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THERMO CREEP TRANSITION IN A THICK-WALLED CIRCULAR CYLINDER UNDER INTERNAL PRESSURE

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Creep stresses and strain rates have been obtained for a thick-walled circular cylinder made of compressible/incompressible materials under internal pressure and temperature by using Seth's transition theory. It has been observed that the circumferential stress has maximum value at the external surface of the cylinder made of incompressible material as compared to compressible material. Introduction of thermal effects reduces the stresses at the external surface of the cylinder in comparison to pressure effects only. Strain rates are found to be maximum at the internal surface of the cylinder made of compressible material and they decrease with the radius. With the introduction of thermal effects, the creep rates have higher values at the internal surface but lesser values at the external surface as compared to a cylinder subjected to pressure only.

Key Words : Thermal; Creep; Stresses; Strain Rates; Cylinder; Internal Pressure

List of Symbols

$\overset{A}{e}_{ii}$ - principal finite strain component.

a, b - internal and external radii of the cylinder.

u, v, z - displacement components.

r, θ, z - radial, circumferential and axial directions respectively.

e_{ij}, T_{ij} - strain and stress tensors respectively.

δ_{ij} - kronecker's delta.

ε_{ij} - creep strain rate tensr.

E - Young's modulus.

Dimensionless Quantities

R = r/b; $R_0 = a/b$; $E_1 = E/p$

$$\sigma_r = \frac{T_{rr}}{p};\ \sigma_\theta = \frac{T_{\theta\theta}}{p};\ \sigma_z = \frac{T_{zz}}{p}$$

INTRODUCTION

Thick-walled circular cylinders are used commonly either as pressure vessels intended for storage in industrial gases or a media transportation of high pressurised fluids. Creep of thick-walled cylinder under internal pressure has been discussed by many authors[1-5]. Rimrott[3] analysed the above problem by considering large strains. These authors made the following assumptions

1. The volume of the material remains constant, or

$$\dot{\varepsilon}_r + \dot{\varepsilon}_\theta + \dot{\varepsilon}_z = 0.$$

2. The ratios of the principal shear strain rates to the principal shear stresses are equal, i.e.,

$$\frac{\dot{\varepsilon}_{\theta\theta} - \dot{\varepsilon}_{rr}}{\sigma_{\theta\theta} - \sigma_{rr}} = \frac{\dot{\varepsilon}_{rr} - \dot{\varepsilon}_{zz}}{\sigma_{rr} - \sigma_{zz}} = \frac{\dot{\varepsilon}_{zz} - \dot{\varepsilon}_{\theta\theta}}{\sigma_{zz} - \sigma_{\theta\theta}}.$$

3. The axial strain rate is zero, i.e, $\dot{\varepsilon}_z = 0$.

4. There is a significant stress-versus-rate of true strain relationship which coincides with the true stress-versus-creep rate relationship in simple tension, for example Norton's Law.

5. The creep deformation is infinitesimally small.

Seth's transition theory[6] does not require any assumptions stated above and thus poses and solves a more general problem from which cases pertaining to these assumptions can be worked out. It utilises the concept of generalised strain measure and the asymptotic solution at turning points or transition points of the governing equation defining the deformed field. It has successfully been applied to a number of creep problems[8-12].

Seth[7] has defined the generalised principal strain measure as

$$e_{ii} = \int_0^{e_{ii}^A} \left[1 - 2 e_{ii}^A \right]^{(n/2) - 1} d e_{ii}^A = \frac{1}{n} \left[1 - \left(1 - 2e_{ii}^A \right)^{n/2} \right] (i = 1, 2, 3). \quad \text{... (1.1)}$$

where n is a measure.

In this paper, we calculated thermal creep stresses and strain rates for a thick-walled circular cylinder under internal pressure using Seth's transition theory. We, not only, obtained the solution for compressible material but also showed, as a particular case that assumptions (2) and (3) stated above come out from the solution itself.

GOVERNING EQUATIONS

Consider a thick-walled circular cylinder of internal and external radius a and b respectively subjected to internal pressure p and to a temperature θ_0 at the internal surface.

In cylindrical co-ordinates the components of displacement are given by[7]

$$u = r(1 - \beta), \; v = 0, \; w = dz, \quad \text{... (2.1)}$$

where β is a function of $r = \sqrt{x^2 + y^2}$ only and d is a constant.

The finite components of strain are[7] :

where
$$\left.\begin{aligned}
e^A_{rr} &= \frac{1}{2}[1-(r\beta'+\beta)^2],\\
e^A_{\theta\theta} &= \frac{1}{2}[1-\beta^2],\\
e^A_{zz} &= \frac{1}{2}[1-(1-d)^2],\\
e^A_{r\theta} &= e^A_{\theta z} = e^A_{zr} = 0,\\
\beta' &= \frac{d\beta}{dr}.
\end{aligned}\right], \qquad \dots (2.2)$$

Substituting eqs. (2.2) in eq. (1.1), the generalised components of strain are

$$\left.\begin{aligned}
e_{rr} &= \frac{1}{n}[1-(r\beta'+\beta)^n],\\
e_{\theta\theta} &= \frac{1}{n}[1-\beta^n],\\
e_{zz} &= \frac{1}{n}[1-(1-d)^n]\\
\text{and } e_{r\theta} &= e_{\theta z} = e_{zr} = 0.
\end{aligned}\right] \qquad \dots (2.3)$$

