Indian J. pure appl. Math., 32(2) : 191-198, February 2001

ON THE SPECTRUM OF THE RHALY OPERATORS ON l_p

M. YILDIRIM

Department of Mathematics, Faculty of Science,
Cumhuriyet University Sivas 58140, *Turkey*

(*Received* 23 *July* 1999; *after Revision* 20 *July* 2000; *accepted* 17 *October* 2000)

In 1989, Rhaly determined the spectrum of R_a, the Rhaly operator which is represented by the matrix :

$$R_a = \begin{pmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & \dots \\ a_2 & a_2 & a_2 & \dots \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix}$$

regarded as an operator on the Hilbert space l_2 normed by $\| x \| = \left(\sum_n | x_n |^2 \right)^{1/2}$. It is the purpose of this paper to determine the spectrum of Rhaly operator R_a as an operator on the spaces l_p of all sequence x such that $\sum_n | x_n |^p < \infty$ holds.

Key Words and Phrases : Rhaly Operator; Cesàro Operator; Spectrum and Point Spectrum

1. INTRODUCTION[1]

Assume $a = (a_n)$ is a sequence of real numbers such that the Rhaly matrix R_a is a bounded linear operator on l_p. For $a = \left(\frac{1}{n+1} \right)$, the spectra on c_0 and c of Cesàro matrix are studied respectively in [5] and [8]. In [6] taking $L = \lim_n \, (n+1)\, a_n$ Rhaly showed that R_a is a bounded operator on l_2, and he also determined its l_2-spectrum and eigenvalues. In previous work [9] and [10], the spectrum of Rhaly operator R_a on c_0 and c are calculated.

The set of all eigenvalues and the spectrum of a bounded operator T on a Banach space X are denoted by $\pi_0(T, X)$ and $\sigma(T, X)$, respectively.

By Goldberg[2], if X is a Banach space, $B(X)$ denotes the collection of all bounded linear operators on X and $T \in B(X)$, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$;

(II) $\overline{R(T)} = X$, but $R(T) \neq X$; and

(III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous and

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by : $I_1, I_2, I_3, II_2, II_3, III_1, III_2, III_3$. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous.

Applying Goldberg's classification to the operator, $A := \lambda I - T$, where $\lambda \in \sigma(T, X)$ the spectrum of T, considered as an operator in $B(X)$ where $X = l_p$, we have

(I) $A = \lambda I - T$ is subjective

(II) $\overline{R(A)} = X$, but $R(A) \neq X$ and

(III) $\overline{R(A)} \neq X$

and three possibilities for A^{-1}:

(1) $A = \lambda I - T$ is injective and A^{-1} is bounded,

(2) $A = \lambda I - T$ is injective and A^{-1} is unbounded and

(3) $A = \lambda I - T$ is not injective.

If λ is a complex number such that $A = \lambda I - T \in I_1$ or $A = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $I_2\, \sigma(T, X), I_3\, \sigma(T, X), II_2\, \sigma(T, X), II_3\, \sigma(T, X), III_1\, \sigma(T, X), III_2\, \sigma(T, X), III_3\, \sigma(T, X)$. For example, if $A = \lambda I - T$ is in a given state, III_2 (say), then we write $\lambda \in III_2\, \sigma(T, X)$.

In this study, we shall assume throughout that (a_n) is a strictly decreasing sequence of positive real numbers and that $\lim_n (n+1)\, a_n = L$. Under these hypotheses, we determine spectrum and the fine Spectrum of R_a on l_p. We write $S = \{a_n : n = 0, 1, \ldots\}$ for the range of a.

2. The Spectrum and the Fine Spectrum of $\boldsymbol{R_a}$ on l_p for $L = 0$

Leibowitz [3, Prop 3.1] shows that

a) if $\{(n+1)\, a_n\}$ is bounded then $\boldsymbol{R_a}$ acts boundedly on l_p for $p > 1$, and $\| \boldsymbol{R_a} \| \leq (p/(p-1)) \sup_n |(n+1)|\, a_n|$.

b) If $\lim_n |(n+1)\, a_n| = 0$, then $\boldsymbol{R_a}$ acts boundedly on l_p for every $p > 1$.

c) If $\lim_n |(n+1)\, a_n| = \infty$, then $\boldsymbol{R_a}$ is not bounded on any l_p for every $p > 1$.

Leibowitz [3, Lemma] further shows that

If $\alpha > 0$, then $|1 + h|^{\alpha} = 1 = \alpha \operatorname{Re}(h) + O(|h|^2)$ as $h \to 0$ in C.

