\rho \; \leq | x_2 |/\rho \leq 1/2\,\rho.$$

Now (2.24) is satisfied if $\delta > 1/2\,\rho$. Since $p_1 \geq 2, p_2 \geq 3$ and $p_3 \geq 5, \rho = (64\, d^6/7)^{1/4}$ and $d \geq \sqrt{7}/32$ we find that

$$\delta\rho \geq 2d + 35\,\rho/4 > 1/2.$$

Thus (2.24) and hence (1.2) is solvable with strict inequality.

Lemma 10 — If $n = 6$, $c_2 \not\equiv 0 \pmod 1$, Q is of the form (2.11) and $\phi \sim \rho\phi'$ then (1.2) is solvable with strict inequality.

PROOF : Since $\phi \sim \rho\phi'$, by Lemma 5, ϕ and hence Q represents a satisfying

$$0 < |a| \leq (4|\Delta|/15)^{1/4} = (64d^6/15)^{1/4}.$$

If $0 < |a| < d/2$, then (1.2) follows from Lemma 3. Therefore, we can suppose that

$$d/2 \leq |a| \leq (64\, d^6/15)^{1/4}. \qquad \text{... (2.25)}$$

Also if $2d > 1/2$, then by Lemma 4, (2.14) and hence (1.2) is solvable with strict inequality. Thus we have from (2.25),

$$\sqrt{15}/32 \leq d \leq 1/4. \qquad \text{... (2.26)}$$

Further, if $|a| \geq 2d/3$, we have

$$\sqrt{15}/18 \leq d \leq 1/4. \qquad \text{... (2.27)}$$

Case 1 — $a > 0$.

Since ϕ represents a, we can write

$$\phi(x_3, \ldots, x_6) = a(x_3 + \ldots)^2 + \xi(x_4, x_5, x_6),$$

where ξ is a non zero ternary quadratic form of type (2.1). Thus

$$Q = (x_1 + a_2 x_2 + a_3 x_3 + \ldots)\, x_2 + a\,(x_3 + \ldots)^2 + \xi(x_4, x_5, x_6)$$

$$= a\,(x_3 + a_3 x_2/2a + \ldots)^2 + (x_1 + a_2' x_2 + a_4' x_4 + \ldots)\, x_2 + \xi(x_4, x_5, x_6),$$

where a_2', a_4', a_5', a_6' are some real numbers.

Let p be an integer such that $p < 2d/a \leq p + 1$. Since from (2.25) and (2.26)

$$4 \geq 2d/a \geq (15/4d^2)^{1/4} \geq (60)^{1/4} > 2,$$

we have $p = 2$ or 3. Now by Lemma 6, (2.14) and hence (1.2) is solvable with strict inequality if there exist $x_i \equiv c_i \pmod 1$, $i = 1, 2, 4, 5$ and 6 such that

$$0 < F = (x_1 + a_2' x_2 + \ldots)\, x_2 + \xi(x_4, x_5, x_6) + p^2 a/4 + d \;<\; 2d + (p^2 - 1)\, a/4. \qquad \text{... (2.28)}$$

Since $c_2 \not\equiv 0 \pmod 1$, by Lemma 4, we find that (2.28) is solvable if

$$2d + (p^2 - 1)\, a/4 > 1/2. \qquad \text{... (2.29)}$$

If $p = 2$, we have $a \geq 2d/3$, then (2.29) follows using (2.27).

If $p = 3$, then $a \geq d/2$ and so $2d + 2a \geq 3d > 1/2$ if $d > 1/6$.

Thus (2.28) and hence (1.2) is solvable unless

$$p = 3 \text{ and } \sqrt{15}/32 \leq d \leq 1/6. \qquad \text{... (2.30)}$$

$\xi(x_4, x_5, x_6)$ is a non-zero ternary form of type (2.1), therefore from a result of Barnes[1], it represents a number b such that

$$0 < b \leq (4 \,|\det \xi|/3)^{1/3} = (64\, d^6/3a)^{1/3} \qquad \text{... (2.31)}$$

and we can write

$$\xi(x_4, x_5, x_6) = b(x_4 + \dots)^2 + \eta(x_5, x_6),$$

where $\eta(x_5, x_6)$ is a binary quadratic form. We can rewrite

$$F = b\,(x_4 + a_4' x_2/2b + \dots)^2 + (x_1 + a_2'' x_2 + a_5'' x_5 + a_6'' x_6)\, x_2$$

