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THE COMPRESSION OF SLANT TOEPLITZ OPERATOR TO $H^2(\partial D)$

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A slant Toeplitz operator A_φ with symbol φ in $L^\infty(\partial D)$ is an operator whose representing matrix $M = (a_{ij})$ is given by $a_{ij} = \langle \varphi, z^{2i-j} \rangle$ where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(\partial D)$. The operator B_φ denotes the compression of A_φ to $H^2(\partial D)$. In addition to some algeraic properties of B_φ, it is proved that a non-zero hyponormal operator cannot be the compression of a slant Toeplitz operator.

Key Words : Toeplitz Operator; Slant Operator; Hyponormal Operator

INTRODUCTION

Let $\varphi \in L^\infty(\partial D)$, where D is the open unit disc. Then $\varphi(z) = \sum_{i=-\infty}^{\infty} a_i z^i$-where $a_i \langle \varphi, z^i \rangle$ is the ith Fourier coefficient of φ, $\{z^i : i \in Z\}$ is the usual basis and Z is the set of integers. The slant Toeplitz operator A_φ is defined as follows :

For $k \in Z$

$$A_\varphi(z^k) = \sum_{i=-\infty}^{\infty} a_{2i-k} z^i.$$

Furthermore, by [2, Proposition 1] $A_\varphi = WM_\varphi$, where M_φ is the multiplication operator and $Wz^{2n} = z^n$, $Wz^{2n-1} = 0$ for $n \in Z$. Since for $i, j = 0, 1, 2, \ldots$

$\langle B_\varphi z^j z^i \rangle = \langle PA_\varphi z^j z^i \rangle = \langle A_\varphi, z^j, z^i \rangle = a_{2i-j}$, the matrix of B_φ is given as

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 \\ a_6 & a_5 & a_4 & a_3 \end{bmatrix}$$

Therefore, $B_\varphi(z^k) = \sum_{i=0}^{\infty} a_{2i-k} z^i$, $k = 0, 1, 2, 3, \ldots$

The adjoint of B_φ, denoted by B^*_φ, has the property that for i, j = 0, 1, 2, ...

$$\langle B^*_\varphi z^j, z^i \rangle = \langle z^j, B_\varphi z^i \rangle = \bar{a}_{2j-i}$$

Therefore, for each $k \geq 0$

$$B^*_\varphi (z^k) = \sum_{i=0}^{\infty} \bar{a}_{2k-i} z^i.$$

Since P reduces W, it follows that

$$B_\varphi = PA_\varphi|_{H^2} = PWM_\varphi|_{H^2} = WPM_\varphi|_{H^2} = WT_\varphi$$

where T_φ is a Toeplitz operator on $H^2(\partial D)$.

DISCUSSION

The correspondence $\varphi \to B_\varphi$ is linear and one-one. Linearity is obvious. To show it is one-one, suppose $B_\varphi = B_\psi$. Then $B_{\varphi-\psi} = 0$. Therefore, for i, j = 0, 1, 2, ..., $\langle B_{\varphi-\psi} z^i, z^i \rangle = 0$.

Equivalently

$$\langle WT_{\varphi-\psi} z^j, z^i \rangle = \langle T_{\varphi-\psi} z^j, z^{2i} \rangle = \langle (\varphi-\psi) z^j, z^{2i} \rangle = \langle \varphi-\psi, z^{2i-j} \rangle = 0.$$

Therefore, $\varphi - \psi = 0$.

We give below a characterization of B_φ, whose proof is adopted from that of a Toeplitz operator on H^2 given by Halmos[1].

Theorem 1.1 — *A bounded operator B on H^2 is the compression of a slant Toeplitz operator to H^2 if and only if $B = T_z B T_{z^2}$, where $T_{\bar{z}}$ and T_{z^2} are Toeplitz operators on H^2 induced by $\bar{z}$ and z^2 respectively.*

PROOF : Suppose B is the compression of a slant Toeplitz operator. Then for i, j = 0, 1, 2, ...

$$\langle Bz^j, z^i \rangle = \langle Bz^{j+2}, z^{i+1} \rangle = \langle BT_{z^2} z^j, T_z z^i \rangle = \langle T_{\bar{z}} BT_{z^2} z^j, z^i \rangle.$$

