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RANGE AND GROUP MONOTONICITY OF OPERATORS

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A real square matrix A is called *range monotone* if $Ax \geq 0, x \in R(A) \Rightarrow x \geq 0$, where $R(A)$ denotes the range space of A. A real square matrix A is called *group monotone* if $A^{\#}$ exists and $A^{\#} \geq 0$.

In this paper, we extend these concepts to operators on infinite dimensional Hilbert spaces and provide characterizations. We also discuss the relationships between group, range and weak monotonicity.

Key Words : Group Inverse; Group Monotonicity; Range Monotonicity; Weak Monotonicity

1. INTRODUCTION

A real square matrix A is called *range monotone* if $Ax \geq 0, x \in R(A) \Rightarrow x \geq 0$. It is well known that a square matrix A is range monotone if and only if $A^{\#}$, the group inverse of A exists and $A^{\#}$ is nonnegative on $R(A)$[3 or 6]. A real square matrix A is called *group monotone* if $A^{\#}$ exists and $A^{\#} \geq 0$. Berman and Plemmons[2] have shown that A is group monotone if and only if $Ax \in N(A) + I\!R^n_+, x \in R(A) \Rightarrow x \geq 0$.

In this paper, we give sufficient conditions under which both the above results can be generalized to operators on Hilbert spaces. This continues our earlier work on monotonicity in the setting of infinite dimensional Hilbert spaces[5 & 8]. The paper is organized as follows. In section 2, we review some of the preliminary concepts that will be useful in the sequel. In section 3, we extend the concept of range monotonicity to operators on infinite dimensional Hilbert spaces and present a characterization. The main result in this section is Theorem 3.7. In section 4, we generalize the notion of group monotonicity and characterize the same for operators between Hilbert spaces. This is done in Theorem 4.4. In section 5, we discuss the relationships between group, range and weak monotonicity. A new class of operators for which group and weak monotonicity are equivalent, is introduced. The main results in this section are Theorems 5.4 and 5.7.

2. PRELIMINARIES AND NOTATION

In this section, we briefly review some of the concepts that will be used in the rest of the paper.

Definition 2.1 — A nonemtpy set S in a real vector space V is called a cone if $\lambda S \subseteq S$ for all $\lambda \geq 0, S$ is convex and $S \cap -S = (0)$. S is said to be *generating* if, in addition $V = S - S$.

Definition 2.2 — Let V be a real inner product space and P a cone. The *dual cone* of P, denoted by P^*, is defined by $P^* = \{y \in P : \langle x, y \rangle \geq 0 \text{ for all } x \in P\}$. The cone P is called *self-dual* if $P = P^*$.

Examples 2.3 — (*i*) $I\!R^n_+$ the nonnegative orthant of $I\!R^n$; then n-dimensional real Euclidean space is a self-dual closed generating cone.

(*ii*) let l^2 denote the real Hilbert space of all square summable real sequences and $P := \{x \in l^2 ; x_i \geq 0 \text{ for all } i\}$. Then P is a self-dual closed generating cone. Henceforth, this cone will be called the *natural cone* of l^2.

Definition 2.4 — Let P be a cone of a real Hilbert space H. Let $\pi(P) := \{T ; H \to H : T(P) \subseteq P\}$. We say that T is *positive* if $T \in \pi(P)$.

The following notation will be used throughout the paper. For a Hilbert space $H, BL(H)$ denotes the space of bounded linear maps from H into H. Let $R(A)$ and $N(A)$ denote the range space and the null space of A, respectively.

Definition 2.5 — Let H be a real Hilbert space and $A \in BL(H)$. The *group universe* of A denoted by $A^{\#}$ is the unique solution, if there is a solution, of the following equations :-

$$AXA = A \quad \ldots (1)$$

$$XAX = X \quad \ldots (2)$$

$$AX = XA \quad \ldots (3)$$

If $H = I\!R^n$, the n-dimensional real Euclidean space then $A^{\#}$ exists if and only if $R(A) = R(A^2)$. (See for instance [1]). However, if H is infinite dimensional, then it is well known that $A^{\#}$ exists if and only if $R(A) = R(A^2)$ and $N(A) = N(A^2)$. We refer the reader to Robert[7] for other equivalent conditions. It also follows that $A^{\#}$ is bounded if and only if $R(A)$ is closed[4].

The following lemma will be useful in the sequel :-

Lemma 2.6 — Let P be a generating set in a real Hilbert space H and $A \in BL(H)$ such that $P \subseteq AP$. Then $R(A) = R(A^2)$.

PROOF : Clearly, $R(A)^2) \subseteq R(A)$. On the other hand, let $x \in R(A)$.

