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NEW CHARACTERIZATIONS OF EULERIAN AND BIPARTITE BINARY MATROIDS

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We show that binary matroid is Eulerian if and only if every element of it is contained in an odd number of circuits. It is proved that a binary matroid is Eulerian if and only if the ground set has an odd number of partitions into circuits. Corresponding results for bipartite binary matroids are derived.

Key Words : Binary Matroid; Eulerian Matroids; Bipartite Matroids; Circuit; Cutset; Partitions

1. INTRODUCTION

A matroid M is a pair $(S, \mathcal{F})$ where S is a finite set and $\mathcal{F}$ is a collection of subsets of S, called independent sets of M, with the following properties :

(1) $\phi \in \mathcal{F}$,

(2) if $X \in \mathcal{F}$ and $Y \subseteq X$ then, $Y \in \mathcal{F}$,

(3) if $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ and $|X| > |Y|$ then there exists an element $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

The set S is known as ground set of M. A maximal independent set of M is a base of M. A subset of S not belonging to $\mathcal{F}$ is said to be dependent. A minimal dependent subset of S is called a circuit of M. The set of all bases of M will be denoted by $\mathcal{B}$. The matroid M^* whose ground set is S and whose set of bases is $\mathcal{B}^* = \{S - B : B \in \mathcal{B}\}$ is called the dual of a matroid M. A circuit of M^* is called a cutset (or co-circuit) of M.

Generalizing graph-theoretic concepts we call a matroid $M = (S, \mathcal{F})$ an Eulerian matroid if there exist disjoint circuits $C_1, C_2, \ldots, C_n$ such that

$$S = C_1 \cup C_2 \cup \ldots \cup C_n.$$

We define M to be a bipartite matroid if every circuit of M has even number of elements.

A matroid $M = (S, \mathcal{F})$ is defined to be binary if the symmetric difference of any set of circuits of M is a union of disjoint circuits of M. The dual M^* of a binary matroid M is also a binary matroid (see [3]). We have the alternative definition that a matroid is binary if and only if for every circuit C and a cutset D of M $|C \cap D|$ is even. If $M = (S, \mathcal{F})$ is a matroid and $T \subset S$ then the deletion of T from M denoted by $M_{/T}$ is a matroid $(S-T, \mathcal{F}')$ where a subset X of S-T is in $\mathcal{F}'$ if and only if $X \in \mathcal{F}$. We follow the notation and terminology of [3] and [7].

We need the following results.

Proposition 1.1[2] — Let $M = (S, \mathcal{F})$ be a binary matroid and $X \subseteq S$. Then X is disjoint union of circuits of M if and only if X intersects each cutset evenly.

Result 1.2[6] — A binary matroid is Eulerian if and only if its dual matroid is bipartite.

2. Main Results

***Theorem* 2.1** — *A binary matroid $M = (S, \mathcal{F})$ is Eulerian if and only if every element of S is contained in an odd number of circuits of M.*

This result is not true for non-binary matroids as shown by the following example.

Example — Consider the uniform matroid $U_{6,2}$ of rank 2 on a six element set. This is a non-binary Eulerian matroid. Every 3-element set is a circuit. The number of circuits containing an element x is equal to the number of ways to choose remaining two elements in a circuit from te remaining 5-elements of $U_{6,2}$. Thus, every element of $U_{6,2}$ is contained in ten circuits of $U_{6,2}$. On the other hand, $U_{4,2}$ is non-binary matroid. Every 3-element set in it is a circuit. So number of circuits containing an arbitrary element x is equal to the number of ways to choose remaining 2 elements in a circuit from remaining 3-elements of $U_{4,2}$. This number is 3, an odd number, but $U_{4,2}$ is not Eulerian.

