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BAKER DOMAINS AND SINGULARITIES FOR CERTAIN MEROMORPHIC FUNCTIONS

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In this paper, we studied the relation between Baker domains and sing (f^{-1}) or $P(f)$ for certain meromorphic functions and proved that every cycle of multiply-connected Baker domains of a meromorphic function f which has finitely many poles must contain a singularity of f^{-1}.

Key Words : Meromorphic Function; Fatou Set; Julia Set; Baker Domain; Singularity

1. INTRODUCTION

Let f be a nonlinear meromorphic function defined in the complex plane $\mathbb{C}$. The Fatou set $F(f)$ of f is the largest subset of $\hat{\mathbb{C}}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ is called the Julia set and denoted by $J(f)$. If U is a component of $F(f)$, then $f^n(U)$ is contained in some component of $F(f)$ which we denote by U_n. If $U_n \cap U_m = \phi$ for all $n \neq m$, then U is called wandering. Otherwise U is called eventually periodic. In particular, if $U_p = U$ for some $p \in \mathbb{N}$ and $U_m \neq U$ for $0 \leq m < p$, then U is called periodic with period p, and $\{U, U_1, \ldots, U_{p-1}\}$ is called a (periodic) cycle of components. More details about Fatou and Julia sets can be found in [5, 6, 9, 14, 17].

Let U be a periodic component of period p, that is $f^p(U) \subset U$. Then we have one of the following possibilities[4-9]:

- U contains an attracting periodic point z_0. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, and U is called the immediate attractive basin of z_0.

- ∂U contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. (For $z_0 = \infty$ we have $(g^p)'(0) = 1$ where $g(z) = 1/f(1/z)$.) In this case, U is called a parabolic domain.

- There exists an analytic homeomorphism $\phi : U \to D$ where D is the unit disc such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in R\ Q$. In this case, U is called a Siegel disc.

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- There exists an analytic homeomorphism $\phi : U \to A$ where A is an annulus, $A = \{z : 1 < |z| < r\}$, $r > 1$, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha} z$ for some $\alpha \in R \setminus Q$. In this case, U is called a Herman ring.
- There exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined. In this case, U is called a Baker domain.

The periodic domains are closely related to the set of singularities of the inverse function f^{-1} of f, that is, the set of the critical and finite asymptotic values of f and (finite) limit points of these values. We denote the set of all singularities of f^{-1} by sing (f^{-1}) and define

$$P(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathit{sing}(f^{-1}))}.$$

Here and in what follows, $\bar{E}$ denotes the closure of a set $E \subset \mathbb{C}$.

Theorem A[9] — *Let f be a meromorphic function, and let $C = \{U_0, U_1, \ldots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$.*

- *If C is a cycle of immediate attractive basis or parabolic domains, then $U_j \cap \mathit{sing}(f^{-1}) \neq \phi$ for some $j \in \{0, 1, \ldots, p-1\}$.*
- *If C is a cycle of Seigel discs or Herman rings, then $\partial U_j \subset P(f)$ for all $j \in \{0, 1, \ldots, p-1\}$.*

Naturally, one may ask whether Baker domains are also related to singularities of f^{-1} for a meromorphic function f. Bergweiler proved.

Theorem B[8] — *Let $f(z)$ be a meromorphic function of finite order and $f'(z) = q(z)\, e^{p(z)} (f(z) - z)$, where $q(z)$ is rational and $p(z)$ is a polynomial. Then every cycle of Baker domains of f contains a singularity of f^{-1}.*

Theorem C[9] — *Let*

$$f(z) = z + r(z)\, e^{p(z)},$$

where $r(z)$ is rational and $p(z)$ is a polynomial. Then every cycle of Baker domains of f contains a singularity of f^{-1}.

In this paper, we have obtained

Theorem 1 — *Let $f(z)$ be of finitely many poles. Then every cycle of multiply-connected Baker domains of $f(z)$ contains a singularity of f^{-1}.*

Remark 1 : Dominguez[14] proved that $f(z) = z + 2 + e^z + 10^{-2}(z - (1 + i\pi))^{-1}$ has a multiply-connected invariant component in which $f^n(z) \to \infty$. Hence there exists function $f \in M_F$ such that f has a multiply-connected Baker domain.

