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STABILITY OF SUPERPOSED VISCOUS-VISCOELASTIC (WALTERS *B*) FLUIDS IN THE PRESENCE OF SUSPENDED PARTICLES THROUGH POROUS MEDIUM

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The Rayleigh-Taylor instability of a Newtonian viscous fluid overlying Walters B' viscoelastic fluid containing suspended particles in a porous medium is considered. For the stable configuration the system is found to be stable or unstable if $v'_1 \langle \text{ or } \rangle \frac{k_1}{\varepsilon \alpha_1}$, depending on kinematic viscoelasticity, permeability of the medium and density of the medium. The system is found to be unstable in the potentially unstable case. The effects of a variable horizontal magnetic field and a uniform rotation are also considered. For the stable configuration, in the hydromagnetic case also, the system is found to be stable or unstable if $v'_1 \langle \text{ or } \rangle \frac{k_1}{\varepsilon \alpha_1}$. However, for the unstable configuration, the magnetic field and viscoelasticity have got stabilizing effects. The system is found to be unstable for the potentially unstable case, for highly viscous fluids, in the presence of a uniform rotation.

Key Words : Rayleigh-Taylor Instability; Walters *B'* Elastico-Viscous Fluid; Suspended Particles; Porous Medium

1. INTRODUCTION

When two fluids of different densities are superposed one over the other (or accelerated towards each other, the instability of the plane interface between the two fluids, when it occurs, is called Rayleigh-Taylor instability. Chandrasekhar[1] has given a detailed account of the instability of the plane interface between two incompressible and viscous fluids of different densities when the lighter is accelerated into the heavier. The influence of viscosity on the stability of the plane interface separating two electrically conducting, incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field, has been studied by Bhatia[2]. He has carried out the stability analysis for two fluids of high viscosities and different uniform densities. Sharma and Kumar[3] have studied the stability of two superposed Walters B' viscoelastic fluids and the analysis has been carried out, for mathematically simplicity, for two highly viscoelastic fluids of equal kinematic viscosities and equal kinematic viscoelasticities. It is found that for stable configuration, the system is stable or unstable under a certain condition, however, the system is found to be unstable for the unstable configuration. Recently Sharma and Kumar[4] have studied the Rayleigh-Taylor instability of two superposed conducting Walters B' elastico-viscous fluids in hydromagnetics.

In geophysical situations, the fluid is often not pure but contains suspended particles. Scanlon and Segel[5] have considered the effects of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number is reduced because of the heat capacity of the particles. The suspended particles were thus found to destabilize the layer. Palaniswamy and Purushotham[6] have studied the stability of shear flow of stratified fluids with fine dust and found the effects of fine dust to increase the region of instability. The medium has been considered to be non-porous in all the above studies.

The flow through porous media is of considerable interest for petroleum engineers and for geophysical fluid dynamicists. When the fluid slowly percolates through the pores of a macroscopically homogeneous and isotropic porous medium, the gross effect is represented by Darcy's law according to which the usual viscous term in the equations of fluid motion is replaced by the resistance term $\left[-\left(\frac{\mu}{k_1}\right)\mathbf{v}\right]$, where μ is the viscosity of the fluid, k_1 is the permeability of the medium and $\mathbf{v}$ is the Darcian (filter) velocity of the fluid. Lapwood[7] has studied the stability of convective flow in hydromagnetics in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through porous medium has been considered by Wooding[8]. The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma[9]. The suspended particles and the permeability of the medium were found to destabilize the layer. Sharma and Kumar[10] have studied the Rayleigh-Taylor instability of fluids in porous media in the presence of suspended particles and variable magnetic field. Kumar[11] has studied the instability of two viscoelastic superposed fluids with suspended particles and variable magnetic field in porous medium and found that the stability criterion is independent of the effects of viscoelasticity, medium porosity and suspended particles but is dependent on the orientation and magnitude of the magnetic field. The stability of two superposed Walters B' viscoelastic fluids in the presence of suspended particles and variable magnetic field in porous medium has been studied by Sharma and Kango[12].

The instability in a porous medium of a plane interface between viscous (Newtonian) and viscoelastic (Walters B') fluids containing suspended particles may be of interest in geophysics, chemical technology and bio-mechanics and is therefore studied in the present paper. The effects of a variable horizontal magnetic field and uniform rotation, bearing relevancy in geophysics, are also considered.

