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PSEUDOCONVEX COMPLEX MINIMAX PROGRAMMING

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Necessary and sufficient optimality conditions are derived for a general complex minimax problem. A dual problem is also defined and weak and strong duality proved. This paper gives a different approach to those of previous treatments of the complex minmax problem, as the results do not depend on certain complicated generalized convexity conditions being imposed on the objective function.

Key Words : Complex Minimax Programming; Pseudoconvexity; Duality; Analytic Function

1. INTRODUCTION

Mathematical programming in complex space originated from Levinson's discussion of linear problems[7]. Several authors have recently been interested in the optimality conditions and the duality theorems for complex nonlinear programming. For details, the readers are devised to consult[4-6, 8-10, 12-14] and chapter 7 of Stancu-Minasian[12].

Craven and Mond[3] have given Fritz John type necessary conditions for a class of nonlinear programming problems in complex space over polyhedral cones. Datta and Bhatia[4] established the necessary and sufficient conditions for the static minmax problems in complex space as extensions of the corresponding real space conditions of [11].

In this paper, we derive the necessary and sufficient conditions for minmax problems in complex space which are extensions of the corresponding real space conditions of [13]. These conditions are then used to extend some duality results of [13] to complex space over arbitrary polyhedral cones.

Throughout this paper, we follow the notations similar to Liu *et al*[9].

Let C^n denote an n-dimensional complex space, $C^{m \times n}$ the collection of $m \times n$ complex matrices. For $z \in C^n$, let the real vectors $Re\ (z)$ and $Im\ (z)$ denote real and imaginary parts of each component of z, respectively, and write $\bar{z} = Re\ (z) - i\, Im\ (z)$ as the conjugate of z. Given a matrix $A = [a_{ij}] \in C^{m \times n}$, we use $\bar{A} = [\bar{a}_{ij}]$ to express its conjugate matrix, and $A^H = [\bar{a}_{ji}]$ to express its conjugate transpose. The inner product of $x, y \in C^n$ is $\langle x, y \rangle = y^H x$.

A nonempty subset S of C^m is said to be a polyhedral cone if there is an integer r and a matrix $K \in C^{r \times m}$ such that

$$S = \{Z \in C^m : Re\ (Kz) \geq 0\}.$$

The dual (also polar) of S is

$$S^* = \{w \in C^m : z \in S \Rightarrow Re\ \langle z, w \rangle \geq 0\}.$$

It is clear that $S = S^{**}$ if S is a polyhedral cone. Let S be a polyhedral cone in C^m and $s_0 \in S$. Then $S\ (s_0)$ is defined to be the intersection of those closed half spaces determining S which include s_0 in their boundaries, or equivalently, if S is specified by $K \in C^{r \times m}$ then $S\ (s_0)$ is specified by a submatrix K_1 of K, i.e.,

$$S\ (s_0) = \{z \in C^m : Re\ K_1 z \geq 0\}.$$

For a complex function $f : C^n \times C^n \to C$ analytic in the $2n$ variales (w_1, w_2) at the point $(z_0, \bar{z}_0) \in C^n \times C^n$ define the gradients by

$$\nabla_z f(z_0, \bar{z}_0) = \left[\frac{\partial f}{\partial (w_1)\, i} (z_0, \bar{z}_0) \right], \ i = 1, 2, \ldots, n;$$

$$\nabla_{\bar{z}} f(z_0, \overline{z_0}) = \left[\frac{\partial f}{\partial (w_2)_i} (z_0, \overline{z_0}) \right], \ i = 1, 2, \ldots, n;$$

and
$$\nabla_\zeta f(z_0, \bar{z}_0) = \left[\begin{array}{c} \nabla_z f(z_0, \overline{z_0} \\ \nabla_z f(z_0, \bar{z}_0) \end{array} \right], \text{ where } \zeta = \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \text{ and } \zeta_0 = \begin{pmatrix} z_0 \\ \bar{z}_0 \end{pmatrix}.$$

