Indian J. pure appl. Math., 32(1) : 9-16, January 2001

ITERATIVE SOLUTIONS OF SINGULAR FOURTH-ORDER BOUNDARY VALUE PROBLEMS

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(*Received* 22 *June* 1999; *after Revision* 23 *March* 2000; *accepted* 17 *October* 200)

In this article, some integral schemes for solving a class of singular fourth-order boundary value problems $\frac{d^4 u(x)}{dx^4} + \frac{\gamma}{x}\frac{d^3 u(x)}{dx^3} = f(x, u(x)), 0 < x < 1$ are derived for various boundary conditions and different values of γ. The sufficient conditions for existence and uniqueness of iteration solutions are obtained. Some numerical examples supporting the theoretical results are given.

Key Words : Integral Method; Singular Fourth-Order Boundary Value Problem; Convergence of Iteration Sequence

1. INTRODUCTION

The singular fourth-order boundary value (SFBV) problems arise in the modeling of various engineering and physical applications. For that, we shall consider SFBV problem

$$\frac{d^4 u(x)}{dx^4} + \frac{\gamma}{x}\frac{d^3 u(x)}{dx^3} = f(x, u(x)),\ x \in (0, 1), \qquad \text{.. (1.1)}$$

with the boundary conditions

$$u(0) = A_1, u'(0) = A_2, u(1) = A_3, u'(1) = A_4, \qquad \text{... (1.2)}$$

or

$$u(0) = A_1, u''(0) = A_2, u(1) = A_3, u'(1) = A_4. \qquad \text{... (1.3)}$$

The values $A_i = i = 1, 2, 3, 4$ are given constants and $f(x, u)$ is a continuous real valued-function for every $x \in [0, 1]$ and $u \in R$ is differentiable for every u.

Several finite difference methods for solving SFBV problem were studied in [2-3, 7-9] and the references therein. The resulting system of these methods, at least, is pentadiagonal and must use fictious points near both boundaries. In [6], a multi-integral method was introduced for solving problem (1.1) without singularity. Also, [4] used multi-integral method for solving nonlinear singular two-point boundary value problem, while [5] solved the same problem in the general case by using single-integral methods.

The organization of the article is as follows. In Section 2, we obtain the integral form for solving the problem (1.1) with different values of γ and the boundary conditions (1.2) and (1.3). The convergence of the iterative sequences of the solutions are given and proved in Section 3. In Section 4, some numerical examples are given.

2. Derivation of the Integral Methods

In this section, we establish the integeral solutions of the problem (1.1) with different boundary conditions and various values of γ. The following two theorems describe these integral slutions of the problem (1.1) with (1.2) and (1.3).

***Theorem* 2.1** — *If $u(x)$ is the solution of (1.1) with (1.2), the real valued-function $f(x, u)$ is continuous for every $x \in [0, 1]$ and $u \in R$ is differentiable for every u, then*

$$u(x) = A_1 + xA_2 - \frac{x^2}{(\gamma-1)}\Big[(\gamma-3)A_1 + (\gamma-2)A_2 - (\gamma-3)A_3 - A_4 + \frac{1}{2}\int_0^1 t(1-t)^2 f(t,u)\,dt \Big] + \frac{x^{3-\gamma}}{2}\int_0^x\int_0^t st^{\gamma-4}(t-s)^2 f(s,u)\,ds\,dt,$$

for $\gamma \in (3, \infty)$, ... (2.1)

and

$$u(x) = x^{3-\gamma}\Big[\Big[(\gamma-3)A_1 + (\gamma-2)A_2 - (\gamma-3)A_3 - A_4 + \frac{1}{2}\int_0^1 t(1-t)^2 f(t,u)\,dt \Big]\int_x^1 t^{\gamma-2}\,dt\,(\gamma-3)A_1\int_x^1 t^{\gamma-4}\,dt - (\gamma-2)A_2\int_x^1 t^{\gamma-3}\,dt + A_3 - \frac{1}{2}\int_x^1\int_0^t st^{\gamma-4}(t-s)^2 f(s,u)\,ds\,dt\Big],$$

for $\gamma \varepsilon (-\infty, 3]$. ... (2.2)