Proof of the Theorem : Suppose that a binary matroid $M = (S, \mathcal{F})$ is Eulerian. So by result 1.2, every cutset of M has even cardinality. Let $x \in S$. If x is a loop then it belongs to exactly one cycle i.e., to an odd number of cycles and the result is proved. Assume therefore that x is not a loop. We show that x is contained in an odd number of circuits of M. Let $\mathcal{C}_x$ denote the set of circuits containing x and $\Delta \mathcal{C}_x$ denote the symmetric difference of members of $\mathcal{C}_x$. Note that $\Delta \mathcal{C}_x$ consists of those element which are contained in an odd number of circuits containing x. Let $C(x_1, x_2, \ldots, x_n)$ denote the number of circuits containing $x_1, x_2, \ldots, x_n$. Firstly, we prove that $X = \Delta \mathcal{C}_x \cup \{x\}$ intersects each cutset of M in an even number of elements. Let D be a cutset of M. If $x \notin D$, then $D \cap X = D \cap (\Delta \mathcal{C}_x \cup \{x\}) = D \cap \Delta \mathcal{C}_x$ is even by binarity of M. Now, let $x \in D$ and $D \cap X$ has odd number of elements. Then we will arrive at a contradiction. Suppose $D = \{x_1, x_2, \ldots, x_n\}$. So by Result 1.2, n is odd. The set $(D \cap X) - x$ is even and therefore the sum $\sum_{i=1}^{n} C(x, x_i)$ is also even. So, at least one of the n terms must be even. Let $C(x, x_1)$ be even and let D_2 be any cutset containing x but not x_1. Let $C(x, x_1)$ be even and let D_2 be any cutset containing x but not x_1. Let $D_2 = \{x, y_1, y_2, \ldots, y_{n'}\}$. By Result 1.2, n' is odd. Now $C(x, x_1) = \sum_{i=i}^{n'} C(x, x_1, y_i)$ is even, therefore at least one of the n' terms must be even; say $C(x, x_1, y_1)$ is even.

Let D_3 be any cutset containing x but neither x_1 nor y_1 etc. Continue this process until cutset D_i can not be chosen. Then the i elements $x_1, y_1, \ldots$ together with x form a circuit and so $C(x, x_1, y_1, \ldots) = 1$. But we also have $C(x, x_1, y_1, \ldots)$ even, which is a contradiction. Therefore, $D \cap X$ must have an even number of elements.

Now by Proposition 1.1, X is a disjoint union of circuits of M. Also, $\Delta \mathscr{b}_x$ is a disjoint union of circuits of M and $X = \mathscr{b}_x \cup \{x\}$. So, we must have $x \in \Delta \mathscr{b}_x$. Consequently, the number of circuits containing x must be odd.

Conversely, suppose that a matroid M $(S, \mathcal{F})$ is binary and every element is contained in an odd number of circuits. In order to prove that M is Eulerian, we prove that every cutset of M is of even size.

Let D be any cutset of M. As mentioned earlier, $C(x)$ denotes the number of circuits containing the element x. Since M is binary every circuit of M intersects D evenly the sum $\Sigma\, C(x)$ over all $x \in D$ will count each circuit an even number of times. So, the sum $\sum_{x \in D} C(x)$ will be even. By assumption each term is odd, so there must be an even number of terms, thereby showing D to be even. Consequently, M is Eulerian.

As an immediate consequence of the above theorem, we have the following characterization of Eulerian binary matroid.

Corollary 2.2 — A binary matroid $M = (S, \mathcal{F})$ is Eulerian if and only if symmetric difference of all circuits of M equals S.

Combining Theorem 2.1 with the Result 1.2, we get a corresponding characterisation for bipartite matroids.

Corollary 2.3 — A binary matroid $M = (S, \mathcal{F})$ is bipartite if and only if every element is contained in an odd number of cutsets of M.

The following theorem gives a characterisation of binary Eulerian matroids in terms of the number of partitions of a ground set into circuits of a matroid. This generalizes to the binary matroids the characterization of Eulerian graphs due to Bondy and Halberstan[1].

Theorem 2.4 — *A binary matroid M $(S, \mathcal{F})$ is Eulerian if and only if S can be partitioned into circuits of M in odd number of ways.*

PROOF : If in a binary matroid M the ground set S can be partitioned into circuits of M in odd number of ways then it surely has at least one circuit partition, therefore M is Eulerian.

Now suppose that M $(S, \mathcal{F})$ is Eulerian and binary. Let $x \in S$ and $C_1, C_2, \ldots, C_k$ be the circuits containing x. Then by Theorem 2.1, $k \equiv 1 \pmod 2$. We proceed by induction on $|S|$. If $|S| = 1$ then S has a trivial partition consisting of a loop.

Let $|S| = n > 1$. If $S_i = S - C_i = \phi$ for some i, then $k = 1$ and S is a circuit with $S = \{C_1\}$ as its unique circuit partition. Assume therefore that $S_i \neq \phi$, $1 \leq i \leq k$. By induction the ground set $S - C_i$ of the matroid $M_{\setminus C_i} = (S - C_i, \mathcal{F}_i)$ has an odd number of circuit partitions. This yields an odd number of circuit partitions of S in M; containing the circuit C_i. Denote this number by $\tau(C_i)$ and denote by $\tau(S)$ the number of all circuit partitions of S in M.