Similarly, for a function $g : C^n \times C^n \to C^m$ analytic in the $2n$ variables (w_1, w_2) at the point $(z_0, \bar{z}_0) \in C^n \times C^n$ define the gradients by

$$\nabla_z\, g(z_0, \bar{z}_0) = \left[\frac{\partial g_i}{\partial (w_i)_j} (z_0, \bar{z}_0) \right] \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n;$$

$$\nabla_{\bar{z}}\, g\ (z_0, \bar{z}_0) = \left[\frac{\partial g_i}{\partial (w_2)_j} (z_0, \bar{z}_0) \right] \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n;$$

and
$$\nabla_\zeta\, g\ (z_0, \bar{z}_0) = \left[\begin{array}{c} \nabla_z g\ (z_0, \bar{z}_0) \\ \nabla_{\bar{z}} g\ (z_0, \bar{z}_0) \end{array} \right].$$

Then, we have

$$\nabla_z\, y^H g\ (z_0, \bar{z}_0) = \nabla_z g\ (z_0, \bar{z}_0) \bar{y} \ \text{ for } y \in C^m$$

The following definitions from [1, 9, 14] will be needed in the sequel :

Definition 1.1 — The real part of f is said to be convex with respect to R_+, if, for all $z_1, z_2 \in C^n$,

$$Re\ [f(z_2, \bar{z}_2) - f(z_1, \bar{z}_1) - (z_2 - z_1)^T\ \nabla_z f(z_1, \bar{z}_1) - (z_2 - z_1)^H\ \nabla_{\bar{z}} f(z_2 - z_1)] \geq 0.$$

The function $-\boldsymbol{g}$ is said to be convex with respect to the polyhedral cone S if, for all $\mu \in S$ and all $z_1, z_2 \in C^n$

$$Re\ \langle \mu, -\boldsymbol{g}\ (z_2, \bar{z}_2) + \boldsymbol{g}\ (z_1, \bar{z}_1) + \nabla_z \boldsymbol{g}\ (z_2, \bar{z}_1)\ (z_2 - z_1)\ +\ \nabla_{\bar{z}}\, g\ (z_1, \bar{z}_1)\ \overline{(z_2 - z_1)}\ \rangle \geq 0$$

Definition 1.2 — The real part of f is said to be pseudoconvex with respect to R_+,

$$Re\ [(z_2 - z_1)^T\ \nabla_z\, g(z_1, \bar{z}_1) + (z_2 - z_1)^H\ \nabla_{\bar{z}} f(z_1, \bar{z}_1)] \geq 0$$

$$\Rightarrow Re\ [f(z_2, \bar{z}_2) - f(z_1, \bar{z}_1)] \geq 0, \text{ for all } z_1, z_2 \in C^n.$$

The function $-g$ is said to be quasiconvex with respect to the polyhedral cone S if

$$Re\ \langle \mu, -g\ (z_2, \bar{z}_2) + g\ (z_1, \bar{z}_1)\ \rangle \leq 0$$

$$\Rightarrow Re\ \langle \mu, \nabla_z\, g\ (z_1, \bar{z}_1)\ (z_2 - z_1) + \nabla_{\bar{z}}\, g(z_1, \bar{z}_1)\ \overline{(z_2 - z_1)}\ \rangle \geq 0$$

for all $\mu \in S$ and all $z_1, z_2 \in C^n$

The complex minmax problem that we consider seeks to choose $\zeta \in \overset{\circ}{S} =. \{\zeta \in C^{2n} : -g\ (\zeta) \in S\}$, which minimizes $f(\zeta) = \sup_{\eta \in W} Re\ \phi\ (\zeta, \eta)$, where $\zeta = (z, \bar{z})$ and $\eta = (w, \bar{w})$ for $z \in C^n$ and $w \in C^m$ $\phi\ (\cdot, \cdot) : C^{2n} \times C^{2m} \to C$ is analytic with respect to ζ, W is a specified compact subset of C^{2m}, S is the polyhedral cone in C^p and $g : C^{2n} \to C^p$ is analytic.

2. Necessary and Sufficient Conditions for General Complex Minimax Problem

For $\zeta = (z, \bar{z}) \in \overset{\circ}{S}$, we define

$$W(\zeta)\ =\ \{\eta \in W : \left\{ Re\ \phi\ (\zeta, \eta) = \sup_{\zeta \in W}\ \phi\ (\zeta, \xi) \right\},$$

and note that $W(\zeta)$ is compact and nonempty. Datta and Bhatia[4] established the following Fritz John necessary optimality conditions : Let $\phi\ (\cdot, \cdot) : C^{2n} \times C^{2m} \to C$ be analytic with respect to ζ, for each $\eta \in W$, $g : C^{2n} \to C^p$ be analytic with respect to ζ and $S \subset C^p$ be a polyhedral cone with nonempty interior. Let ζ^* be a solution of the minimax problem, then there exist a positive integer s, scalars $\lambda_i \geq 0$, $i = 1, 2, ..., s$, $0 \neq \mu \in S^*$ and vectors $\eta_i \in W(\zeta^*)$, $i = 1, 2, ..., s$ such that