Then

$$x = Ay \text{ for some } y \in H.$$

$$= A(y_1 - y_2), \text{ with } y_1, y_2 \in P \text{ as } P \text{ is generating.}$$

$$= Ay_1 - Ay_2$$

$$= A^2 z_1 - A^2 z_2, \text{ where } y_i 0 = Az_1 0, i = 1, 2, \text{ as } P \subseteq AP.$$

$$= A^2 z, z = z_1 - z_2.$$

Thus $x \in R(A^2)$. This completes the proof.

3. RANGE MONOTONICITY

We begin with the definition of range monotonicity of operators in infinite dimensional Hilbert spaces.

Definition 3.1 — Let $\boldsymbol{H}$ be a real Hilbert space with a cone $\boldsymbol{P}$ and $A \in \boldsymbol{BL}(\boldsymbol{H})$. We say that A is *range monotone* if

$$Ax \in P \ \& \ x \in R(A) \Rightarrow x \in P.$$

If $H = I\!R^n$ and $P = I\!R^n_+$, then the following theorem of Peris and Subiza[6] characterizes range monotonicity.

Theorem 3.2 — *A is range monotone if and only if $A^{\#}$ exists and $A^{\#}$ is nonnegative on $R(A)$.*

The following example shows that the situation is different in infinite dimensional spaces.

Example 3.3 — Let $H = l^2$ with the natural cone P and $A : l^2 \to l^2$ be the right shift operator, i.e., $A(x) = (0, x_1, x_2, x_3, \ldots)$. Then $Ax \in P \Rightarrow x \in P$ and hence A is range monotone. However, $R(A) \neq R(A^2)$. Hence, $A^{\#}$ does not exist.

Remark 3.4 : For a map $A \in BL(H)$ it follows that if $A^{\#}$ exists and $A^{\#}$ is positive on $R(A)$, then A is range monotone. Further, if A is range monotone and A exists then A is positive on $R(A)$ (See Theorem 3.7).

We now give a necessary condition for A to be range monotone.

Lemma 3.5 — Let A be range monotone. Then $R(A^2) \cap P \subseteq AP$.

PROOF : Let $y \in R(A^2) \cap P$. Then $y = A^2 x \in P$ for some $x \in H$. Let $z = Ax$. Then $z \in R(A)$ and $Az = A^2 x \in P$. Since A is range monotone, $z \in P$. Thus $y = Az \in AP$.

Remark 3.6 : The following example shows that the converse of lemma 3.5 is not true. Consider the left shift operator A on l^2, i.e., $A(x) = (x_2, x_3, x_4, \ldots)$. Then A is onto so that $R(A) = R(A^2) = l^2$. Thus $R(A^2) \cap P = P$ and $x = (0, y_1, y_2, y_3, \ldots)$, then $y = Ax$ with $x \in P$. Thus $R(A^2) \cap P \subseteq AP$. However, if $x^0 = (-1, 0, 0, \ldots)$, thn $x^0 \notin P$, but $Ax^0 = 0 \in P$. Thus A is not range monotone.

The next theorem gives a sufficient condition under which Theorem 3.2 can be generalized.

Theorem 3.7 — *Let H be a real Hilbert space, P be a generating cone and $A \in BL(H)$ such that $P \subseteq AP$. Then A is range monotone if and only if $A^{\#}$ exists and $A^{\#}$ is positive on $R(A)$.*

PROOF : Let A be range monotone. Let $A^2 y = 0$ and $z = Ay$. Then $z \in R(A)$ and $Az = 0$. So $z \in P$. Similarly $-z \in P$. Since $P \cap -P = \{0\}$, we have $z = 0$. Thus $Ay = 0$. So $N(A^2) = N(A)$. Since $P \subseteq AP$, by lemma 2.6, we have $R(A^2) = R(A)$. Thus $A^{\#}$ exists. We next show that $A^{\#}$ is positive on $R(A)$. Let $x \in P \cap R(A)$. Then $x = Az$ for some $z \in H$. Let $u = A^{\#} x = A^{\#} Az = AA^{\#} z \in R(A)$. Then, $Au = AA^{\#} x = AA^{\#} Az = Az = x \in P$. Thus $Au \in P$ and $u \in R(A)$. So $u = A^{\#} x \in P$, since A is range monotone. Thus $A^{\#}$ is positive on $R(A)$.

Conversely, suppose that $A^{\#}$ exists and is positive on $R(A)$. Let $Ax \in P$. Then $A^{\#} AX \in P$. Now $x \in R(A) = R(AA^{\#})$ implies that $x = AA^{\#} x \in P$. Thus A is range monotone.