$$+\ \eta(x_5, x_6) + p^2 a/4 = d.$$

Let q be an integer such that

$$q < (2d + 2a)/b \leq q + 1. \qquad \text{... (2.32)}$$

Using (2.31), (2.30) and $a \geq d/2$, we find that $q \geq 2$ and infact $q \geq 3$ if $d < \sqrt{3/128}$. Now working as above we can see that (2.28) is solvable if

$$d_1 = 2d + 2a + (q^2 - 1)\, b/4 > 1/2. \qquad \text{... (2.33)}$$

Since $b \geq (2d + 2a)/(q + 1)$ and $a \geq d/2$ we find that $d_1 \geq (2d + 2a)\,(q + 3)/4 \geq 3d\,(q + 3)/4$ which exceeds 1/2 for $q \geq 3$ and $d > \sqrt{15}/32$ or if $q = 2$ and $d \geq \sqrt{3/128}$. This proves (2.33) and hence (1.2) is solvable with strict inequality in this case.

Case (ii) — $a < 0$.

Let $a = -a'$ $(a' > 0)$. Since ϕ represents $-a'$, we can write

$$\phi(x_3, \dots, x_6) = -a'\,(x_3 + \dots)^2 + \xi(x_4, x_5, x_6),$$

where ξ is a positive definite ternary quadratic form.

Therefore, by a result of Gauss and Seeber, see Cassels [4, Thm 3, Page 33], it represents a number b' such that

$$0 < b' \leq (2\,|\det \xi|)^{1/3} = (32\, d^6/a')^{1/3}.$$

So we can write

$$\xi(x_4, x_5, x_6) = b'\,(x_4 + \dots)^2 + \eta(x_5, x_6),$$

where $\eta(x_5, x_6)$ is a positive definite binary quadratic form.

Further by a classical result of Lagrange, we can write

$$\eta\,(x_5, x_6) = c(x_5 + \lambda\, x_6)^2 + tx_6^2,$$

where $0 < c < (4\,|\det \eta\,|/3)^{1/2} = (64\, d^6/3a'\, b')^{1/2}$,

λ and t are some real numbers. Thus

$$Q = (x_1 + a_2\, x_2 + a_3 x_3 + \ldots)\, x_2 - a'\,(x_3 + ..)^2 + b'\,(x_4 + ..)^2 + c\,(x_5 + \ldots)^2 + tx_6^2$$

$$= -a'\,(x_3 + \ldots)^2 + b'\,(x_4 + \ldots)^2 + c(x_5 + \ldots)^2 + (x_1 + a_2{}'\, x_2 + a_6{}'\, x_6)\, x_2 + tx_6^2.$$

Now the proof can be completed by proceeding as in case (i).

2.2 When Q is of the Form (2.12)

Here we have

$$Q = (x_1 + \ldots)\, x_2 + m_2\,(x_3 + \ldots)\, x_4 + \ldots + m_{k+1}\,(x_{2k+1} + \ldots.)\, x_{2k+2} + \psi\,(x_{2k+3}, x_{2k+4})$$

where ψ is a positive definite binary quadratic form of determinant $2^n\, d^n/m_2^2 \ldots. m_{k+1}^2$. Further, we can write

$$\psi = A\,(x_{2k+3} + \lambda\, x_{2k+4})^2 + tx_{2k+4}^2,$$

where $0 < A \le \sqrt{4\,|\det \psi|/3}$, $0 \le \lambda \le 1/2$ and $t > 0$.

If $0 < A < d/2$, (1.2) is solvable by Lemma 3, as ψ and hence Q represents A. Let therefore

$$d/2 \le A \le (4\,|\det \psi|/3)^{1/2} = (2^{n+2}\, d^n/3m_2^2 \ldots m_k^2)^{1/2}. \qquad \ldots (2.34)$$

This gives

$$2d \ge (3m_2^2 \ldots m_k^2/64)^{1/(n-2)} \ge (3/64)^{1/(n-2)}. \qquad \ldots (2.35)$$

Lemma 11 — If $c_2 \not\equiv 0 \pmod 1$ and Q is of the form (2.12), then (1.2) is solvable with strict inequality.