Therefore, $B = T_{\bar{z}} BT_{z^2}$

Conversely, suppose B is bounded and $B = T_{\bar{z}} BT_{z^2}$. Then, for i, j = 0, 1, 2, ... $\langle Bz^j, z^i \rangle = \langle Bz^{j+2}, z^{i+1} \rangle$. Consider for each non-negative integer n, the operator on L^2 given by

$$B_n = U^{*n} BPU^{2n}$$

where U is the bilateral shift. If i, j, ≥ 0, then

$$\langle B_n z^j, z^i \rangle = \langle BPz^{j+2n}, z^{i+n} \rangle = \langle Bz^{j+2n}, z^{i+n} \rangle = \langle Bz^j, z^i \rangle.$$

Now if j or i is negative, then for n sufficiently large both $j + 2n$ and $i + n$ are positive, and from then on $\langle BPz^{j+2n}, z^{i+n} \rangle$ is independent on n. Consequently, if p and q are trigonometric polynomials, then the sequence $\{\langle B_n p, q \rangle\}$ is convergent. Since $\| B_n \| \leq \| BP \| = \| B \|$ and trigonometric polynomials are dense in L^2, it follows that, for $f, g \in L^2$, the sequence $\{\langle B_n f, g \rangle\}$ is Cauchy and hence converges. Therefore, the sequence $\{B_n\}$ converges weakly to a bounded operator B_∞ on L^2.

Since, for all i and j,

$$\langle B_\infty z^j, z^i \rangle = \lim_{n \to \infty} \langle U^{*n} BPU^{2n} z^j, z^i \rangle = \lim_{n \to \infty} \langle U^{*n+1} BPU^{2n+2} z^j, z^i \rangle$$

$$= \lim_{n \to \infty} \langle U^{*n} BPU^{2n} z^{j+2}, z^{i+1} \rangle = \langle B_\infty z^{j+2}, z^{i+1} \rangle.$$

It follows that the operator B_∞ is a slant Toeplitz operator on L^2. If f and g are in H^2, then

$$\langle PB_\infty f, g \rangle = \langle B_\infty f, g \rangle = \lim_{n \to \infty} \langle U^{*n} BPU^{2n} f, g \rangle = \langle BP f, g \rangle = \langle Bf, g \rangle,$$

so that $PB_\infty f = Bf$ for each f in H^2. We conclude that B is the compression to H^2 of a slant Toeplitz operator.

The function φ that induces B can be obtained from the matrix of B as follows :

If $i, j \geq 0$, then

$$\langle B z^j, z^i \rangle = \langle B_\infty z^j, z^i \rangle = \langle \varphi, z^{2i-j} \rangle.$$

This implies that

$$\langle \varphi, z^{-j} \rangle = \langle B z^j, z^0 \rangle, \text{ for } j \geq 0,$$

$$\langle \varphi, z^j \rangle = \langle B z^j, z^i \rangle, \text{ for } j \geq 0.$$

Hence φ is the function whose backward Fourier coefficients (the ones with negative index) are the entries of the zero row of the matrix of B and whose forward Fourier coefficients are the entries of the main diagonal of the matrix.

Remark 1.2 : If $\varphi \in H^\infty$, then B_φ is the restriction of A_φ to H^2. Therefore, similar to [2, Proposition 3], we have the following characterization of B_φ : A bounded operator B on H^2 is the restriction of a slant Toeplitz operator with symbol φ in H^∞ if and only if $T_z B = BT_{z^2}$.

Corollary 1.3 — The set of all compressions of slant Toeplitz operators is closed.

PROOF : Suppose $\langle B_n f, g \rangle \to \langle B f, g \rangle$ for $f, g \in H^2$. Since $B_n = T_{\bar{z}} B_n T_{z^2}$ by Theorem 1.1, it follows that as $n \to \infty$,

$$\langle B_n f, g\rangle = \langle B_n T_{z^2} f, T_z g\rangle \to \langle BT_{z^2} f, T_{z^2} f, T_z g\rangle = \langle T_{\bar{z}} BT_{z^2} f, g\rangle.$$

Therefore, $T_{\bar{z}} B_n T_{z^2} \to T_{\bar{z}} BT_{z^2}$ weakly. Hence, $B = T_{\bar{z}} BT_z^2$. Thus B is the compression of a slant Toeplitz operator.

Corollary 1.4 — The only compact B_φ is 0.