Consequently,

$$\tau(S) = \sum_{i=1}^{k} \tau(C_i) = k \equiv 1 \pmod 2,$$

i.e., $$\tau(S) \equiv 1 \pmod 2.$$

Remark 2.5 : This also does not hold for non-binary matroids as shown by the following example.

Consider the uniform matroid $U_{6,2}$ of the above example. Since every 3-element set is a circuit, there are ${}^6C_3$, i.e., 20 circuits. Now any circuit and its complement in the 6-element set which is also a circuit form a circuit partition of the ground set of $U_{6,2}$. Thus, in all there are ten, an even number of circuit partitions of the ground set of $U_{\sigma,2}$.

Combining Theorem 2.4 and the Result 1.2, we give another characterization of binary bipartite matroids in terms of the number of partitions of ground set into cutsets of a matroid.

Corollary 2.6 — A binary matroid $\boldsymbol{M} = (S, \mathcal{F})$ is bipartite if and only if S can be partitioned into cutsets of $\boldsymbol{M}$ in odd number of ways.

3. Algorithm for Circuit Partitions of a Ground Set of an Eulerian Matroid

Here, we present a description of an algorithm for the construction of the set $\mathcal{P}(S)$, of all circuit partitions of S in a binary Eulerian matroid $\boldsymbol{M} = (S, \mathcal{F})$. Every Eulerian matroid has at least one circuit, C (say) and $\boldsymbol{M}_{\setminus C} = (S-C\ \mathcal{F}')$ is also Eulerian. We use this to describe an algorithm to obtain circuit partitions of arbitrary binary Eulerian matroid.

Suppose $\boldsymbol{M} = (S, \mathcal{F})$ is a Eulerian matroid and $S = \{x_1, x_2, \ldots, x_n\}$. Choose an arbitrary element $x \in S, x = x_{i_1}$ say. A first subroutine produces in lexicographic order the set $\mathcal{C}_x$ of all circuits containing x. For $C_i \in \mathcal{C}_x$, the matroid $\boldsymbol{M}_{\setminus C_i} = (S-C_i, \mathcal{F}')$ is Eulerian. A second subroutine produces the set $\mathcal{P}_i$ of all circuit partitions of S in the matroid $\boldsymbol{M}$ which have circuit C_i in common. This is achieved by determining $\mathcal{P}(S\text{-}C_i)$ and forming

$$\mathcal{P}_i = \{\{C_i\} \cup P \mid P \in \mathcal{P}(S - C_i)\}.$$

Then we note that $\mathcal{P}_i\, U\, \mathcal{P}_j = \phi$ for $i \neq j$, $1 \leq i, j \leq |\mathcal{C}_x| = \gamma_x$ where γ_x is the number of circuits containing x.

Now, it follows that

$$\mathcal{P}(S) = \bigcup_{i=1}^{\gamma_x} \mathcal{P}_i$$

is the required set. Here each $\mathcal{P}_i$ is a partition of S into circuits of $\boldsymbol{M}$.

Remark 3.1 : (1) We have $|\mathcal{P}(S)| = 1$ for a matroid $\boldsymbol{M}$ each of whose component is a circuit.

(2) By Theorem 2.4, the number of partitions of S into circuits of $\boldsymbol{M}$ is odd if and only if $\boldsymbol{M} = (S, \mathcal{F})$ is binary Eulerian matroid. Hence,

$$|\mathcal{P}(S)| \equiv 1 \pmod 2$$

in case of binary Eulerian matroid.

REFERENCES

1. J. A. Bondy and F. Y. Halberstan, *J. Graph Theory*, **10** (1986), 107-15.
2. T. A. Mckee, *Discrete Math.* **43** (1983), 215-22.
3. J. G. Oxley, *Matroid Theory*, Oxford University Pres, New York, 1992.
4. M. M. Shikare and T. T. Raghunathan, *Indian J. Pure Appl. math.* **27**(2) (196) 153-55.
5. D. J. Wilde, *J. comb. Theory (B)* **18** (1975) 260-64.
6. D. J. A. Welsh, *J. comb. Theory* **6** (1969) 375-77.
7. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.