PROOF : From (2.35) we find that $2d > 1/2$ if $n \ge 8$. Then (2.14) is solvable by Lemma 4. Let now

$$n = 6,\ (3/64)^{1/4} \le 2d \le 1/2. \qquad \ldots (2.36)$$

$$Q = (x_1 + a_2\, x_2 + \ldots)\, x_2 + m_2\,(x_3 + b_4\, x_4 + \ldots)\, x + A\,(x_5 + \lambda\, x_6)^2 + tx_6^2$$

$$= A\left(x_5 + \frac{a_5\, x_2}{2A} + \frac{m_2\, b_5}{2A}\, x_4 + \lambda x_6\right)^2 + (x_1 + a_2'\, x_2 + \ldots)\, x_2 + m_2\,(x_3 + \ldots)\, x_4 + t\, x_6^2.$$

Now by Lemma 6, (2.14) and hence (1.2) is solvable with strict inequality if there exist $x_i \equiv c_i \pmod 1$, $i = 1, 2, 3, 4, 6$, such that

$$0 < (x_1 + a_2' x_2 + \ldots) x_2 + m_2 (x_3 + \ldots) x_4 + t x_6^2 + d + m^2 A/4$$

$$< 2d + (m^2 - 1) A/4, \qquad \ldots. (2.37)$$

where m is an integer such that $m < 2d/A \leq m + 1$. From (2.34) and (2.36), we have

$$4 \geq 2d/A \geq (3/64d^4)^{1/2} \geq (3.4^4/64)^{1/2} > 3.$$

Therefore, $m = 3$. Now by Lemma 4, (2.37) and hence (1.2) is solvable with strict inequality if

$$2d + (m^2 - 1) A/4 = 2d + 2A > 1/2$$

which can easily be verified for $A \geq d/2$ and $2d > (3/64)^{1/4}$.

Lemma 12 — Let $c_2 \equiv 0$ (nod 1), Q of the form (2.12). If we write $Q = (x_1 + a_2 x_2 + \ldots) x_2 + Q_{n-2}(x_3, \ldots, x_n)$, then (1.2) is solvable except when $Q_{n-2} \sim \rho F_1$ and $(c_3, \ldots, c_n)$ is equivalent to

$$P_1' = (0, \ldots, 0, 1/2, 1/2) \text{ where } \rho \neq 0 \text{ and}$$

$$F_1 = x_3 x_4 + \ldots. + x_{n-3} x_{n-2} + x_{n-1}^2/2 + 3x_n^2/2.$$

Further (1.2) is solvable with strict inequality unless $d = 1/2$, $Q_{n-2} \sim \rho F_2, \rho F_3$ or ρF_4 and $(c_3, \ldots, c_n)$ is equivalent to P_2', P_3', or P_4' respectively, where $\rho \neq 0$ and

$$F_2 = x_3 x_4 + \ldots. + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2 + P_2' = (0, \ldots., 0, 1/2, 1/2),$$

$$F_3 = \left(x_3 + \frac{1}{2} x_4 \right) x_4 + x_5 x_6 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2,$$

$$P_3' = (1/2, 0, \ldots., 0, 1/2, 1/2),$$

$$F_4 = \left(x_3 + \frac{1}{2} x_4 + \frac{1}{2} x_{n-1} \right) x_4 + x_5 x_6 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2/2 + 2x_n^2,$$

$$P_4' = (1/2, 0, \ldots, 0, 1/2).$$

PROOF : By Lemma 4, we can assume $2d \leq 1$. Q_{n-2} is an indefinite quadratic form in $(n - 2)$ variables of determinant $4|D|$, in absolute value. By induction hypothesis there exist $x_i \equiv c_i \pmod 1$, $3 \leq i \leq n$ such that

$$|Q_{n-2}| \le (4 \cdot |D|/4)^{1/(n-2)} = (4d^n)^{1/(n-2)}. \quad \text{... (2.38)}$$

except when $Q_{n-2} \sim \rho F_1$ and $(c_3, \ldots, c_n)$ is equivalent to P_1'. Further equality holds in (2.38) if and only if $Q_{n-2} \sim \rho F_2, \rho F_3$ or ρF_4 and $(c_3, \ldots, c_n)$ is equivalent to P_2', P_3' or P_4' respectively. Now the Lemma follows on taking $x_1 = c_1, x_2 = 0$ and observing that $(4d^n)^{1/(n-2)} \le d$ with strict inequality unless $d = 1/2$.

3. The Critical Forms

The above proof shows that (1.2) holds with strict inequality unless $Q \sim (x_1 + \ldots) x_2 + \rho F_i, i = 1, 2, 3, 4; c_2 \equiv 0 \pmod 1$, $(c_3, \ldots, c_n) \sim P_i'$ where F_i and P_i' are as given Lemma 12.