Proof : Multiply both sides of (1.1) by η and integrate it three times. We get

$$\int_0^x\int_0^t\int_0^s [\eta u^{(4)}(\eta) + \gamma u^{(3)}(\eta)]\,d\eta\,ds\,dt = \int_0^x\int_0^t\int_0^s \eta f(\eta, u)\,d\eta\,ds\,dt.$$

We compute the integral on the right hand side of the above equality by changing the variables η, s and t as follows :

$$\int_0^x\int_0^t\int_0^s \eta f(\eta, u)\,d\eta\,ds\,dt = \int_0^x\int_\eta^x\int_\eta^t \eta f(\eta, u)\,ds\,dt\,d\eta$$

$$= \int_0^x \int_\eta^x \eta\,(t-\eta) f(\eta, u)\, dt\, d\eta = \frac{1}{2} \int_0^x \eta\,(x-\eta)^2 f(\eta, u)\, d\eta.$$

From the above and using (1.2), we obtain

$$xu'(x) + (\gamma-3)\,u(x) = (\gamma-3)\,A_1 + x\,(\gamma-2)\,A_2 + \frac{x^2(\gamma-1)}{2} u''(0)$$

$$+ \frac{1}{2} \int_0^x t(x-t)^2 f(t, u)\, dt. \qquad \text{... (2.3)}$$

From the boundary condition at $x = 1$, we have

$$\frac{(\gamma-1)}{2} u''(0) = -(\gamma-3)\,A_1 - (\gamma-2)\,A_2 + (\gamma-3)\,A_3 + A_4 - \frac{1}{2} \int_0^1 t(1-t)^2 f(t, u)\, dt. \qquad \text{... (2.4)}$$

By substituting from (2.4) into (2.3), we get

$$x^{4-\gamma}(x^{\gamma-3}\,u(x))' = (\gamma-3)\,A_1 + x\,(\gamma-2)\,A_2 - x^2\Big[(\gamma-3)\,A_1 + (\gamma-2)\,A_2 - (\gamma-3)\,A_3$$

$$- A_4 + \frac{1}{2} \int_0^1 t(1-t)^2 f(t, u)\, dt\Big] + \frac{1}{2} \int_0^x t(x-t)^2 f(t, u)\, dt. \qquad \text{... (2.5)}$$

For $\gamma \in (3, \infty)$, integrate both sides of (2.5) from 0 to x, then we have the solution in the form (2.1). For $\gamma \in (-\infty, 3]$, integrate both sides of (2.5) from x to 1, then we have the solution in the form (2.2). This completes proof of the Theorem.

Theorem 2.2 — *If $u(x)$ is the solution of (1.1) under (1.3) and the other assumption in Theorem 2.1 is satisfied, then*

$$u(x) = A_1 + \frac{x^2}{2} A_2 - \frac{x}{(\gamma-2)}\Big[(\gamma-3)\,A_1 + \frac{(\gamma-1)}{2} A_2 - (\gamma-3)\,A_3 - A_4$$

$$+ \frac{1}{2} \int_0^1 t(1-t)^2 f(t, u)\, dt\Big] + \frac{x^{3-\gamma}}{2} \int_0^x \int_0^t s t^{\gamma-4} (t-s)^2 f(s, u)\, ds dt,$$

for $\gamma \in (3, \infty)$, ... (2.6)

and

$$u(x) = x^{3-\gamma}\Big[\Big[(\gamma-3)\,A_1 + \frac{(\gamma-1)}{2} A_2 - (\gamma-3)\,A_3 - A_4$$

$$+ \frac{1}{2} \int_0^1 t(1-t)^2 f(t, u)\, dt\Big] \int_x^1 t^{\gamma-3}\, dt - (\gamma-3)\,A_1 \int_x^1 t^{\gamma-4}\, dt - \frac{(\gamma-1)}{2} A_2 \int_x^1 t^{\gamma-2}\, dt$$

$$+ A_3 - \frac{1}{2} \int_x^1 \int_0^t st^{\gamma-4} (t-s)^2 f(s,u)\, ds\, dt \Bigg], \text{ for } \gamma \in (-\infty, 3]. \qquad \ldots (2.7)$$