***Theorem* 2.1** — *If $L = \lim_n (n+1) a_n = 0$, then $\pi_0(\boldsymbol{R_a}, l_p) = S$ for $p \geq 2$.*

PROOF : If $\boldsymbol{R_a} x = \lambda x$ then $(a_0 - \lambda) x_0 = 0$ and $(\lambda a_n^{-1} - 1) x_n = a_{n-1}^{-1} \lambda x_{n-1}$ for $n \geq 1$. Since each $a_n > 0$, $0 \notin \pi_0(\boldsymbol{R_a}, l_p)$. If m is the smallest integer for which $x_m \neq 0$, then $\lambda = a_m$ and

$$x_n = \prod_{j=m+1}^{n} \frac{\lambda a_{j-1}^{-1}}{\lambda a_j^{-1} - 1} x_m \qquad \text{... (1)}$$

for $n \geq m+1$. From (1) we conclude that the eigenvalues of $\boldsymbol{R_a}$ are simple. Let us search for the condition that $x \in l_p$ when $\lambda = a_m$.

For $n \geq m$, since

$$\left| \frac{x_n}{x_{n+1}} \right|^p = \frac{|\lambda - a_{n+1}|^p}{|\lambda|^p} \frac{a_n^p}{a_{n+1}^p}$$

the ratio test is inconclusive, we turn of Kummer's test, [4, p. 395] with $p_n = \frac{1}{n a_n^p}$. Hence together with Leibowitz's lemma above, we have

$$\lim_{n \to \infty} \left(p_n \left| \frac{x_n}{x_{n+1}} \right|^p - p_{n+1} \right) = \lim_{n \to \infty} \left[\frac{1}{n a_n^p} \frac{|\lambda - a_{n+1}|^p}{|\lambda|^p} \frac{a_n^p}{a_{n+1}^p} - \frac{1}{(n+1) a_{n+1}^p} \right]$$

$$= \lim_{n \to \infty} \frac{1 - p(n+1) a_{n+1} Re(1/\lambda) + (n+1) a_{n+1}^2 O(1/|\lambda|^2)}{n(n+1) a_{n+1}^2 a_{n+1}^{p-2}} = \infty,$$

for $p > 2$. Rhaly[6] showed that $\pi_0(R_a, l_2) = S$. Hence $\pi_0(R_a, l_p) = S$ for $p \geq 2$.

$\boldsymbol{R_a^*}$ adjoint operator of R_a regarded as operator on the space l_p is the transpose of R_a [9].

***Theorem* 2.2** — *If $L = \lim_n (n+1) a_n = 0$, then $\pi_0(\boldsymbol{R_a^*}, l_p^* \cong l_q) = S$ for $p \geq 2$.*

PROOF : Since R_a^* is transpose of the matrix of R_a, if $\boldsymbol{R_a^*} x = \lambda x$, we have $\lambda a_n^{-1} x_{n+1} = (\lambda a_n^{-1} - 1) x_n$. Since $\lambda = 0$ implies that all $x_n = 0$, $0 \notin \pi_0(\boldsymbol{R_a^*}, l_q)$ and hence $x_{n+1} = \left(1 - \frac{a_n}{\lambda}\right) x_n$ for all $n \geq 0$. It follows that every $a_m \in S$ is an eigenvalue of $\boldsymbol{R_a^*}$, that every eigenvalue λ is simple, and that if $\lambda \neq a_m$ then

$$x_n = \prod_{j=0}^{n-1} \left(1 - \frac{a_j}{\lambda}\right) x_0. \qquad \text{... (2)}$$

Now we show that if λ is not onle of the a_n's then λ is not an eigenvalue. If x satisfies (2), then

$$\lim_{n\to\infty}\frac{|x_{n+1}|^q}{|x_n|^q}=\lim_{n\to\infty}\frac{|\lambda-a_n|^q}{|\lambda|^q}. \qquad \dots (3)$$

Since the ratio test is inconclusive, we turn to Raabe's test, by (3) and Leibowitz's lemma we have

$$\lim_{n\to\infty} n\left(\left|1-\frac{a_n}{\lambda}\right|^q-1\right)=\lim_{n\to\infty} n\,[-a_n\, qRe\,(1/\lambda)+a_n^2\, O\,(1/|\lambda|^2)]=0.$$

Therefore the series diverges, and the only eigenvalues are the a_n.