We can suppose without loss of generality that

$$Q = (x_1 + a_2 x_2 + \ldots = a_n x_n) x_2 + \rho F_i, \; i = 1, 2, 3, 4$$

and

$$-1/2 < a_j \le 1/2, \text{ for } 2 \le j \le n. \quad \text{... (3.1)}$$

We need the following Lemma 6 of Macbeath[12]. (Also see Lemma 11 of Dumir and Sehmi[10]).

Lemma 13 — Let α, β, γ be real numbers with $\alpha > 0, \gamma > 0$. Let $2h$ ad k be integers such that

$$|h - k^2 \alpha| + 1/2 < \gamma.$$

Suppose that either $\alpha \ne h/k^2$ or $\beta \ne (h/k) \pmod{(1/k, 2\alpha)}$ then for any real number v, there exist integers x and y satisfying

$$0 < \pm x + \beta y \pm \alpha y^2 + v < \gamma.$$

Lemma 14 — When $Q = (x_1 + a_2 x_2 + \ldots + a_n x_n) x_2 + \rho F_2, c_2 \equiv 0 \pmod 1$, $(c_3, \ldots, c_n) \equiv P_2' = (0, \ldots, 0, 1/2, 1/2) \pmod 1$, $d = 1/2$ then (1.1) is solvable with strict inequality unless $Q \sim Q_2$ or Q_3 and $(c_1, \ldots, c_n) \sim P_2$ or P_3 respectively.

Proof : Here

$$1/2^n = d^n = |D|/4 = \rho^{(n-2)}/4^2 \cdot 2^{(n-4)} = \rho^{(n-2)}/2^n$$

so that $\rho = 1$ and therefore

$$Q = (x_1 + a_2 x_2 + \ldots + a_n x_n) x_2 + x_3 x_4 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2.$$

By (*ii*) and (*iii*) of Remark 1,

$$a_3 = a_4 = \ldots = a_{n-2} = 0.$$

Take $(x_1, x_5, \ldots, x_n) \equiv (c_1, c_5, \ldots, c_n) \pmod 1$ arbitrarily, $x_4 = \pm 1$, $x_3 = x, x_2 = y$, so that

$$Q + d = \pm x + \beta y \pm |a_2| y^2 + v$$

where v is some real number and $\beta = c_1 + a_{n-1} x_{n-1} + a_n x_n$.

If $a_2 \neq 0$, then

$$|1/2 - |a_2|| + 1/2 = 1 - |a_2| < 1 = 2d.$$

Therefore, on taking $h = 1/2$, $k = 1$, $\alpha = |a_2| \neq 0$, it follows from Lemma 13, that (2.14) is solvable with strict inequality except perhaps when $a_2 = 1/2$ and

$$c_1 + a_{n-1} x_{n-1} + a_n x_n \equiv 1/2 \pmod 1. \quad \ldots (3.2)$$

Taking $x_{n-1} = c_{n-1}$ and $1 + c_{n-1}$ in (3.2) we get $a_{n-1} = 0$.

Similarly, taking $x_n = c_n$ and $1 + c_n$ we get $a_n = 0$. Then from (3.2) we have $c_1 \equiv 1/2 \pmod 1$. Thus in this case $Q = Q_3$ and $(c_1, \ldots, c_n) \equiv P_3 \pmod 1$.

If $a_2 = 0$, then $a_i = 0$ for all i, (By (i) of Remark 1) so that $Q = Q_2$. Further if $c_1 \not\equiv 0 \pmod 1$, choose $x_1 \equiv c_1 \pmod 1$ such that $0 < |x_1| \leq 1/2$ and then choose integer x_2 such that

$$0 < Q + d = x_1 x_2 + v \leq |x_1| \leq 1/2 < 1 = 2d,$$

so that (2.14) and hence (1.2) is solvable with strict inequality. Thus we must have $Q = Q_2$ and $(c_1, \ldots, c_n) \equiv P_2 \pmod 1$.

Lemma 15 — When $Q = (x_1 + a_2 x_2 + \ldots + a_n x_n) x_2 + \rho F_3, d = 1/2$, $c_2 \equiv 0 \pmod 1$, $(c_3, \ldots, c_n) \equiv P_3' \pmod 1$, then (1.2) is solvable with strict inequality unless $Q \sim Q_3$ and $(c_1, \ldots, c_n) \sim P_3$.