PROOF : The proof is similar to the proof of Theorem 3.1

3. CONVERGENCE OF THE PRESENT METHODS

We will discuss next the convergence of the sequences generated in the above Section. It's shown that, under suitable upper bounds for $sup\ |f_u|$, all sequences are convergent. To prove the convergence of the sequence (2.1), it's enough to show that the iterative sequence $\{u^k\}$ is Cauchy sequence.

Lemma 3.1 — If $sup\ |f_u| < \frac{12(\gamma^2-1)}{\gamma}$ for $\gamma \in (3, \infty)$ and the sequence $\{u^k\}$ generated from the iterative process

$$u^{k+1}(x) = A_1 + xA_2 - \frac{x^2}{(\gamma-1)} \Bigg[(\gamma-3)A_1 + (\gamma-2)A_2 - (\gamma-3)A_3 - A_4$$

$$+ \frac{1}{2} \int_0^1 t(1-t)^2 f(t, u^k)\, dt \Bigg] + \frac{x^{3-\gamma}}{2} \int_0^x \int_0^t st^{\gamma-4} (t-s)^2 f(s, u^k)\, ds\, dt,$$

for $\gamma \in (3, \infty)$, ... (3.1)

is bounded by constant C, then it's Cauchy sequence.

PROOF : For $\varepsilon > 0$ and for arbitrary $n, m \geq N$ such that $K^N < \frac{\varepsilon}{2C}$, we have :

$$(u^n - u^m)(x) = \frac{x^2}{2(\gamma-1)} \int_0^1 t(1-t)^2 \Big[f(t, u^{m-1}) - f(t, u^{n-1}) \Big] dt$$

$$+ \frac{x^{3-\gamma}}{2} \int_0^x \int_0^t st^{\gamma-4} (t-s)^2 \Big[f(t, u^{n-1}) - f(t, u^{m-1}) \Big] ds\, dt. \qquad \ldots (3.2)$$

By taking absolute values on both sides of (3.2) and using the mean value theorem, we have:

$$|(u^n - u^m)(x)| \leq |f_u| \Bigg[\frac{x^2}{2(\gamma-1)} \int_0^1 t(1-t)^2\, dt$$

$$+ \frac{x^{3-\gamma}}{2} \int_0^x \int_0^t st^{\gamma-4} (t-s)^2\, ds\, dt \Bigg] \| u^{n-1} - u^{m-1} \|_\infty$$

$$= Sup\ |f_u| \left[\frac{x^2}{24(\gamma-1)} + \frac{x^4}{24(\gamma+1)} \right] \| u^{n-1} - u^{m-1} \|_\infty.$$

Let $K = Sup\ |f_u| \frac{\gamma}{12(\gamma^2-1)}$, then we have

$$\| u^n - u^m \|_\infty \leq K \| u^{n-1} - u^{m-1} \|_\infty \leq 2CK^m \leq \varepsilon.$$

Then $\{u^k\}$ is Cauchy sequence and the present Lemma is proved.

Now, we will discuss the convergence of the solution (1.1) with (1.2) when $\gamma \in (-\infty, 3]$ by the following Lemma.

Lemma 3.2 — The sequence $\{u^k\}$ generated from the iterative process

$$u^{k+1}(x) = x^{3-\gamma} \Big[\Big[(\gamma-3) A_1 + (\gamma-2) A_2 - (\gamma-3) A_3 - A_4$$

$$+ \frac{1}{2} \int_0^1 t(1-t)^2 f(t, u^k)\, dt \Big] \int_x^1 t^{\gamma-2}\, dt - (\gamma-3) A_1 \int_x^1 t^{\gamma-4}\, dt$$

$$\left\{ - (\gamma-2)A_2 \int_x^1 t^{\gamma-3}\, dt + A_3 - \frac{1}{2} \int_x^1 \int_{to}^t s t^{\gamma-4} (t-s)^2 f(s, u^k)\, ds\, dt \right\},$$

$$\text{for } \gamma \in (-\infty, 3], \quad \ldots (3.3)$$

is Cauchy sequence when it's bounded by a constant C and

(i) $Sup\ |f_u| < 111.1$ for $\gamma = -1, 1$.