PROOF : Observe that if φ is a bounded measurable function, and if both $2n + k$ and $n + k$ are non-negative integers, then

$$\langle B_\varphi z^{k+2n}, z^{n+k}\rangle = \langle WT_\varphi z^{k+2n}, z^{n+k}\rangle = \langle T_\varphi z^{k+2n}, z^{2(n+k)}\rangle$$

$$= \langle \varphi \cdot z^{k+2n}, z^{2(n+k)}\rangle = \langle \varphi, z^k\rangle.$$

This, implies that $|\langle \varphi, z^k\rangle| \le \| B_\varphi z^{k+2n}\|$, for each $k \in Z$. Now since $z^{k+2n} \to 0$ weakly as $n \to \infty$ and B_φ is compact, it follows that $B_\varphi z^{k+2n} \to 0$ strongly as $n \to \infty$. Therefore, $\langle \varphi, z^k\rangle = 0$ for each k, consequently $\varphi = 0$.

Remark 1.5 : An alternative proof of Corollary 1.4 is given here in Remark 1.7 following Theorem 1.6. This proof is similar to that of [2, Proposition 7].

Theorem 1.6 — *(a)* $WB^*_\varphi = T_{w\bar{\varphi}}$; *(b)* $B_\varphi T_\psi = B_{\varphi\psi}$ *if* $\bar{\varphi}$ *or* ψ *is analytic; (c)* $B_\varphi B^*_\psi = T_{W(\varphi, \bar{\psi})}$, *if* $\bar{\varphi}$ *or* $\bar{\psi}$ *is analytic; and (d)* $T_\psi B_\varphi = B_\varphi T_{\psi_{z^2}} = B_{\varphi(z)\cdot\psi(z^2)}$, *if* φ *is analytic.*

PROOF : (a) Since P reduces W, by [2, p. 850] we get

$WB^*_\varphi = WPA^*_\varphi|_{H^2} = PWA^*_\varphi|_{H^2} = PM_{w\,\varphi^*}|_{H^2} = T_{w\,\bar{\varphi}}$.

(b) If $\bar{\varphi}$ or ψ is analytic, then by [1, Problem 195]

$$B_\varphi T_\psi = WT_\varphi T_\psi - WT_{\varphi, \psi} = B_{\varphi\cdot\psi}$$

(c) If $\bar{\varphi}$ or $\bar{\psi}$ is analytic, then by [1, Problem 195] and (a)

$$B_\varphi B^*_\psi = WT_\varphi T_{\bar{\psi}} W^*|_{H^2} = WT_{\varphi\bar{\psi}} W^*|_{H^2} = B_{\varphi\bar{\psi}} W^*|_{H^2} = (WB^*_{\varphi\bar{\psi}})^*|_{H^2}$$

$$= T_{w\,(\bar{\varphi}\psi)} = T_{w\,(\varphi, \bar{\psi})}$$

(d) If φ is analytic, then by [3, Theorem 2],

$$T_\psi B_\varphi = PM_\psi WM_\varphi|_{H^2} = PA_{\psi(z^2)} M_\varphi|_{H^2} = PA_{\psi(z^2)\,\varphi(z)}|_{H^2} = B_{\psi(z^2)\,\varphi(z)}.$$

Remark 1.7 : Suppose B_φ is compact. Then by Theorem 1.6 (a), we have $WB^*_\varphi = T_{w\,\bar{\varphi}}$ is compact and hence $T_{w\,\bar{\varphi}}$ is compact, and this implies that $W\bar{\varphi} = 0$ by [1, Corollary 2, Problem 194]. Therefore, we have for each $n \in Z$.

$$\langle W\bar{\varphi}, z^n\rangle = \langle \bar{\varphi}, z^{2n}\rangle = \bar{a}_{2n} = 0$$

On the other hand, consider $B_{\varphi}T_z$. Since B_{φ} is compact by assumption, it follows that $B_{\varphi}T_z$ is compact. By Theorem 1.6 (*b*), $B_{\varphi}T_z = B_{\varphi(z)\cdot z}$. Since $WB_{\varphi(z)\cdot z} = T_{w(\overline{\varphi}(z)\cdot\overline{z})}$ by Theorem 1.6 (*a*), it follows that $T_{w(\overline{\varphi}(z)\cdot\overline{z})}$ is compact and hence $W(\overline{\varphi}(z)\cdot\overline{z}) = 0$. Therefore, for each integer n,

$$\langle W(\overline{\varphi}(z)\cdot\overline{z}), z^n\rangle = \langle (\overline{\varphi}(z)\cdot\overline{z}, z^{2n}\rangle = \langle \overline{\varphi}, z^{2n+1}\rangle = \overline{a}_{(2n+1)} = 0.$$

Hence, we have $\varphi = 0$.