Theorem 2 — *Let f be a transcendental meromorphic function and U a simply-connected Baker domain of period p. If the space of quasiconformal deformations of f is finitely dimensional, then the cycle of Baker domains $\{U, U_1, U_2, \ldots, U_{p-1}\}$ of f contains a singularity of f^{-1}.*

From Theorems 1, 2 and Reference [1], we have

Corollary 1 — Let $q(z)$ be a rational function, $p(z)$, $p_1(z)$ be two polynomials and $m, n \in I\!N$ be two positive integers with $n \geq \deg p(z) + m$. Suppose that f is a meromorphic function of finite order, satisfying the following differential equation.

$$(f'(z))^n = q(z)\, p(f(z))\, e^{p_1(z)}\, (f(z) - z)^m. \qquad \dots (1)$$

Then every cycle of Baker domains of f contains a singularity of f^{-1}.

From Corollary 1 we obtain Theorem B.

Corollary 2 — Let $q(z)$ be a rational function, $p(z)$ be a polynomial and $m, n \in I\!N$ with $n \geq \deg p(z) + m$. Suppose that f is a meromorphic function and satisfies the following differential equation

$$(f'(z))^n = q(z)\, p(f(z))\, (f(z) - z)^m. \qquad \dots (2)$$

Then every cycle of Baker domains of f contains a singularity of f^{-1}.

Remark 2 : The following equation

$$f'(z) = 2z\,(f(z) - z)$$

has a meromorphic solution $f(z) = z - e^{z^2}\left(\int_0^z e^{-t^2}\,dt + \frac{\pi^{\frac{1}{2}}}{2}\right)$ which has an invariant Baker domain (see [8]).

From Theorems 1, Theorem 2 and [16], we obtain Theorem C.

2. Proof of Theorem 1

For the proof of Theorem 1, we need the following lemmas.

Lemma 1[9] — Let $f(z)$ be a meromorphic function, and let $\{U_0, U_1, \dots, U_{p-1}\}$ be a periodic cycle of Baker domains of $f(z)$. Denote by z_j the limit corresponding to U_j, and define $z_p = z_0$. Then $z_j \in \bigcup_{n=0}^{p-1} f^{-n}(\infty)$ for all $j \in \{0, 1, 2, \dots, p-1\}$, and $z_j = \infty$ for at least one $j \in \{0, 1, 2, \dots, p-1\}$. If $z_j = \infty$, then z_{j+1} is an asymptotic value of f. In particular, if $p = 1$, then $f^n|_U \to \infty$ as $n \to \infty$.

Lemma 2[1 & 9] — Let G be an unbounded open set in $\mathbb{C}$ with at least two finite boundary points, and let f be analytic in G. Let D be a domain contained in G, and suppose that $f^n(D) \subset G$ for all n and that $f^n|_D \to \infty$ as $n \to \infty$. Then, for any compact subset K of D, there exist constants C and n_0 such that

$$|f^n(z')| \leq |f^n(z)|^C$$

for all $z, z' \in K$ and $n \geq n_0$. If, in addition, $f(D) \subset D$, then there exist a constant $A > 1$ and a curve $\gamma \subset D$ tending to ∞ which satisfies $f(\gamma) \subset \gamma$ such that

$$|z|^{\frac{1}{A}} \leq |f(z)| \leq |z|^A$$

for all $z \in \gamma$.

Lemma 3[8 & 16] — Let f be a meromorphic function with finitely many poles. If f is not entire, then support that $O^-(\infty) = \{z : z \text{ is a pole of } f^n \text{ for some } n \in \mathbb{N}\}$ is infinite and let U be a component of $F(f)$ such that U and all the $U_{n,n} = 1, 2, \ldots$ are multiply-connected and do not contain singularities of f^{-1}. Suppose also that all limit functions of $\{f^n|_U\}$ are constant (possibly ∞). Let γ be a Jordan curve in U which is not null-homotopic in U. Then, for any $M > 0$, there exist $L = L(M) \in \mathbb{N}$ and a continuum $\delta_L \subset f^L(\gamma)$ such that $\delta_L \subset \{z : |z| > M\}$, with one of the components of the complement of δ_L being bounded and containing $\{z ; |z| < M\}$.