(ii) $Sup\ |f_n| < \frac{24 |\gamma^2 - 1|}{(2|\gamma| + |\gamma+1| + \gamma - 1|)}$, for $\gamma \varepsilon (-\infty, 3] - \{-1, 1\}$.

PROOF : The proof is similar to the proof Lemma 3.1

The convergence of the sequences $\{u^k\}$ generated from (2.6) and (2.7) are investigated by the following Lemma.

Lemma 3.3 — The sequences $\{u^k\}$ generated from the interative processes

$$u^{k+1}(x) = A_1 + \frac{x^2}{2} A_2 - \frac{x}{(\gamma-2)} \Big[(\gamma-3)A_1 + \frac{(\gamma-1)}{2} A_2 - (\gamma-3)A_3 - A_4$$

$$+ \left[\frac{1}{2} \int_0^1 t(1-t)^2 f(t, u^k) dt \right] + \frac{x^{3-\gamma}}{2} \int_0^x \int_0^t s t^{\gamma-4} (t-s)^2 f(s, u^k)\, ds\, dt,$$

$$\text{for } \gamma \in (3, \infty], \quad \ldots (3.4)$$

and
$$u^{k+1}(x) = x^{3-\gamma}\left\{\left[(\gamma-3)A_1 + \frac{(\gamma-1)}{2}A_2 - (\gamma-3)A_3 - A_4\right.\right.$$
$$\left. + \frac{1}{2}\int_0^1 t(1-t)^2 f(t,u^k)\,dt\right]\int_x^1 t^{\gamma-3}\,dt - (\gamma-3)A_1\int_x^1 t^{\gamma-4}\,dt$$
$$-\frac{(\gamma-1)}{2}A_2\int_x^t t^{\gamma-2}\,dt$$
$$\left. + A_3 - \frac{1}{2}\int_x^1\int_0^t st^{\gamma-4}(t-s)^2 f(s,u^k)\,ds\,dt\right], \text{ for } \gamma\in(-\infty,3], \ldots \quad (3.5)$$

are Cauchy sequences when they are bounded by a constant C and

(i) $Sup\ |f_u| < 48.4$, for $\gamma = 2$.

(ii) $Sup\ |f_u| < 100.4$, for $\gamma = -1$.

(iii) $Sup\ |f_u| < \dfrac{24(\gamma-1)(\gamma+1)}{(2\gamma-1)}$, for $\gamma\in(3,\infty)$.

and (iv) $Sup\ |f_u| < \dfrac{|\gamma+1| + |\gamma-2|}{12|\gamma+1||\gamma-2|}$, for $\gamma\in(-\infty,3] - \{-1, 2\}$.

PROOF : The proof is similar to the proof Lemma 3.1.

4. NUMERICAL RESULTS

In this Section, we conclude this paper by reporting some numerical results, obtained from a set of test examples. These numerial results describe the performance of the algorithms. Tables indicate the convergence pattern of the iterative sequence of approximate solution and compare the present method with centred finite difference method and the multi-integral method given in [8]. In all these examples Simpson's method is used to approximate the integrals. In order that, we divide the interval [0, 1] into a uniform mesh $x_j = jh, j = 0(1)N$ and $h = 1/N$. We solve the test problems for each $N = 2^i$, $i = 3, 4, \ldots, 8$. The maximum absolute error, E^N, is defined by

$$E^N = max_j\ |u(x_j) - u^{k+1}(x_j)|, j = 1(1)N-1,$$

where $u(x_j)$ is the exact solution at x_j. The experimental rate of convergence, see [1], is

$Rate = log_2(E^N/E^{2N})$.