2. Formulation of the Problem and Perturbation Equations

Let T_{ij}, τ_{ij}, e_{ij} δ_{ij}, v_i, x_i, p, μ and μ' denote the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, velocity vector, position vector, isotropic pressure, viscosity and viscoelasticity, respectively. The constitutive relations for the Walters B' elastico-viscous fluid are

$$\left.\begin{aligned} T_{ij} &= -p\,\delta_{ij} + \tau_{ij}, \\ \tau_{ij} &= 2\left[\mu - \mu'\frac{\partial}{\partial t}\right]e_{ij}, \\ e_{ij} &= \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_1}\right]. \end{aligned}\right\} \quad \ldots (1)$$

Here we consider a static state in which an incompressible Walters B' viscoelastic fluid containing suspended particles is arranged in horizontal strata in a porous medium. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and by following its further evolution.

Let $\boldsymbol{v}$ (u, v, w), ρ and p denote respectively the velocity of the pure fluid, the density and the pressure; $\boldsymbol{u}$ $(\bar{x}, t)$ and $N(\bar{x}, t)$ denote the velocity and number density of the suspended particles, respectively.

$K = 6\pi\mu\eta$, where η is the particle radius, is the Stokes' drag coefficient. $\boldsymbol{u} = (l, r, s), \bar{x} = (x, y, z)$ and $\boldsymbol{\lambda} = (0, 0, 1)$. Let ε, k_1 and g stand for medium porosity, medium permeability and acceleration due to gravity, respectively.

Then the equations of motion and continuity for the Walters B' viscoelastic fluid containing suspended particles in a porous medium are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{\varepsilon}(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right] = -\nabla p - \rho g \boldsymbol{\lambda} + \frac{KN}{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}) - \frac{1}{k_1}\left(\mu - \mu' \frac{\partial}{\partial t}\right)\boldsymbol{v}, \quad \ldots (2)$$

$$\nabla \cdot \boldsymbol{v} = 0. \quad \ldots (3)$$

Since the density of a fluid particle remains unchanged as we follow it with its motion, we have

$$\varepsilon\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = 0. \quad \ldots (4)$$

In the equations of motion (2), by assuming a uniform spherical particle shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term proportional to the velocity difference between the particles and the fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion of the particles. The distances between particles are assumed quite large compared with their diameter, so that interparticle reactions are ignored. The effects of pressure, gravity and Darcian force on the suspended particles are negligibly small and therefore ignored. If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions are

$$mN\left[\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{\varepsilon}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] = KN(\boldsymbol{v} - \boldsymbol{u}) \quad \ldots (5)$$

and

$$\varepsilon\frac{\partial N}{\partial t} + \nabla \cdot (N\boldsymbol{u}) = 0. \quad \ldots (6)$$

Let $\boldsymbol{v}$ (u, v, w), $\delta\rho$, δp, and $\boldsymbol{u}$ (l, r, s) denote respectively the perturbations in fluid velocity (0, 0, 0), fluid density ρ, fluid pressure p, and particle velocity (0, 0, 0). Then the linearized perturbation equations of the fluid-particle layer are

$$\frac{\rho}{\varepsilon}\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\delta p + \boldsymbol{g}\,\delta\rho + \frac{KN}{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}) - \frac{1}{k_1}\left(\mu - \mu'\frac{\partial}{\partial t}\right)\boldsymbol{v}, \quad \ldots (7)$$

$$\nabla \cdot \boldsymbol{v} = 0, \quad \ldots (8)$$

$$\varepsilon\frac{\partial}{\partial t}(\delta\rho) = -w\,(D\rho), \quad \ldots (9)$$

$$\left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)\boldsymbol{u}=\boldsymbol{v}, \quad \text{... (10)}$$

and

$$\frac{\partial M}{\partial t}+\nabla\cdot\boldsymbol{u}=0, \quad \text{... (11)}$$

where $M=\frac{\varepsilon N}{N_0}$ and N_0, N stand for initial uniform number density and perturbation in number density, respectively, $\boldsymbol{g}$ $(0, 0, -g)$ is the acceleration due to gravity and $D=\frac{d}{dz}$.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp\,(ik_x x+ik_y y+nt), \quad \text{... (12)}$$

where k_x, k_y are horizontal wave numbers, $k^2=k_x^2+k_y^2$, and n is a complex constant.