Lemma 4[2] — Let f be a transcendental entire function. Then any multiply-connected component of $F(f)$ must be a wandering domain.

Lemma 5[3] — Suppose that f is the form: $f(z) = \dfrac{e^{g(z)}}{z^m}$, where $m \in \mathbb{N}$ and $g(z)$ is non-constant entire function and that U is component of $F(f)$. Then if γ is a simple closed curve in U, either (i) γ separates $0, \infty$ or (ii) complement of γ has a compact component which belongs to U.

Lemma 6[3] — Let f is of the form: $f(z) = \dfrac{e^{g(z)}}{z^m}$, where $m \in \mathbb{N}$ and $g(z)$ is non-constant entire function. Suppose that γ_1, γ_2 are disjoint Jordan curves in $F(f)$ which separates $0, \infty$. Then the region Δ bounded by γ_1, γ_2 contains no points of $J(f)$.

Lemma 7[3] — If f is the form: $f(z) = \dfrac{e^{g(z)}}{z^m}$, where $m \in \mathbb{N}$ and $g(z)$ is non-constant entire function, then the components of $F(f)$ are simple or doubly-connected. Furthermore, there is at most one doubly-connected component.

Now we prove Theorem 1.

By Lemma 4, we know that $f(z)$ is a meromorphic function with one pole at least. We consider two cases in the following.

Case 1 — $f(z)$ is not of the form: $f(z) = \alpha + \dfrac{e^{g(z)}}{(z-\alpha)^m}$, where $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$ and $g(z)$ is non-constant entire function. Hence $O^-(\infty)$ is infinite.

Let U be a Baker domain of f with period p and suppose that $U, U_1, \ldots, U_{p-1}$ do not contain singularities of f^{-1}. It follows that U and all U_n are multiply-connected, so that U satisfies the hypothesis of lemma 3. We choose γ as in Lemma 3. Next we consider two subcases:

Case 1.1 — $p = 1$. By Lemma 1 and Lemma 2 we can show that if $z_1, z_2 \in \gamma$, then

$$|f^n(z_2)| \le |f^n(z_1)|^A \quad \ldots (3)$$

for some positive constant A and all sufficiently large $n \in \mathbb{N}$. We deduce from Lemma 3 thet for any $M > 0$ there exist L, $R \ge M$ such that

$$\delta_L \subset f^L(\gamma) \subset \{z : R \le |z| < R^A\} \quad \ldots (4)$$

for some $L \in \mathbb{N}$.

By Lemma 2, there exists a curve $\sigma \subset U$ tending to ∞ such that

$$|f(z)| \leq |z|^C \qquad \text{... (5)}$$

for all $z \in \sigma$.

We now choose R and L such that (4) holds. If R is sufficiently large, then δ_L intersects σ. Hence by (4) and (5), we obtain

$$\min_{z \in f^L(\gamma)} |f(z)| \leq \min_{z \in \delta_L} |f(z)| \leq R^{AC}. \qquad \text{... (6)}$$

Applying (3) for $n = L + 1$, we deduce that

$$\max_{z \in \delta_L} |f(z)| \leq \max_{z \in f^L(\gamma)} |f(z)| \leq R^{A^2C},$$

provided R is large enough. Since f has only infinitely many poles, there exists a rational function $T(z)$ such that $T(z) \to 0$ as $z \to \infty$ and $f - T$ is an entire function. We deduce that

$$\max_{z \in \delta_L} |f(z)| = \max_{z \in \delta_L} |f(z) - T(z)| + o(1) \geq \max_{|z|=R} |f(z) - T(z)| + o(1)$$

as $R \to \infty$. It follows that

$$\max_{|z|=R} |f(z) - T(z)| \leq R^{A^2C}$$

for arbitrary large R. This implies that $f - T$ is a polynomial. Hence f is rational, contradicting our assumption.