PROOF : As in Lemma 14, we get $\rho = 1$. So that

$$Q = (x_1 + a_2 x_2 + \ldots) x_2 + \left(x_3 + \frac{1}{2} x_4 \right) x_4 + x_5 x_6 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2.$$

By (ii) and (iii) of Remark 1,

$$a_3 = a_5 = a_6 = \ldots = a_{n-2} = 0.$$

Take $(x_1, x_5, \ldots, x_n) \equiv (c_1, c_5, \ldots, c_n) \pmod 1$ arbitrarily, $x_4 = \pm 1$, $x_3 = x, x_2 = y$ so that

$$Q + d = \pm x + \beta y \pm |a_2| y^2 + v,$$

where $$\beta = c_1 \pm a_4 + a_{n-1} x_{n-1} + a_n x_n. \quad \ldots (3.3)$$

Working as in Lemma 14, we find that (1.2) is solvable with strict inequality unless $a_2 = 0$ or $a_2 = 1/2$.

If $a_2 = 0$, we must have $a_i = 0$ for every i by (i) of Remark 1 and $c_1 \equiv 0 \pmod 1$ as in Lemma 14 and so

$$Q = x_1 x_2 + \left(x_3 + \frac{1}{2} x_4 \right) x_4 + x_5 x_6 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2$$

$\sim Q_3$ and $(c_1, \ldots, c_n) \sim P_3$.

If $a_2 = 1/2$, we must have as in Lemma 14, $a_{n-1} = a_n = 0$ and $\beta \equiv 1/2 \pmod 1$. Then from (3.3) we have $c_1 \pm a_4 \equiv 1/2 \pmod 1$ which gives $(c_1, a_4) = (0, 1/2)$ or $(1/2, 0)$.

If $(c_1, a_4) = (0, 1/2)$, $Q\ (-1, 1, 1/2, 0, \ldots, 0, 1/2, 1/2) = 0$ gives a solution of (1.2).

If $(c_1, a_4) = (1/2, 0)$

$$Q = \left(x_1 + \frac{1}{2} x_2 \right) x_2 + \left(x_3 + \frac{1}{2} x_4 \right) x_4 + x_5 x_6 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2 + x_n^2 \sim Q_3$$

and $(1/2, 0, 1/2, 0, \ldots, 0, 1/2, 1/2) \sim P_3$ by means of unimodular transformation

$$x_1 \to x_1, x_2 \to x_2 + x_4, x_3 \to -x_1 - x_2 + x_3 - x_4, x_i \to x_i \text{ for } i \geq 4.$$

Lemma 16 — When $Q = (x_1 + a_2 x_2 + \ldots + a_n x_n) x_2 + \rho F_4$, $d = 1/2$. $c_2 \equiv 0 \pmod 1$, $(c_3, \ldots, c_n) \equiv P_4' \pmod 1$ then (1.2) is solvable with strict inequality unless $Q \sim Q_4$ and $(c_1, \ldots, c_n) \sim P_4$.

PROOF : Working as in Lemma 14, we find that (1.2) is solvable with strict inequality unless

$$Q = x_1 x_2 + \left(x_3 + \frac{1}{2} x_4 + \frac{1}{2} x_{n-1} \right) x_4 + x_5 x_6 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2/2 + 2x_n^2,$$

$\sim Q_4$ and $(c_1, \ldots, c_n) \sim P_4$

or $$Q = \left(x_1 + \frac{1}{2} x_2 + a_4 x_4 \right) x_2 + \left(x_3 + \frac{1}{2} x_4 + \frac{1}{2} x_{n-1} \right) x_4 + x_5 x_6 + \ldots + x_{n-3} x_{n-2} + x_{n-1}^2/2 + 2x_n^2,$$

where $(c_1, a_4) = (0, 1/2)$ or $(1/2, 0)$ and $(c_2, c_3, \ldots, c_n) \equiv (0, 1/2, 0, \ldots, 0, 1/2) \pmod 1$,

When $(c_1, a_4) = (0, 1/2)$, $Q\ (-1, 1, 1/2, 0, \ldots, 0, 1/2) = 0$, and

when $(c_1, a_4) = (1/2, 0)$, $Q\ (-3/2, 1, 1/2, 0, \ldots, 0, 1, 1/2) = 0$; giving thereby a solution of (1.2).