***Theorem* 2.3** — *If $L = 0$, then $\sigma(\boldsymbol{R}_a, l_p)=S\cup\{0\}$ for $p\geq 2$.*

PROOF : For $L = 0$, since $\boldsymbol{R}_a$ is a compact operator, $0\in\sigma(\boldsymbol{R}_a, l_p)$ [7, p. 99] and if $0\neq\lambda\in\sigma(\boldsymbol{R}_a, l_p)$ then $\lambda\in\pi_0(\boldsymbol{R}_a, l_p)\cup\pi_0(\boldsymbol{R}_a^*, l_q)$. Combining these results with Theorem 2.1 and 2.2 we have the proof.

***Theorem* 2.4** — *If $L = 0$, then $0\in H_2\,\sigma(\boldsymbol{R}_a, l_p)$ for $p\geq 2$.*

PROOF : The proof can easily be done following the reasoning in [9, Theorem 6].

***Theorem* 2.5** — *If $L = 0$, then $(m = 0, 1, \dots)$, $\lambda=a_m\in III_3\,\sigma(\boldsymbol{R}_a, l_p)$ for $p\geq 2$.*

PROOF : Can easily be done following the reasoning in [9, Theorem 7].

Finally when $0< L=\lim_n (n+1)\,a_n<\infty$, let us determine the spectrum of the $\boldsymbol{R}_a$ Rhaly operator.

3. THE SPECTRUM OF $\boldsymbol{R}_a$ ON l_p FOR $0<L<\infty$

Lemma 3.1 — If $Re\,\frac{1}{\lambda}=\alpha$ and $0\neq\lim_n (n+1)\,a_n=l<\infty$, then

$$\prod_{k=0}^{N-1}\left|1-\frac{a_k}{\lambda}\right|^q=\frac{1}{N^{\alpha L_q}}.$$

as $N\to\infty$.

We use the notation $a_n \simeq b_n$ in the sense that $\left(\frac{a_n}{b_n}\right),\left(\frac{b_n}{a_n}\right)$ are both bounded.

PROOF : Since $\lim_n (n+1)\,a_n=L, a_n\simeq\frac{L}{n}$. By using $e^x\geq 1+x\,(x\in R)$ we have

$$\prod_{j=0}^{N-1}\left|1-\frac{a_j}{\lambda}\right|^q=\prod_{j=0}^{N-1}\left(\left|1-\frac{a_j}{\lambda}\right|^2\right)^{\frac{q}{2}}=\prod_{j=0}^{N-1}\left\{|1-a_j(\alpha+i\beta)|^2\right\}^{\frac{q}{2}}$$

$$=\prod_{j=0}^{N-1}\left(1-2a_j\alpha+(\alpha^2+\beta^2)\,a_j^2\right)^{\frac{q}{2}}$$

$$\leq \prod_{j=0}^{N-1} \exp\left(-qa_j\,\alpha + \frac{(\alpha^2+\beta^2)\,q}{2}\,a_j^2 \right)$$

$$= \leq \exp \sum_{j=0}^{N-1} \left((-q\alpha)\,a_j + \frac{(\alpha^2+\beta^2)\,q}{2}\,a_j^2 \right)$$

$$\leq \exp\left[(-\alpha qL) \sum_{j=0}^{N-1} \frac{1}{n+1} + O(1) \right] \leq \exp\left[-\alpha qL \log N + O(1) \right] = \frac{O(1)}{N^{\alpha qL}}.$$

Theorem 3.1 — *If* $0 < l < \infty$, *then* $S \cap (2L, \infty) \subset \pi_0\,(\boldsymbol{R}_a, l_p) \subset S \cap [2L, \infty)$.

PROOF : Since $S \cap (2L, \infty) \subset \pi_0\,(R_a, l_p) \subset S \cap [2L, \infty)$ from [10] and $l_p \subset C_0$, the proof is obvious.

Theorem 3.2 — *If* $0 < L < \infty$, *then*

$$\pi_0\,(\boldsymbol{R}_a^*, l_q) = \left\{ \lambda : \left| \lambda - \frac{q^L}{2} \right| < \frac{q^L}{2} \right\} \cup S.$$

PROOF : If $\boldsymbol{R}_a^*\,x = \lambda x$, we have $\lambda x_{n+1} = (\lambda - a_n)\,x_n$. From the proof of Theorem 2.2, $0 \notin \pi_0\,(\boldsymbol{R}_0^*, l_q)$. Since $x_{n+1} = \left(1 - \frac{a_n}{\lambda} \right) x_n$, if $\lambda = a_m$, then $\lambda \in \pi_0\,(\boldsymbol{R}_a^*, l_q)$. Hence now

$$x_n = \prod_{j=0}^{n-1} \left(1 - \frac{a_j}{\lambda} \right) x_0$$

is obtained. For other eigenvalues, with $\alpha = Re\,\frac{1}{\lambda}$, we have the expression

$$|\,x_n\,|^q = \left(\prod_{j=0}^{n-1} \left| 1 - \frac{a_j}{\lambda} \right| |\,x_0\,| \right)^q = |\,x_0\,|^q \prod_{j=0}^{n-1} \left| 1 - \frac{a_j}{\lambda} \right|^q \simeq \frac{|\,x_0\,|^q\,O(1)}{n^{\alpha Lq}}.$$

Therefore we need $\alpha Lq > 1$, should read which is equivalent to $\left| \lambda - \frac{qL}{2} \right| < \frac{qL}{2}$ is obtained.