$$\sum_{i=1}^{s} \lambda_i \overline{\nabla_z \phi(\zeta^*, \eta_i)} + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta^*, \eta_i) + \mu^T \overline{\nabla_z g(\zeta^*)} + \mu^H \nabla_{\bar{z}} g(\zeta^*) = 0,$$

$$Re\ \mu^H g(\zeta^*) = 0.$$

We now extend the regularity conditions proposed by Weir[13] to the complex case.

Assuming that g_i, $i = 1, 2, \ldots, p$ are pseudoconvex at ζ^* and there exists a feasible ζ' for (P) with

$Re\ [g_i(\zeta')] < 0$, $i = 1, 2, \ldots, p$. If $\lambda_i^* = 0$, $i = 1, 2, \ldots, s$, then $(\mu_i^*\ i = 1, 2, \ldots, p) \neq 0$. Then also by assumption,

$$Re\ [\mu_i^* g_i(\zeta^*)] = 0,\ i = 1, 2, \ldots, p \text{ and}$$

$$\overline{\mu^T \nabla_z g(\zeta^*)} + \mu^H \nabla_{\bar{z}} g(\zeta^*) = 0.$$

As well, for $i \notin I(\zeta^*)$, $\mu_i^* = 0$. For $i \in I(\zeta^0)$ Re $[g_i(\zeta) - g_i(\zeta^*)] < 0$, and by pseudoconvexity,

$$Re\ [(z - z^*)^T \overline{\nabla_z g}(\zeta^*) + (z - z^*)^H \nabla_{\bar{z}} g(\zeta^*)] < 0.$$

For $i \notin I(\zeta^0)$, $\mu^T \overline{\nabla_z g(\zeta^*)}(z - z^*) + \mu^H \nabla_{\bar{z}} g(\zeta^*), \overline{(z - z^*)} = 0$.

Noting that $(\mu_i^*, i = 1, 2, \ldots, p) \neq 0$ and hence $I(\zeta^*) \neq 0$, it follows that

$$(z - z^0)\left[\sum_{i=1}^{p} \mu_i^* \overline{\nabla_z g_i}(\zeta^*) + \sum_{i=1}^{p} \mu_i^* \nabla_{\bar{z}} g_i(\zeta^*)\right] < 0, \text{ contradicting}$$

$$\sum_{i=1}^{p} \mu_i^* \nabla_z g_i(\zeta^*) + \sum_{i=1}^{p} \mu_i^* \nabla_{\bar{z}} g_i(\zeta^*) = 0$$

Hence, $(\lambda_i^*, i = 1, 2, \ldots, s) \neq 0$. The assumption that g is pseudoconvex at ζ^0 and that there exists a feasible ζ for which $Re\ [g_i(\zeta)] < 0$, $i = 1, 2, \ldots, p$ is Slater's weak constraint qualification. More generally, the condition

$$\sum_{i=1}^{p} \mu_i^* \overline{\nabla_z g_i}(\zeta^*) + \sum_{i=1}^{p} \mu_i^* \nabla_{\bar{z}} g_i(\zeta^*) = 0 \text{ and}$$

$$Re\left[\sum_{i=1}^{p} \mu_i^* g_i(\zeta^*) = 0\right] \Rightarrow \mu^* = (\mu_i^*, i = 1, 2, \ldots, p) = 0,$$

will imply $\lambda^* = (\lambda_i^*, i = 1, 2, \ldots, s) \neq 0$; for if $\lambda^* = 0$ then $\mu^* = 0$ will contradict the Fritz John conditions. Note that this condition holds if the constraints $Re\ g_i(\zeta^*) \leq 0$ which are active at ζ^* have linearly independent gradients. In view of the above discussion, the problem (P) will be said to satisfy a constraint qualification at ζ^* if :