For perturbations of the form (12), eqs. (7)-(10) after eliminating $\boldsymbol{u}$ give

$$\frac{1}{\varepsilon}\left[\rho+\frac{mN}{\tau n+1}\right]nu=-ik_x\,\delta p-\frac{1}{k_1}[\mu-\mu' n]\,u, \quad \text{... (13)}$$

$$\frac{1}{\varepsilon}\left[\rho+\frac{mN}{\tau n+1}\right]nv=-ik_y\,\delta p-\frac{1}{k_1}[\mu-\mu' n]\,v, \quad \text{... (14)}$$

$$\frac{1}{\varepsilon}\left[\rho+\frac{mN}{\tau n+1}\right]nw=-D\,\delta p-g\delta\rho-\frac{1}{k_1}[\mu-\mu' n]\,w, \quad \text{... (15)}$$

$$ik_x u+ik_y v+Dw=0, \quad \text{... (16)}$$

and

$$\varepsilon n\,\delta\rho=-w\,D\,\rho, \quad \text{... (17)}$$

where

$$\tau=m/K.$$

Eliminating δp between eqs. (13), (15) and using (16) and (17), we obtain

$$\frac{n}{\varepsilon}[D(\rho\,Dw)-k^2\rho w]+\frac{n}{\varepsilon(\tau n+1)}[D\,(mNDw)-k^2 mNw]$$
$$+\frac{1}{k}[D\,\{(\mu-\mu' n)\,Dw\}-k^2\,\{\mu-\mu' n\}\,w]+\frac{1}{\varepsilon n}gk^2(D\,\rho)\,w=0. \quad \text{... (18)}$$

3. Two Uniform Viscous and Viscoelastic (Walters B') Fluids Separated by a Horizontal Boundary

Consider the case of two uniform fluids of densities, viscosities, suspended particles number densities; ρ_2, μ_2, N_2 (upper Newtonian fluid) and ρ_1, μ_1, N_1 (lower Walters B' viscoelastic fluid) separated by a horizontal boundary at $z=0$. Then, in each region of constant ρ, constant μ, μ' and constant mN, eq. (18) becomes

$$(D^2 - k^2)\, w = 0. \qquad \text{... (19)}$$

The general solution of eq. (19) is

$$w = Ae^{+kz} + B^{-kz}, \qquad \text{... (20)}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are :

(i) The velocity w should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).

(ii) $w(z)$ is continuous at $z = 0$.

(iii) The jump condition at the interface $z = 0$ between the fluids is obtained by integrating eq. (18) over an infinitesimal element of z including 0, and is

$$\frac{n}{\varepsilon}[\rho_2 Dw_2 - \rho_1 Dw_1]_{z=0} + \frac{mn}{\varepsilon(\tau n + 1)}[N_2 Dw_2 - N_1 Dw_1]_{z=0}$$

$$+ \frac{1}{k_1}[\mu_2 Dw_2 - (\mu_1 - n\mu'_1]_{z=0} = -\frac{gk^2}{\varepsilon n}[\rho_2 - \rho_1]\, w_0, \qquad \text{... (21)}$$

remembering that upper fluid is Newtonian and lower Walters B' viscoelastic. Here w_0 is the common value of w at $z = 0$.

Applying the boundary conditions (i) and (ii), we can write

$$w_1 = Ae^{+kz}, \ (z < 0) \qquad \text{... (22)}$$

and

$$w_2 = Ae^{-kz}, \ (z > 0), \qquad \text{... (23)}$$

where the same constant A has been chosen to ensure the continuity of w at $z = 0$.

Applying the condition (21) to the solution (22) and (23), we obtain

$$n^3\left[\tau - \frac{\varepsilon\tau\nu'_1\alpha_1}{k_1}\right] + n^2\left[1 + \frac{m(N_1 + N_2)}{\rho_2 + \rho_1} + \frac{\varepsilon\tau}{k_1}(\alpha_2\nu_2 + \alpha_1\nu^1)\right]$$

$$+ n\left[\frac{\varepsilon}{k_1}(\nu_2\alpha_2 + \nu_1\alpha_1) - gk\tau(\alpha_2 - \alpha_1)\right] - [gk(\alpha_2 - \alpha_1)] = 0, \qquad \text{... (24)}$$

where

$$\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1 + \rho_2},\ \nu_1 = \frac{\mu_1}{\rho_1},\ \nu_2 = \frac{\mu_2}{\rho_2},\ \nu'_1 = \frac{\mu'_1}{\rho_1}.$$

For the potentially stable arrangement ($\alpha_1 > \alpha_2$), the system is stable or unstable according as

$$\nu_1' \langle or \rangle \frac{k_1}{\varepsilon\alpha_1}. \qquad \text{... (25)}$$

For the potentially unstable configuration ($\alpha_2 > \alpha_1$), there is atleast one change of sign in eq. (24) and so this equation has one positive root. The occurrence of positive root implies that the system is unstable.