Case 1.2 — $p \geq 2$. If there exists $z_j \neq \infty$ for some $j \in \{0, 1, ..., p-1\}$. Then by Lemma 1, there exists a curve σ tending to ∞ such that

$$|f(z)| \leq B \qquad \text{... (7)}$$

for all $z \in \sigma$, where B is a finite constant. We now choose R and L such that (4) holds. In the following, by using the same argument to case 1.1, we can deduce a contradiction.

If $z_j = \infty$ for all $j \in \{0, 1, ..., p-1\}$. Take $a_j \in U_j$, $j = 1, 2, ..., p-1$ and $M > \max\{|a_0|, |a_1|, ..., |a_{p-1}|\}$. Then there exist $L, R \geq M$ such that $\delta_L \subset f^L(\gamma) \subset \{z : R \leq |z| \leq R^A\}$ for some $L \in \mathbb{N}$ by Lemma 3. Without loss of generality, we assume that $\delta_L \subset U_0$, then $U_0 \cap U_j \neq \phi$ for all $j \in \{1, 2, ..., p-1\}$, a contradiction.

Case 2 — $f(z)$ is of the form: $f(z) = \alpha + \dfrac{e^{g(z)}}{(z-\alpha)^m}$, where $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$ and $g(z)$ is non-constant entire function. Without loss of generality we assume that $\alpha = 0$. Hence $f(z) = \dfrac{e^{g(z)}}{z^m}$. Next we also consider two subcases:

Case 2.1 — $p = 1$. Obviously, $f^n|_U \to 0$ or $f^n|_U \to \infty$ as $n \to \infty$. Suppose that U is an invariant multiply-connected Baker domain and that U does not contain a singularity of f^{-1}. Let

γ be a simple closed curve in U which separates 0, ∞, then there exists a curve $\gamma_n \subset f^n(\gamma)$ separates 0, ∞ by Lemma 5. If $f^n|_U \to 0$ as $n \to \infty$, then by Lemma 6 we deduce that $0 \in J(f)$ is an isolated point of $J(f)$, a contradiction. Hence, $f^n|_U \to \infty$. By Lemma 7, U is doubly-connected Let $c_1 u, c_2 u$ be components of $\hat{\mathbb{C}}$. We claim that $c_i u \neq \{\infty\}$, i = 1, 2. Suppose that there exists $c_i u = \{\infty\}$ for $i \in \{1, 2\}$. Without loss of generality we assume that $c_2 u = \{\infty\}$, then there exists $B > 0$ such that $f(z)$ is analytic in $\{|z| \geq B\}$ and $\{|z| \geq B\} \subset U$. By Lemma 2, there exist a curve σ tending to ∞ and $A > 0$ such that

$$|f(z)| \leq |z|^A$$

for any $z \in \sigma$.

As U does not contain a singularity of f^{-1} and the curve $\gamma_B : |z| = B$ is not null-homotopic, we know that $f^n(\gamma B)$ is not hull-homotopic. Hence there exists a curve $\gamma_n \subset f^n(\gamma B)$ such that γ_n separates 0, ∞. By Lemma 1 for any $M > 0$, there exist $R \geq M, N \in \mathbb{N}$ and $C > 0$ such that

$$R \leq |f^N(\gamma B)| \leq R^C.$$

In the following by using the same argument to Case 1.1, we can deduce that $f(z)$ is a rational function, a contradiction. Hence we obtain that $c_i u \neq \{\infty\}$, $i = 1, 2$. Without loss of generality we assume that $c_2 u$ is an unbounded component. Taking $z_0 \in c_2 u$ such that $|z_0| = \min_{z \in c_2 u} \{|z|\}$, and $l = \{z : z = t z_0, 0 \leq t \leq 1\}$, then $f^n(\gamma) \cap l \neq \phi$ for large n by Lemma 6, which contradicts $f^n(\gamma) \to \infty$ as $n \to \infty$.

Case 2.2 — $p \geq 2$. Suppose that a cycle of Baker domains $\{U, U_1, \ldots, U_{p-1}\}$ do not contain a singularity of f^{-1} and U is doubly-connected . Let γ be a simple closed curve which is not null-homotopic in U, then $f(\gamma)$ is not null-homotopic in U_1. Hence U_1 is not simply-connected, thus by Lemma 7 we deduce a contradiction. The proof of Theorem 1 is complete.