(a) Slater;s weak constraint qualification holds, namely g is pseudoconvex at ζ^* and there is a feasible ζ for (P) with $Re\ (g_i(\zeta)) < 0$, $i = 1, 2, ..., p$, or

$$\left[\sum_{i=1}^{p} \mu_i^* \overline{\nabla_z g_i}(\zeta^*) + \sum_{i=1}^{p} \mu_i^* \nabla_{\bar{z}} g_i(\zeta^*) = 0 \text{ and } Re\left(\sum_{i=1}^{p} \mu_i^* g_i(\zeta^*) \right) = 0 \right] \Rightarrow \mu^* = (\mu_i^*, i = 1, 2, \ldots, p) = 0.$$

The above considerations lead to the following Kuhn-Tucker type theorem :

Theorem 1 — *Let ζ^* be an optimal solution for (P) at which a constraint qualification is satisfied. Then, there exists a positive interger s, $1 \leq s \leq n+1$, scalars $\lambda_i^* \geq 0$, i = 1, 2, ..., s, not all zero, scalars $\mu_i^* \geq 0$, i = 1, 2, ..., p, vectors $\eta_i \in W(\zeta^*)$, i = 1, 2, ..., s, such that*

$$\sum_{i=1}^{s} \lambda_i^* \overline{\nabla_2 \phi(\zeta^*, \eta_i)} + \sum_{i=1}^{s} \lambda_i^* \nabla_{\bar{z}} \phi(\zeta^*, \eta_i) + \sum_{i=1}^{p} \mu_i^* \overline{\nabla_z g_i}(\zeta^*)$$

$$+ \sum_{i=1}^{p} \mu_i^* \nabla_{\bar{z}} g_i(\zeta^*) = 0$$

$$\text{Re } (\mu_i^* g_i(\zeta^*)) = 0, i = 1, 2, \ldots, p.$$

We now extend Theorem 2 of Weir[13] to the complex case.

Theorem 2 — *Let there be a positive integer s, $1 \leq s \leq n+1$, scalars $\lambda_i^* \geq 0$, i = 1, 2, ..., s not all zero, scalars $\mu^* \in S^*$, vectors $\eta_i \in W(\zeta^*)$ such that*

$$\sum_{i=1}^{s} \lambda_i^* \overline{\nabla_z \phi}(\zeta^*, \eta_i) + \sum_{i=1}^{s} \lambda_i^* \nabla_{\bar{z}} \phi(\zeta^* \eta_i) + \mu^T\ \nabla_z g(\zeta^*) + \mu^H\ \nabla_{\bar{z}} g(\zeta^*) = 0$$

$$\text{Re } \mu^H g(\zeta^*) = 0.$$

If $\phi(\cdot, \eta)$ is analytic and have pseudoconvex real part with respect to R_+ for every $\eta \in W$ and g (•) is analytic and quasiconvex with respect to a polyhedral cone $S \subset C^p$, then ζ^* is a minimax solution to (P).

PROOF : Suppose that the conditions of the theorem are satisfied but ζ^* is not a minmax solution. Then there exists a feasible $\overset{\circ}{\zeta}$ such that

$$Re\left[\sup_{\eta\in W} \phi(\zeta^*, \eta)\right] = Re\left[\sup_{\eta\in W} \phi(\zeta^*, \eta)\right].$$

Now

$$Re\left[\sup_{\eta\in W} \phi(\overset{\circ}{\zeta}, \eta)\right] = Re\,[\phi(\zeta^*, \eta_i)],\ i = 1, 2, \dots, s \quad \text{and}$$

$$Re\ [\sup \phi(\overset{\circ}{\zeta}\ \eta_i)] \le Re\left[\sup_{\eta\in W} \phi(\zeta^*, \eta_i)\right],\ i = 1, 2, \dots, s,$$

therefore,

$$Re\ [\phi(\overset{\circ}{\zeta}, \eta_i)] < Re\,[\phi(\zeta^*, \eta_i)],\ i = 1, 2, \dots, s.$$

Pseudoconvexity of $Re\ \phi(\cdot, \eta_i)$ then implies

$$Re\ [(\overset{\circ}{z} - z^*)^T\, \nabla_z \phi(\zeta^*, \eta_i) + (\overset{\circ}{z} - z^*)^H\, \nabla_{\bar{z}} \phi(\zeta^*, \eta_i)] < 0,$$

$$i = 1, 2, \dots, s.$$

Hence,

$$Re\ [(\overset{\circ}{z} - z^*)^T \lambda_i^*\, \nabla_z \phi(\zeta^*, \eta_i) + (\overset{\circ}{z} - z^*)^H \lambda_i^*\, \nabla_{\bar{z}} \phi(\zeta^*, \eta_i)] \le 0,\ i = 1, 2, \dots, s.$$