4. Effect of a Variable Horizontal Magnetic Field

Consider the motion of an incompressible, infinitely conducting Newtonian and Walters B' viscoelastic fluids in a porous medium in the presence of suspended particles and a variable horizontal magnetic field $\boldsymbol{H}(z), 0, 0)$. Let $\boldsymbol{h}(h_x, h_y, h_z)$ denote the perturbation in the magnetic field, then the linearized perturbation equations are

$$\frac{\rho}{\varepsilon}\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \delta p + \boldsymbol{g}\,\delta\rho + \frac{KN}{\varepsilon}(\boldsymbol{u}-\boldsymbol{v}) +$$

$$\frac{\mu_e}{4\pi}\{(\nabla\times\boldsymbol{H})\times\boldsymbol{h} + (\nabla\times\boldsymbol{h})\times\boldsymbol{H}\} - \frac{1}{k_1}\left(\mu - \mu'\frac{\partial}{\partial t}\right)\boldsymbol{v}, \qquad \dots (26)$$

$$\nabla\cdot\boldsymbol{h} = 0. \qquad \dots (27)$$

$$\varepsilon\frac{\partial \boldsymbol{h}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{H}), \qquad \dots (28)$$

together with eqs. (8)-(10). Assume that the perturbation $\boldsymbol{h}(h_x, h_y, h_z)$ in the magnetic field has also a space and time dependence of the form (12). Here μ_e stands for the magnetic permeability. Following the procedure as in Section 3, we obtain

$$n^3\left[\tau - \frac{\varepsilon\tau\nu_1'\alpha_1}{k_1}\right] + n^2\left[1 - \frac{\varepsilon\nu_1'\alpha_1}{k_1} + \frac{m(N_1+N_2)}{\rho_2+\rho_1} + \frac{\varepsilon\tau}{k_1}(\alpha_2\nu_2 + \alpha_1\nu_1)\right]$$

$$+ n\left[\frac{\varepsilon}{k_1}(\nu_2\alpha_2 + \nu_1\alpha_1) - gk\tau(\alpha_2-\alpha_1) + k_x^2 V_A^2\tau\right]$$

$$+ [k_x^2 V_A^2 - gk(\alpha_2-\alpha_1)] = 0, \qquad \dots (29)$$

where for the sake of simplicity, we have considered that the Alfvén velocities of the two fluids are the same, so that

$$V_A^2 = \frac{\mu_e H_1^2}{4\pi\rho_1} = \frac{\mu_e H_2^2}{4\pi\rho_2}.$$

For the potentially stable arrangement ($\alpha_2 < \alpha_1$), the system is stable or unstable according as

$$\nu_1' \langle\, or \,\rangle \frac{k_1}{\varepsilon\alpha_1}. \qquad \dots (30)$$

For the potentially unstable configuration $(\alpha_2 > \alpha_1)$, if

$$k_x^2 V_A^2 > gk(\alpha_2 - \alpha_1) \text{ and } \nu_1' < \frac{k_1}{\varepsilon \alpha_1}, \qquad \ldots (31)$$

Eq. (29) does not admit any change of sign and so has no positive root. Therefore, the system is stable. However, if

$$k_x^2 V_A^2 < gk(\alpha_2 - \alpha_1), \qquad \ldots (32)$$

the constant term in eq. (29) is negative. Equation (29), therefore, allows one change of sign and so has one positive root. The occurrence of a positive root implies that the system is unstable.

Thus for the unstable case $(\alpha_2 > \alpha_1)$, the magnetic field and viscoelasticity have got stabilizing effects and completely stabilize the system for all wave numbers which satisfy the inequalities

$$k_x^2 V_A^2 > gk(\alpha_2 - \alpha_1) \text{ and } \nu_1' < \frac{k_1}{\varepsilon \alpha_1}.$$

5. Effect ɔf Uniform Rotation

Here we consider the motion of an incompressible Walters B' viscoelastic fluid containing suspended particles in a porous medium in uniform rotation $\boldsymbol{\Omega}(0, 0, \Omega)$. Then the linearized perturbation equations are

$$\frac{\rho}{\varepsilon}\frac{\partial \mathbf{v}}{\partial t} = -\nabla \delta p + \mathbf{g}\,\delta\rho + \frac{KN}{\varepsilon}(\mathbf{u} - \mathbf{v}) + \frac{2\rho}{\varepsilon}(\mathbf{v} \times \boldsymbol{\Omega}) - \frac{1}{k_1}\left(\mu - \mu' \frac{\partial}{\partial t}\right)\mathbf{v}, \qquad \ldots (33)$$

together with eqs. (8)-(10).