Also, we denote the smallest number of iterations by NIT, required to get

$$\|u^{k+1}(x_j) - u^k(x_j)\|_\infty < 10^{-10}.$$

Example 1 — $\dfrac{d^4u}{dx^4} + xe^{-u^2}u = 24 + x^3(x-1)^2 e^{-x^4(x-1)^4}$, $x\in(0,1)$,

$$u(0) = u'(0) = u(1) = u'(1) = 0.$$

The exact solution is $u(x) = x^2 (x-1)^2$ and the initial guess is $u^0(x) = 0$. For this example the minimum absolute error in [6] is 1.47E—03 when N = 10. The numerical results are summarized in Table I.

In this example, we can evaluate $f_u(x, u)$ in the form

$$f_u(x, u) = x(2u^2 - 1)e^{-u^2}, \quad 0 \leq x \leq 1,$$

then $$\sup |f_u| = 1, \quad 0 \leq x \leq 1.$$

Since $\gamma = 0$, then the parameter K in Lemma 3.2 is $K = \frac{1}{12}$ and therefore for any bounded constant C, we can choose N such that $2CK^N < \varepsilon$. Hence, this example satisfies the requirements of Lemma 3.2. Also, it's obvious from Table I.

Example 2 —

$$\frac{d^4x}{dx^4} + \frac{4}{x}\frac{d^3u}{dx^3} = 15u^5(1 - x^2u^2)(1 - 7x^2u^2), x \in (0, 1),$$

$$u(0) = \frac{1}{2}, u'(0) = 0, u(1) = \frac{1}{\sqrt{5}}, u'(1) = -\frac{1}{\sqrt{5}}.$$

The exact solution is $u(x) = \frac{1}{\sqrt{4 + x^2}}$ and the initial guess is $u^0(x) = \left(\frac{1}{\sqrt{5}} - \frac{1}{2}\right)x + \frac{1}{2}$. The numerical results are given in Table (2).

In this example, we can evaluate $f_u(x, u)$ in the form

$$f_u(x, u) = 15u^4(5 - 56x^2u^2 + 63x^4u^4), 0 \leq x \leq 1,$$

then $$\sup |f_u| = \frac{75}{16}, 0 \leq x \leq 1.$$

Since $\gamma = 4$, then the parameter K in Lemma 3.1 is $K = \frac{5}{48}$ and therefore for any bounded constant C, we can choose N such that $2CK^N < \varepsilon$. Hence, this example satisfies the requirements of Lemma 3.1. Also, it's obvious from Table II

TABLE I

	Centred finite diff. method		The present method	
N	NIT	E^N *(Rate)*	NIT	E^N *(Rate)*
8	4	1.27E-02(1.84)	4	8.14E-05(4.00)
16	4	3.53E-03(1.93)	4	5.09E-06(4.00)
32	4	9.30E-04(1.96)	4	3.18E-07(4.00)
64	4	2.38E-04(1.98)	4	1.99E-08(4.00)
128	4	6.03E-05(1.99)	4	1.24E-09(4.00)
256	4	1.52E-05	4	7.76E-11

TABLE II

	Centred finite diff. method		The present method	
N	NIT	E^N (*Rate*)	NIT	E^N (*Rate*)
8	6	1.10E-04(2.00)	7	7.86E-06(4.03)
16	6	2.75E-05(2.00)	7	4.80E-07(4.01)
32	6	2.86E-06(2.00)	7	2.99E-08(4.00)
64	6	1.71E-06(2.00)	7	1.86E-09(4.00)
128	6	4.26E-07(2.00)	7	1.16E-10(4.00)
256	6	1.06E-07	7	7.28E-12

We observe from the above tables, the acuracy is better than [6] and acceptale, but we can improve it by using smaller grid size in Simpson's method or by using more accurate methods for integrals.

ACKNOWLEDGEMENT

The authors are indebted to Professor S.E.El-Gendi for various valuable suggestions and constructive criticism.

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