Theorem 3.3 — *If* $0 \neq L < \infty$ *then*

$$\sigma\,(\boldsymbol{R}_a, l_p) = \left\{ \lambda : \left| \lambda - \frac{qL}{2} \right| \leq \frac{qL}{2} \right\} \cup S$$

for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF : From Theorem 3.2. we obtain

$$\left\{\lambda : \left|\lambda - \frac{qL}{2}\right| \le \frac{qL}{2}\right\} \cup S \subseteq \sigma(\boldsymbol{R_a}, l_p).$$

To complete the proof let us show that,

$$\sigma(\boldsymbol{R_a}, l_p) \subseteq \left\{\lambda : \left|\lambda - \frac{qL}{2}\right| \le \frac{qL}{2}\right\} \cup S$$

Now, let $\left|\lambda - \frac{qL}{2}\right| > \frac{qL}{2}$ (which means that $\alpha L < \frac{1}{q}$ where $\alpha = Re\frac{1}{\lambda}$) and $\lambda \ne a_m$ $(m = 0, 1, 2, \ldots)$. If $T_\lambda = \lambda I - \boldsymbol{R_a}$ then T_λ^{-1} exists, and we can write T_λ^{-1} as

$$T_\lambda^{-1} = (b_{nk}) = \begin{cases} \dfrac{1}{\lambda - a_n} & k = n \\ \dfrac{a_n}{\lambda^2 \prod_{j=k}^{n} \left(1 - \frac{a_j}{\lambda}\right)} & k < n \\ 0 & \text{otherwise.} \end{cases}$$

Now, we must show that $T_\lambda^{-1} \in b(l_p)$.

From [1, Theorem 2], if A is a nonnegative triangular matrix and (b_n) is a positive sequence such that $M_1 := sup_n \sum_k |a_{nk}| (b_k/b_n)^{1/p} < \infty$ and $M_2 := sup_k \sum_n |a_{nk}| (b_n/b_k)^{1/q} < \infty$ then $A \in B(l_p)$ and $\| A \| \le M_2^{1/q} M_2^{1/p}$.

In the above expression, let $b_n = a_n > 0$ and $a_{nk} = |b_{nk}|$.

Hence, since $\alpha = Re\frac{1}{\lambda}$, $\alpha L < \frac{1}{q}$ we have, using Lemma 3.1,

$$\sum_{k=0}^{n} a_{nk} (b_k/b_n)^{1/p} = |a_{nn}| (b_n/b_n)^{1/p} + \sum_{k=0}^{n-1} a_{nk} (b_k/b_n)^{1/p}$$

$$= \frac{1}{|\lambda - a_n|} + \sum_{k=0}^{n-1} \frac{a_n^{1-1/p} a_k^{1/p}}{|\lambda|^2 \prod_{j=k}^{n} \left|1 - \frac{a_j}{\lambda}\right|}$$

$$= \frac{1}{|\lambda - a_n|} + \frac{a_n^{1-1/p}}{|\lambda|^2} \sum_{k=0}^{n-1} \frac{a_k^{1/p}}{\prod_{j=k}^{n} \left|1 - \frac{a_j}{\lambda}\right|}$$

$$= \frac{1}{|\lambda - a_n|} + \frac{a_n^{1-1/p}}{|\lambda|^2} \left\{ \frac{a_0^{1/p}}{\prod_{j=0}^{n} \left| 1 - \frac{a_j}{\lambda} \right|} + \sum_{k=1}^{n-1} \frac{a_k^{1/p} \prod_{j=0}^{k-1} \left| 1 - \frac{a_j}{\lambda} \right|}{\prod_{j=k}^{n} \left| 1 - \frac{a_j}{\lambda} \right|} \right\}$$

$$\leq O(1) + O(1)\,(n+1)\, a_n\, a_n^{1/p}\, (n+1)^{\alpha L - 1} \left\{ a_0^{1/p} + \sum_{k=1}^{n-1} \frac{a_k^{1/p}}{k^{\alpha L}} \right\}.$$

Since $\lim_n (n+1)\, a_n = L > 0$, there exists a positive integer N such that, for all $n \geq N$, $L/2 < (n+1)\, a_n < 3L/2$. Also, $\lim_n \ (n+1)\, a_n = L$ implies that $\lim_n \ a_n = 0$.