Theorem 1.8 — (*a*) WB_{φ} is the compression of a slant Toeplitz operator if and only if $\varphi = 0$; and (*b*) Let $\overline{\varphi}$ or ψ is analytic. Then B_{φ}, B_{ψ} is the compression of a slant Toeplitz operator if and only if $\varphi(z^2)$. $\psi(z) = 0$ if and only if $B_{\varphi}, B_{\psi} = 0$. In particular, $B^2_{\varphi} = B_{\varphi}$ if and only if $\varphi = 0$.

PROOF : (a) Suppose WB_{φ} is the compression of a slant Toeplitz operator. Then for $i, j =$ 0, 1, 2, ...,

$$\langle WB_{\varphi} z^j, z^i\rangle = \langle WB_{\varphi} z^{j+2}\, z^{i+1}\rangle.$$

This implies that

$$\left\langle \sum_{k=0}^{\infty} a_{2k-j}\, z^k, z^{2i}\right\rangle = \left\langle \sum_{k=0}^{\infty} a_{2k-j-2}\, z^k, z^{2i+2}\right\rangle.$$

Therefore, $a_{4i-j} = a_{4i-j+2}$. Hence, $\varphi = 0$.

(*b*) If $\overline{\varphi}$ or ψ is analytic, then by [1, Problem 195] and Theorem 1.6 (*d*)

$$B_{\varphi}B_{\psi} = WT_{\varphi}WT_{\psi} = WB_{\varphi(z^2)}T_{\psi} = W^2 T_{\varphi(z^2)\psi(z)} = WB_{\varphi(z^2)\cdot\psi(z)}.$$

By (*a*), we get $B_{\varphi}B_{\psi}$ is the compression of a slant Toeplitz operator if and only if $\varphi(z^2)\psi(z) = 0$ if and only if $B_{\varphi}B_{\psi} = 0$.

Theorem 1.9 — B_{φ} *is hyponormal if and only if* $\varphi = 0$.

PROOF : Suppose B_{φ} is hyponormal. Then $\| B_{\varphi}f\| \geq \| B^*_{\varphi}f\|$, for every $f \in H^2$. This implies that $\| WT_{\varphi}f\| \geq \| T_{\varphi}W^*f\|$ and this in turn implies that

$$\| WP(\varphi f)\| \geq \| P(\overline{\varphi}(z)\cdot f(z^2))\|.$$

Put $f(z) = z^{2n}$, $n = 0, 1, 2, \ldots$ Then

$$\| WP(\varphi(z)\cdot z^{2n})\| \geq \| P(\overline{\varphi}(z)\cdot z^{4n}))\|.$$

Therefore, $\sum_{l=1}^{n} |a_{-2l}|^2 + \sum_{k=0}^{\infty} |a_{2k}|^2 \geq \sum_{l=1}^{4n} |\bar{a}_l|^2 + \sum_{k=0}^{\infty} |\bar{a}_{-k}|^2$

Passing to the limit, we get

$$\sum_{l=1}^{\infty} |a_{-2l}|^2 + \sum_{k=0}^{\infty} |a_{2k}|^2 \geq \sum_{l=1}^{\infty} |\bar{a}_l|^2 + \sum_{k=0}^{\infty} |\bar{a}_{-k}|^2$$

This is equivalent to

$$\sum_{k=-\infty}^{\infty} |a_{2k}|^2 \geq \sum_{k=-\infty}^{\infty} |a_{-k}|^2.$$

Therefore, $a_{2k-1} = 0$, for $k \in Z$.

Put $f(z) = z^{2n+1}$, $n = 0, 1, 2, \ldots$. Then

$$\| \boldsymbol{WP}\ (\varphi(z) \cdot z^{2n+1}) \| \geq \| \boldsymbol{P}\ (\overline{\varphi}(z) \cdot z^{4n+2})) \|$$

$$\sum_{l=0}^{n} |a_{-(2l+1)}|^2 + \sum_{k=0}^{\infty} |a_{2k+1}|^2 \geq \sum_{l=1}^{4n+2} |\bar{a}_l|^2 + \sum_{k=0}^{\infty} |\bar{a}_{-k}|^2$$

Passing to the limit, we get

$$\sum_{k=-\infty}^{\infty} |a_{2k+1}|^2 \geq \sum_{k=-\infty}^{\infty} |\bar{a}_k|^2$$

Therefore, $a_{2k} = 0$, for $k \in Z$. Hence, $\varphi = 0$.

The converse is obvious.

Corollary 1.10 — $\boldsymbol{B}_{\varphi}$ cannot be an isometry.

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