with at least one strict inequality, since $\lambda^* = (\lambda_i^* = i = 1, 2, \dots, s) \neq 0$. It follows that

$$Re\,\langle \overset{\circ}{z} - z^*, \overline{\nabla_z \phi}(\zeta^*, \eta_i)\, \lambda^* + \nabla_{\bar{z}} \phi(\zeta^*, \eta_i)\, \lambda^* \rangle < 0,$$

and so,

$$Re\ \langle z^0 - z^*, \overline{\nabla_z g(\zeta^*)}\, \mu^* + \nabla_{\bar{z}} g(\zeta^*)\, \mu^* \rangle > 0$$

Thus, we have

$$Re\ \langle \mu^*, (z^0 - z^*)^T\, \nabla_z g(z\eta^*) + (z^0 - z^*)^H\, \nabla_{\bar{z}} g(\zeta^*) \rangle > 0$$

However, feasibility of ζ^* for (P) implies

$$Re\ \langle \mu^*, g(\zeta^0) \rangle \le Re\,\langle \mu^*, g(\zeta^*) \rangle$$

and so by quasiconvexity of g we have

$$Re\ \langle \mu^*, (z^0 - z^*)^T\, \nabla_z g(\zeta^*) + (z^0 - z^*)^H\, \nabla_{\bar{z}} g(\zeta^*) \rangle \le 0$$

which is a contradiction. Therefore, ζ^* is optimal for (P).

3. DUALITY

As in the case of the sufficient optimality conditions, a number of different duals to (P) are possible depending on the convexity conditions of the functions involved. We now present a dual for (P) based on the pattern of formulation as given by Weir[13] for which weak duality will hold if only the objective function $\phi, (\bullet, \eta)$ is analytic and have pseudoconvex real part with respect to R_+.

For the minmax problem (P) we define a dual problem s follows :

(D) Maximize t

subject to

$$Re\ [\lambda_i\ \{\phi(\zeta, \eta_i) - t\}] \geq 0,\ i = 1, 2, ..., s,\ 1 \leq s \leq n+1,$$

$$Re\ \langle \mu, g(\zeta) \rangle \geq 0,$$

$$(s, \lambda, \eta) \in \psi$$

and $$(\zeta, \mu) \in \Theta(s, \lambda, \eta),$$

where ψ denotes the set of triplets (s, λ, η), where s ranges over the integers $1 \leq s \leq n+1, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, $\lambda_i \geq 0$, $i = 1, 2, ..., s$, $\lambda \neq 0$, $\eta = \eta_i$, $i = 1, 2, ..., s$. with $\eta_i \in W$ for all $i = 1, 2, ..., s$ and

$$\Theta(s, \lambda, \eta) = \left[(z\eta, \mu) \in C^{2n} \times S^* : \sum_{i=1}^{s} \lambda_o \overline{\nabla_z \phi}(\zeta, \eta_i;) + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta, \eta_i) \right.$$

$$\left. + \mu^T \nabla_z g(\zeta) + \mu^H \nabla_{\bar{z}} g(\zeta) = 0,\ \mu \in S^*, \mu \geq 0 \right]$$

Theorem *(Weak Duality) — Let $\phi(\bullet, \eta)$ be analytic and have pseudoconvex real part with respect to R_+ and g be analytic and quasiconvex with respect to the polyhedral cone S for all feasible ζ for (P) and for all feasible t, $(s, \lambda, \eta) \in \psi, (u, \mu) \in \Theta(s, \lambda, \eta)$ for (D). Then, inf $(P) \geq$ sup (D).*

PROOF : Suppose these exists $\overset{\circ}{\zeta}$ feasible for (P) and feasible $t, (s, \lambda, \eta) \in \psi, (u, \mu) \in \Theta(s, \lambda, \eta)$ for (D) such that $Re \left[\sup_{\xi \in W} \phi(\overset{\circ}{\zeta}\ \xi) - t \right] < 0.$

Then $Re\ [\phi(\overset{\circ}{\zeta}, \xi) - t] < 0.$ for all $\xi \in W$, and hence $Re\ \lambda_i\, \phi(\overset{\circ}{\zeta}\ \xi) - \lambda_i t] \leq 0$ for all $i = 1, 2, ..., s$, with at least one strict inequality since $\lambda \neq 0$. From the dual constraints it then follows that $Re\ [\lambda_i\, \phi(\overset{\circ}{\zeta}, \xi) - \lambda_i\, \phi(u, \eta_i)] \leq \upsilon$ for all $\xi \in W$ and $i = 1, 2, ..., s$ with at least one strict inequality.