3. PROOF OF THEOREM 2 — For the proof of Theorem 2, we need the following lemma :

Lemma 8[17] — Suppose that $f(z)$ is a proper map[17] of D onto G. Then, in any simply-connected subdomain G containing no critical of f, there exists an analytic inverse of f which is uniquely determined by its value at a single point.

Now we prove Theorem 2.

Suppose that $U, U_1, \ldots, U_{p-1}$ do not contain singularities of f^{-1}, then U_k $(k = 0, 1, \ldots, p-1)$ are simply-connected and $f|_{U_k}$ is univalent for all $k \in \{0, 1, \ldots, p-1\}$ by Lemma 8. As observed by Herman [15, p609] (see also [8, p237], this implies that the space of quasiconformal deformations of f is infinite dimensional, a contradiction. Thus the proof of Theorem 2 is complete.

4. PROOFS OF COROLLARIES 1-2

Firstly we prove Corollary 1.

It is easy to see that f has finitely many poles by (1). Let U be Baker domains of f with period p and suppose that $U, U_1, \ldots, U_{p-1}$ do not contain singularities of f^{-1}. Hence, it follows from Theorem 1 that U and all U_n are simply-connected and that $f|_{U_n}$ is univalent for all n. Now we apply the quasi-conformal methods of Sullivan[18]. For a given $K > 1$, we consider K-quasiconformal self-maps ϕ of the sphere which fix 0, 1, ∞, such that $f_\phi = \phi f \phi^{-1}$ is meromorphic. In the following, by using the same argument to [2, 7, 8, 19], we prove that the deformation family $\{f_\phi\}$ depend only on finitely many parameters (also see [19]).

It is clear that a pole z_0 of $f(z)$ corresponds to the pole $\phi(z_0)$ of f_ϕ with the same multiplicity. If we express $p(z)$ in the form: $C_0 \prod_{0<j\leq l} (f-b_j)$, where k is a non-negative integer, and C_0, b_j are complex numbers, $(0 \leq j \leq l)$, then we have from (1) that the critical points of f correspond to the fixpoints, b_j-points, $(0 \leq j \leq l)$, of f, with finitely many exceptions. It is also easy to see that the fixpoints, b_j-points $(0 \leq j \leq l)$ of f correspond to the fixpoints, $\phi(b_j)$-points, $(0 \leq j \leq l)$, of f, with infinitely many exceptions. It is also easy to see that the fixpoints, b_j-points $(0 \leq j \leq l)$ of f correspond to the fixpoints, $\phi(b_j)$-points $(0 \leq j \leq l)$ of f_ϕ with finitely many exceptions. Thus we find that $(f'_\phi(z))^n / \left\{ \prod_{j=0}^{l} [f_\phi(z) - \phi(b_j)] (f_\phi(z) - z)^m \right\}$ has only finitely many zeros and poles, as many as $q(z)$ has. The Hölder continuity of ϕ at ∞ gives that $|\phi(a)| = O(|z|^k)$ and $\phi^{-1}(z) = O(|z|^k)$ as $z \to \infty$. This implies that $\rho(f_\phi) \leq K_\rho(f)$, where $\rho(f)$ denotes the order of f (see [18, 25]). Therefore, the order of $(f'_\phi(z))^n / \left\{ \prod_{j=0}^{l} [f_\phi(z) - \phi(b_j)] (f_\phi)(z) - z)^m \right\}$ is alsmot at most $K_\rho(f)$, we conclude that

$$\frac{(f'_\phi(z))^n}{\prod_{j=0}^{l} [f_\phi(z) - \phi(b_j)] (f_\phi(z) - z)^m} = q_\phi(z)\, e^{r_\phi(z)} \qquad \ldots (8)$$

for some rational function q_ϕ of the same degree as $q(z)$ and some polynomial $r_\phi(z)$ of degree at most $K_\rho(f)$, (8) gives that the deformation family f_ϕ depend only on finitely many parameters, which is a contradiction. Thus by Theorem 2, the proof Corollary 1 is complete.

Similarly, we can prove Corollary 2. We omit the details.

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