4. Group Monotonicity

We begin with the following definition :-

Definition 4.1 — Let $\boldsymbol{H}$ be a real Hilbert space, $\boldsymbol{P}$ be a cone and $\boldsymbol{A} \in \boldsymbol{BL}(\boldsymbol{H})$. We say that $\boldsymbol{A}$ is *group monotone* if $\boldsymbol{A}^{\#}$ exists a $\boldsymbol{A}^{\#}$ is positive.

The following theorem of Berman and Plemmons[2] gives a characterization of group monotonicity of finite matrices.

Theorem 4.2 — *$\boldsymbol{A}$ is group monotone if and only if $\boldsymbol{Ax} \in N(A) + I\!R^n_+$, $\boldsymbol{x} \in \boldsymbol{R(A)} \Rightarrow \boldsymbol{x} \geq 0$.*

Remark 4.3 : The right shift operator of example 3.3 illustrates that the above result is not true in infinite diemsnional spaces.

We now give a characterization of group monotonicity in the infinite setting.

Theorem 4.4 — *Let H be a real Hilbert space, P be a generating cone and $A \in BL(H)$ such that $P \subseteq AP$. Then A is group monotone if and only if $Ax \in N(A) + P, x \in R(A) \Rightarrow x \in P$.*

PROOF : Suppose $\boldsymbol{A}$ is group monotone and $\boldsymbol{Ax} = \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} \in \boldsymbol{N(A)}, \boldsymbol{v} \in \boldsymbol{P}$. $\boldsymbol{x} \in \boldsymbol{R(A)}$ = $\boldsymbol{R}(\boldsymbol{AA}^{\#}) \Rightarrow \boldsymbol{x} = \boldsymbol{AA}^{\#}\boldsymbol{x} = \boldsymbol{A}^{\#}\boldsymbol{Ax} = \boldsymbol{A}^{\#}(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{A}^{\#}\boldsymbol{v} \in \boldsymbol{P}$, since $N(A) = N(A^{\#})$.

Conversely, suppose that $\boldsymbol{Ax} \in \boldsymbol{N(A)} + \boldsymbol{P}, \boldsymbol{x} \in \boldsymbol{R(A)} \Rightarrow \boldsymbol{x} \in \boldsymbol{P}$.

Proceeding as in Theorem 3.7 we can show that $\boldsymbol{N(A)} = \boldsymbol{N(A^2)}$. Also $\boldsymbol{R(A)} = \boldsymbol{R(A^2)}$, as $\boldsymbol{P} \subseteq \boldsymbol{AP}$. Thus $\boldsymbol{A}$ exists. Next, let $\boldsymbol{w} \in \boldsymbol{P}$ and $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} \in \boldsymbol{R(A)}$ and $\boldsymbol{v} \in \boldsymbol{N(A)}$ (This is possible as $\boldsymbol{H} = \boldsymbol{R(A)} * \boldsymbol{N(A)}$. See [7] for more details). Now, $\boldsymbol{AA}^{\#}\boldsymbol{u} = \boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v} \in \boldsymbol{p} + \boldsymbol{N(A)}$ and $\boldsymbol{A}^{\#}\boldsymbol{u} \in \boldsymbol{R(A^{\#})}$ = $\boldsymbol{R(A)}$. By hypothesis, we then have $\boldsymbol{A}^{\#}\boldsymbol{w} = \boldsymbol{A}^{\#}\boldsymbol{u} \in \boldsymbol{p}$. So $\boldsymbol{A}\#$ is positive. This completes the proof.

5. Relationship with Weak Monotonicity

In this section, we discuss the relationship between group, range and weak monotonicity of operators. We need the following definition.

Definition 5.1 — Let $\boldsymbol{H}$ be a real Hilbert space, $\boldsymbol{P}$ be a cone and $\boldsymbol{A} \in \boldsymbol{BL}(\boldsymbol{H})$. We say that $\boldsymbol{A}$ is *weak monotone* if $\boldsymbol{Ax} \in \boldsymbol{P} \Rightarrow x \in \boldsymbol{N(A)} + \boldsymbol{P}$.

Remarks 5.2 : Weak monotonicity of operators on Hilbert spaces was introduced and studied in [8]. The following lemma will be used in this section.

Lemma 5.3 — $\boldsymbol{A} \in \boldsymbol{BL}(\boldsymbol{H})$ is weak monotone if and only if $\boldsymbol{Ax} \in \boldsymbol{P} \Rightarrow \boldsymbol{Ax} = \boldsymbol{Ay}$ for $\boldsymbol{y} \in \boldsymbol{P}$.