Therefore, $Re\ [\lambda_i\, \phi(\overset{\circ}{\zeta}, \eta_i) - \lambda_i\, \phi(u, \eta_i)] \leq 0,\ i = 1, 2, ..., s$ and since $Re\ \phi(\cdot, \xi)$ is pseudoconvex,

$$Re\ [(z_0 - u)^T \nabla_z \phi(u, \eta_i) + (z_0 - u)^H \nabla_{\bar{z}} \phi(u, \eta_i)] \leq 0,\ i = 1, 2, ..., s.$$

Thus $$Re\ \langle z_0 - u, \overline{\nabla_z g}(u)\, \mu + \nabla_{\bar{z}} g(u)\, \mu \rangle > 0,$$

and so $$Re \langle \mu, (z_0 - u)^T \overline{\nabla_z g} (u) + (z_0 - u)^H \nabla_{\bar{z}} g(u) \rangle > 0$$

Since, $z\overset{\circ}{\eta}$ is a feasible solution for (P), it follows that

$$Re \langle \mu, g(\zeta^0) \rangle \leq Re \langle \mu, g(u) \rangle,$$

and quasiconvexity of g implies

$$Re \langle \mu, (z_0 - u)^T \nabla_z g(u) + (z_0 - u)^H \nabla_{\bar{z}} g(u) \rangle \leq 0,$$

which is a contradiction.

***Theorem* 4 (*Strong Duality*)** — *Let ζ^* be optimal for (P) and let a constraint qualification be satisfied. Then there exists $(s^*, \lambda^*, \xi^*) \in \Psi, \mu^* \in S^*, \mu \geq 0$, with $(\zeta^*, \mu^*) \in \Theta(s^*, \lambda^*, \xi^*)$ such that (s^*, λ^*, ξ^*) and (ζ^*, μ^*) are feasible for (D). If also, $\phi(\cdot, \xi)$ is ε alytic and have pseudoconvex real part with respect to R_+ and g is analytic and quasiconvex with respect to the polyhedral cone S for all feasible ζ for (P) and for all feasible t, $(s, \lambda, \eta) \in \Psi, (u, \mu) \in \Theta(s, \lambda, \eta)$ for *D), then (s^*, λ^*, ξ^*) and (ζ^*, μ^*) is an optimal solution for (D).*

PROOF : Since ζ^* is an optimal solution for (P) and a constraint qualification is satisfied, then Theorem 1 guarantees the existence of a positive integer $s^*, 1 \leq s^* \leq n+1$, scalars $\lambda_i^* \geq 0, i = 1, 2, ..., s^*$ not all zeroes, scalars $\mu^* \in S^*, \mu_i^* \geq 0$, $i = 1, 2, ..., p$, vectors $\xi_i \in W(\zeta)$ such that

$$\sum_{i=1}^{\alpha} \lambda_i^* \overline{\nabla_z \phi} (\zeta^*, \eta_i) + \sum_{i=1}^{\alpha} \lambda_i^* \nabla_{\bar{z}} \phi (\zeta^*, \eta_i) + \mu^{*^T} \overline{\nabla_z g} (\zeta^*) + \mu^{*^H} \nabla_{\bar{z}} g(\zeta^*) = 0,$$

$$Re\ [\mu_i^* g_i (\zeta^*)] = 0,\ i = 1, 2, ..., p.$$

Thus, denoting $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_{s^*}^*)$ and $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{s^*}^*)$, $(s^*, \lambda^*, \xi^*) \in \Psi, (\zeta^*, \mu^*) \in \Theta(s^*, \lambda^*, \xi^*)$. Therefore, $(s^*, \lambda^*, \xi^*) \in \Psi, (\zeta^*, \mu^*) \in \Theta(s^*, \lambda^*, \xi)$ and $Ret = Re\ f(\zeta^*, \xi_i)$, $i = 1, 2, ..., s^*$ are feasible for the dual and the values of the primal and dual are equal. The optimality of $(s^*, \lambda^*, \xi^*) \in \Psi, (\zeta^*, \mu^*) \in \Theta(s^*, \lambda^*, \xi^*)$ for (D) follows by weak duality. ■

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