Therefore

$$\sum_{k=0}^{n} a_{nk}\, (b_k/b_n)^{1/p} = O(1) + o(1)$$

$$+ \ O(1)\, [(n+1)\, a_n]^{1-1/p}\, (n+1)^{\alpha L - 1/q} \left\{ \frac{3L}{2} \sum_{k=N}^{n-1} \frac{1}{(k+1)^{1/p}\, k^{\alpha L}} \right\}$$

$$= \ O(1) + O(1)\, (n+1)^{\alpha L - 1/q} \left\{ \sum_{k=N}^{n-1} \frac{1}{k^{\alpha L + 1/p}} \right\}$$

$$= \ O(1)\, (n+1)^{\alpha L - 1/q} \left\{ \int_N^{n-1} x^{-\alpha L - 1/p}\, dx \right\}$$

$$= \ O(1)\, (n+1)^{\alpha L - 1/q}\, n^{-\alpha L - 1/q + 1} = O\,(1).$$

Now, as $\alpha L + \frac{1}{p} - 1 < 0$

$$M_1 = \ \sup_n \sum_k a_{nk}\, |\, (b_k/b_n)^{1/p} < \infty$$

holds.

$$\sum_{n=k}^{\infty} a_{nk}\, (b_n/b_k)^{1/q} = a_{kk} + \sum_{n=k+1}^{\infty} a_{nk}\, (b_n/b_k)^{1/q}.$$

Without loss of generality, we may assume that $k > N$. For otherwise,

$$\sum_{n=k}^{\infty} a_{nk}(b_n/b_k)^{1/q} = \sum_{n=k}^{N} a_{nk}(b_n/b_k)^{1/q} \sum_{m=N+1} a_{nk}(b_n/b_k)^{1/q}$$

$$= O(1) + \sum_{n=N+1}^{\infty} a_{nk}(b_n/b_k)^{1/q}$$

Using Lemma 3.1,

$$\sum_{n=k+1}^{\infty} a_{nk}(b_n/b_k)^{1/q} = \sum_{n=k+1}^{\infty} \frac{a_n^{1+1/q} a_k^{-1/q}}{|\lambda|^2 \prod_{j=k}^{n} |1-\frac{a_j}{\lambda}}$$

$$= O(1)\, a_k^{-1/q} \sum_{n=k+1}^{\infty} \frac{(n+1)^{\alpha L} 3L}{2(n+1)^{1+1/q} k^{\alpha L}}$$

$$= O(1)\left(\frac{2}{L(k+1)^{1/q} k^{\alpha L}}\right) \sum_{n=k+1}^{\infty} (n+1)^{\alpha L-1-1/q}$$

$$= O(1)\,(k+1)^{-1/q-\alpha L} \int_{k}^{\infty} (x+1)^{\alpha L-1-1/q}\, dx$$

$$= O(1)\,(k+1)^{-1/q-\alpha L}(k+1)^{\alpha L-1/q} = O(1)$$

Hence $M_2 = sup_k \sum_k |a_{nk}|(b_n/b_k)^{1/q} < \infty$. For $\left|\lambda - \frac{qL}{2}\right| > \frac{qL}{2}$ and $\lambda \notin S, T^{-1} \in B(l_p)$, yielding $\lambda \in \rho(R_a, l_p)$. This completes the proof.

REFERENCES

1. D. Borwein and A. Jakimowski, *Rocky Mountain J. math.* 4 (1979), 463-77.
2. S. Goldberg, Unbounded Linear Operators, Mc Graw-Hill Book Comp, 1966.
3. G. Leibowitz, *J. math. Anal. Appl.* **128** (1987), 272-86.
4. J. M. H. Olmsted, *Advanced Calculus*, Appleton-Century-Crofts, New York, 1961.
5. J. B. Reade, *Bull. London. math. Soc.* **17** (1985), 263-67.
6. Jr. H. C. Rhaly, *Bull. London math. Soc.* **21** (2989), 399-406.
7. W. Rudin. *Functional Analysis*. (Mc Graw-Hill Book Comp, 1973).
8. R. B. Wenger, *Indian J. Pure. Appl. Math.* **6** (1965), 695-712.
9. M. Yildirim, *Indian J. pure. appl. Math.* **27** (8) (1998), 779-84.
10. M. Yildirim, *Indian J. Pure. Appl. Math.* **29** (12) (1998), 1301-1309.