Lemma 17 — When $Q = (x_1 + a_2 x_2 + \ldots + a_n x_n) x_2 + \rho F_1$, $c_2 \equiv 0 \pmod 1$, $d \leq 1/2$, $(c_3, \ldots, c_n) \equiv P_1' \equiv (0, \ldots, 0, 1/2, 1/2) \pmod 1$, then (1.2) is solvable with strict inequality except when $Q \sim Q_1$ and $(c_1, \ldots, c_n) \sim P_1$.

PROOF : Here we have

$$Q = (x_1 + ...)\, x_2 + \rho\, (x_3 x_4 + \dots + x_{n-3}\, x_{n-2} + x_{n-1}^2/2 + 3x_n^2/2).$$

If $\rho < 2d$ then $Q(c_1, 0, ..., 0, 1/2, 1/2) = \rho/2$ provides a solution of (1.2). Let therefore

$$2d \le \rho = (2^{(n+2)}\, d^n/3)^{1/(n-2)}. \qquad \text{... (3.4)}$$

This gives

$$d \ge \sqrt{3/16}. \qquad \text{... (3.5)}$$

We notice that $Q\,(x + c_1, +1, 0, \dots, 0, y + 1/2, 1/2) + d = x + \beta y + \rho y^2/2 + v$, where $\beta = a_{n-1} + \rho/2$. For ρ and d satisfying (3.4) and (3.5) one find that $|\, 1/2 - \rho/2\, | + 1/2 < 2d$. Therefore, by Lemma 13, with $h = 1/2$ and $k = 1$, (2.14) and hence (1.2) is solvable with strict inequality unless

$$\rho/2 = 1/2 \text{ i.e., } \rho = 1 \text{ and } \beta = a_{n-1} + 1/2 \equiv 1/2 \pmod 1 \text{ i.e. } a_{n-1} = 0.$$

Similarly $Q\,(x + c_1, + 1, 0, ..., 0, 1/2, y + 1/2) + d = x + (a_n + 3/2)\, y + 3y^2/2 + v$. For d satisfying (3.5) we find that $|\, h - 3k^2/2\, | + 1/2 < 2d$ for $h = 3/2$ and $k = 1$. Thus by Lemma 13, (2.14) is solvable with strict inequality unless $a_n + 3/2 \equiv 1/2 \pmod 1$ i.e., unless $a_n = 0$. Thus

$$Q = (x_1 + a_2 x_2 + \dots + a_{n-2}\, x_{n-2})\, x_2 + x_3 x_4 + \dots + x_{n-3}\, x_{n-2} + x_{n-1}^2/2 + 3x_n^2/2.$$

By (ii) and (iii) of Remark 1, $a_3 = a_4 = \dots = a_{n-2} = 0$. And $\rho = 1$ gives

$$d \ge (3/2^{(n+2})^{1/n} \ge (3/256)^{1/6}, \text{ as } n \ge 6. \qquad \text{... (3.6)}$$

If $a_2 = 0$, we get $a_i = 0$ for all i by (i) of Remark 1 and then as usual we must have $c_1 \equiv 0 \pmod 1$.

So $Q = Q_1$ and $(c_1, \dots, c_n) \equiv P_1 \pmod 1$.

If $a_2 \ne 0$, Q represents a_2, therefore by Lemma 3, (1.2) is solvable with strict inequality if $|\, a_2\, | < d/2$. let now

$$|\, a_2\, | \ge d/2. \qquad \text{... (3.7)}$$

Using (3.7) and (3.6) one can easily find that

$$|\, 1/2 - |\, a_2\, |\, | + 1/2 = 1 - |\, a_2\, | < 2d.$$

Now working as in Lemma 14, with $h = 1/2$ and $k = 1$, we see that (2.14) is solvable with strict inequality unless $a_2 = 1/2$, $c_1 \equiv 1/2 \pmod 1$.

Thus $Q = \left(x_1 + \frac{1}{2} x_2 \right) x_2 + x_3 x_4 + \dots + x_{n-3}\, x_{n-2} + x_{n-1}^2/2 + 3x_n^2/2$

$(c_1, \dots, c_n) \equiv (1/2, 0, ..., 0, 1/2, 1/2) \pmod 1.$

But then $Q \sim Q_1$ and $(c_1, \ldots, c_n) \sim P_1$ by the transformation $x_1 \to x_1 - x_2 - x_{n-1}$, $x_{n-1} \to x_{n-1} + x_2, x_i \to x_i$ for $i \neq 1, n-1$.

Lemmas 1-17 complete the proof of the Theorem.

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