Following the same procedure as in Section 3 (and Chandrasekhar [1], p. 443), we obtain

$$1 + \frac{mn(N_1 + N_2)}{(\tau n + 1)\left[\left\{n + \frac{\varepsilon}{k_1}(\nu - \nu' n)\right\}\rho_1 + \left\{n + \frac{\varepsilon\nu}{k_1}\right\}\rho_2\right]} -$$

$$\frac{gk^2(\rho_2 - \rho_1)}{nx\left[\left\{n + \frac{\varepsilon}{k_1}(\nu - \nu' n)\right\}\rho_1 + \left\{n + \frac{\varepsilon\nu}{k_1}\right\}\rho_2\right]} +$$

$$\frac{4(\tau n + 1)\Omega^2(\rho_2 - \rho_1)}{\left[(\tau n + 1)n + \frac{mnN}{\rho} + (\tau n + 1)\frac{\varepsilon}{k_1}(\nu - \nu' n)\right]\left[\left\{n + \frac{\varepsilon}{k_1}(\nu - \nu' n)\right\}\rho_1 + \left\{n + \frac{\varepsilon\nu}{k_1}\right\}\rho_2\right]}$$

$$= 0, \qquad \ldots (34)$$

where

$$\chi = \frac{k}{\left[1 + \dfrac{2\Omega^2 (\tau n + 1)^2}{\left\{ n(\tau n + 1) + \dfrac{nmN}{\rho} + (\tau n + 1) \dfrac{\varepsilon}{k_1} (\nu - \nu' n) \right\}^2} \right]}, \quad \ldots (35)$$

for a highly viscous fluid. The terms $\nu (= \mu/\rho)$ and $\nu' (= \mu'/\rho)$ stand for the kinematic viscosity and kinematic viscoelasticity respectively. Here we assumed the kinematic viscosities and kinematic viscoelasticities of both fluids to be equal i.e., $\nu_1 = \nu_2 = \nu$ (Chandrasekhar[2], p. 443), $\nu_1' = \nu_2' = \nu'$ and $\frac{mN}{\rho} = \frac{mN_1}{\rho_1} = \frac{mN_2}{\rho_2}$, as these simplifying assumptions do not obscure any of the essential features of the problem.

Equation (34), after substituting the value of χ from (35) and simplification, yields

$$A_9 n^9 + A_8 n^8 + A_7 n^7 + \ldots\ldots\ldots + A_2 n^2 + A_1 n + A_0 = 0, \quad \ldots (36)$$

where

$$A_9 = \tau^4 \left\{ 1 + \frac{\varepsilon^2 \nu^2}{k_1^2} \right\} \left\{ 1 - \frac{\varepsilon \nu'}{k_1} \right\} \left\{ \left(1 - \frac{\varepsilon \nu' \alpha_1}{k_1} \right) + \alpha_2 \right\}$$

$$A_0 = -gk [\alpha_2 - \alpha_1] \frac{\varepsilon \nu}{k_1} \left[\frac{\varepsilon^2 \nu^2}{k_1^2} + 2\Omega^2 \right], \quad \ldots (37)$$

and the coefficients $A_1 - A_8$, being quite lengthy and not needed in the discussion of stability, have not been written here. For the potentially stable arrangement $(\alpha_2 < \alpha_1)$ if $\nu_1' < \frac{k_1}{\varepsilon \alpha_1}$, eq. (36) does not allow any positive root as there is no change of sign. The system is therefore stable. Thus when the ordinary (Newtonian) viscous fluid overlies Walters B' viscoelastic fluid in a porous medium in the presence of suspended particles and a uniform rotation, the system is stable for the potentially stable configuration if $\nu'_1 < \frac{k_1}{\varepsilon \alpha_1}$. Otherwise if $\nu'_1 > \frac{k_1}{\varepsilon \alpha_1}$, the system is unstable for potentially stable configuration.

For the potentially unstable arrangement $(\alpha_2 > \alpha_1)$, the constant term is negative and so there is at least one change of sign in eq. (36). Therefore, eq. (36) allows at least one positive root of n, meaning thereby instability of the system.

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