PROOF : Let $\boldsymbol{A}$ be weak monotone and $\boldsymbol{Ax} \in \boldsymbol{P}$. Then by definition, $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{v}$ with $\boldsymbol{u} \in \boldsymbol{N(A)}$ and $\boldsymbol{v} \in \boldsymbol{P}$. So $\boldsymbol{Ax} = \boldsymbol{Av}, \boldsymbol{v} \in \boldsymbol{P}$. To prove the converse, suppose $\boldsymbol{Ax} \in \boldsymbol{P}$. Then $\boldsymbol{Ax} = \boldsymbol{Ay}$ for some $\boldsymbol{y} \in \boldsymbol{P}$. Thus $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{u} \in \boldsymbol{N(A)}$. That is, $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{y}$ with $\boldsymbol{u} \in \boldsymbol{N(A)}$ and $\boldsymbol{y} \in \boldsymbol{P}$. Hence, $\boldsymbol{A}$ is weak monotone.

Theorem 5.4 — *Let $\boldsymbol{A}$ be a group monotone. Then $\boldsymbol{A}$ is weak monotone.*

PROOF : Let $\boldsymbol{A}$ be group monotone. Then $\boldsymbol{A}^{\#}$is positive. Let $\boldsymbol{Ax} \in \boldsymbol{P}$ and $\boldsymbol{y} = \boldsymbol{A}^{\#}\boldsymbol{Ax}$. Then $\boldsymbol{y} \in \boldsymbol{p}$ and $\boldsymbol{Ay} = \boldsymbol{Ax}$. Thus by Lemma 5.3, $\boldsymbol{A}$ is weak monotone. □

Remark 5.5 : The converse of Theorem 5.4 is not true. The right shift operator is weak monotone but does not have a group inverse. However, for a class of operators group monotonicity is equivalent to weak monotonicity. This class is defined below.

Definition 5.6 — Let H be a real Hilbert space and $A \in BL(H)$. Then A is said to be $P_{\#}$ if $A^{\#}$ exists and $AA^{\#}$ is positive.

Theorem 5.7 — *Let A be $P_{\#}$ and weak monotone. Then A is group monotone.*

PROOF : Suppose A is $P_{\#}$ and weak monotone. Let $y \in P$ and $x = A^{\#}y$. Since $AA^{\#}$ is positive, $AA^{\#}\ y \in P$. Since A is weak monotone, $A^{\#}y = u + v, u \in N(A), v \in P$. So $AA^{\#}y = Av$. Now, $x = A^{\#}y = A^{\#}AA^{\#}y = A^{\#}Av \in P$ as $A^{\#}A = AA^{\#}$ is positive. Thus A is group monotone. □

Remark 5.8 : The following example shows that the assumption that $AA^{\#}$ is positive cannot be dropped in Theorem 5.7. Let $A \in BL(l^2)$ be defined by $A\ (x = (x_1, x_3, x_1 + x_3, x_2, x_4, x_5 \ldots)$. It can be verified that $A^{\#}(x) = (x_1 - x_2 + x_3, x_1 - x_2 + x_3, -x_1 + 2x_2 + 2x_2 - x_3, x_4, x_5 \ldots)$. Now let $Ax \in P$. Then $x_1 + x_3, x_2, x_4, x_5.$ are all nonnegative. If $u = (x_1, 0, -x_1, 0, 0, \ldots)$ & $v = (0, x_2, x_1 + x_3 - x_4, x_5, \ldots,)$ then $u \in N(A), v \in P$ and $x = u + v$. Thus A is weak monotone. Further, if $x^* = (0, 1, 0, 0, \ldots) \in P$, then $AA^{\#}(x^*) = (1, 1, -1, 0, 0, \ldots) \in P$. Thus A is not group monotone.

Theorem 5.9 — *Let A be onto and range monotone. Then A is weak monotone.*

PROOF : Straightforward.

Theorem 5.10 — *Suppose H is a real Hilbert space, P a generating cone and $A \in BL(H)$ such that $P \subseteq AP$. If A is range monotone, then A is weak monotone.*

PROOF : Let $z = Ax \in P$. Then $z \in R(A)$ and so it follows from Theorem 3.7, that $A^{\#}z = A^{\#}Ax \in P$. Also $Ax = AA^{\#}Ax = A\ (A^{\#}Ax) = A\ (A^{\#}z)$. Thus A is weak monotone.

Remark 5.11 : In Remark 3.6, we have shown that the left shift operator A is not range monotone. Let $Ax = (x_2, x_3, \ldots) \in P, u = (x_1, 0, 0, \ldots)$ & $v = (0, x_2, x_3, \ldots)$. Then $x = u + v$ with $u \in N(A)$ and $v \in P$. Thus A is weak monotone. Note that $P \subseteq AP$. Thus the converse of Theorem 5.10 is not true. However, we have the following result.

Theorem 5.12 — *Let A be one-one and weak monotone. Then A is range monotone.*

PROOF : If A is one-one and weak monotone, then $Ax \in P \Rightarrow x \in P$. Clearly, this implies that A is range monotone.

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