The stress-strain relation[13] are,

$$T_{ij} = \lambda\,\delta_{ij}\,e_{kk} + 2\mu e_{ij} - \xi\,\theta\,\delta_{ij},\ (i,\ j\ =\ 1,\ 2,\ 3) \qquad \dots (2.4)$$

where λ and μ are Lame's constants and e_{kk} is the first strain invariant $\xi = \alpha(3\lambda + 2\mu)$, α being the coefficient of thermal expansion and θ is the temperature. Further θ has to satisfy

$$\theta_{,ii} = 0. \qquad \dots (2.5)$$

Substituting the strain components from eq. (2.3) in eq. (2.4), we get —

$$\left.\begin{aligned}
T_{rr} &= \lambda I_1 + \frac{2\mu}{n}[1-(r\beta'+\beta)^n - \xi\,\theta,\\
T_{\theta\theta} &= \lambda I_1 + \frac{2\mu}{n}[1-\beta^n] - \xi\,\theta,\\
T_{zz} &= \lambda I_1 + \frac{2\mu}{n}[1-(1-d)^n] - \xi\,\theta\\
\text{and}\quad T_{r\theta} &= T_{\theta z} = T_{zr} = 0,
\end{aligned}\right] \qquad \dots (2.6)$$

where
$$I_1 = \frac{1}{n}[3-(r\beta'+\beta)^n - \beta^n - (1-d)^n] \text{ and } \beta' = d\beta/dr.$$

The temperature field satisfying eq. (2.5) and

$$\theta = \theta_0 \text{ at } r = a$$

$$\theta = 0 \text{ at } r = b,$$

where θ_0 a constant, is given by,

$$\theta = \theta_0 \frac{\log (r/b)}{\log (a/b)}. \qquad \text{... (2.7)}$$

Equations of equilibrium are all satisfied except

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \qquad \text{... (2.8)}$$

Using eqs. (2.6) in eq. (2.8), we get a nonlinear differential equation in β as

$$nP (P = 1)^{n-1} \beta \frac{dP}{d\beta} + nP (P + 1)^n + (1 - C) nP - [1 - (P + 1)^n] C + \frac{nC \xi \overline{\theta}_0}{2\mu \beta^n} = 0, \qquad \text{... (2.9)}$$

where $\quad r\beta' = \beta P, \; C = 2\mu/(\lambda + 2\mu)$ and $\overline{\theta}_0 = \dfrac{\theta_0}{\log (a/b)}$.

The transition points of β in eq. (2.9) are $P \to -1$ and $P \to \pm\infty$. The boundary conditions are

$$T_{rr} = -p \text{ at } r = a$$

$$T_{rr} = 0 \text{ at } r = b. \qquad \text{... (2.10)}$$

The resultant force transmitted by the wall in axial direction is equal to $\pi a^2 p$, that is

$$2\pi \int_a^b r T_{zz} \, dr = \pi a^2 p. \qquad \text{... (2.11)}$$

SOLUTION THROUGH THE PRINCIPAL STRESS DIFFERENCE

It has been shown[8-12] that the transition function through the stress difference at the transition point $P \to -1$ gives the creep stresses. The transition function R is taken as

$$R = T_{rr} - T_{\theta\theta} = \frac{2\mu \beta^n}{n} [1 - (P + 1)^n]. \qquad \text{... (3.1)}$$

Taking logarithmic diferentiation of eq. (3.1) w.r.t. r and using eq. (2.9), we get

$$\frac{d}{dr} (\log R) = \frac{1}{r [1 - (P + 1)^n]} \left[nP (2 - C) - C \left\{ 1 - (P + 1)^n \right\} + \frac{nC \xi \overline{\theta}_0}{2\mu \beta^n} \right]. \qquad \text{... (3.2)}$$

Taking asymptotic value of eq. (3.2) at $P \to -1$ we get after integration,

$$R = A r^{-2n + C(n-1)} \exp f, \qquad \dots (3.3)$$

where $f = \dfrac{\alpha \bar{\theta}_0 (3 - 2C)\, r^n}{D^n}$ and A is a constant of integration.

Asymptotic value of β as $P \to -1$ is D/r, D being a constant.

Substituting the value of $R = T_{rr} - T_{\theta\theta}$ from eq. (3.3) in eq. (2.8) we get

$$T_{rr} = -A \int F dr + B, \qquad \dots (3.4)$$

where $F = r^{-2n + C(n-1) - 1} \exp f$ and B is a constant of integraiton.

The constants A and B are obtained by using boundary conditions given by eq. (2.10) in eq. (3.4) as

$$A = \frac{-p}{\int_a^b F dr} \text{ and } B = A\left[\int F\, dr\right] \text{ at } r = b.$$

Substituting values of A and B in eq. (3.4) we get

$$T_{rr} = -p \int_r^b F dr \Big/ \int_a^b F dr. \qquad \dots (3.5)$$

The values of $T_{\theta\theta}$ and T_{zz} are obtained from eqs. (3.3) and (2.6) respectively as

$$T_{\theta\theta} = T_{rr} + \frac{prF}{\int_a^b F dr}, \qquad \dots (3.6)$$

and
$$T_{zz} = \left(\frac{1-C}{2-C}\right)(T_{rr} + T_{\theta\theta}) + E e_{zz} - E\, \alpha\, \theta \qquad \dots (3.7)$$

where $E = \left[\dfrac{3-2C}{2-C}\right] 2\mu$. The term e_{zz} is obtained by using eq. (3.7) in eq. (2.11) as

$$e_{zz} = \frac{\dfrac{C a^2 p}{(2-C)} + E\, \alpha\, \bar{\theta}_0 \left[a^2 \log (b/a) + \dfrac{a^2 - b^2}{2} \right]}{b^2 - a^2}. \qquad \dots (3.8)$$

Eqs. (3.5)-(3.7) give the thermal creep stresses for a thick-walled circular cylinder under internal pressure.

In non-dimensional form eqs. (3.5)-(3.8) are written as

$$\sigma_r = \frac{-\int_R^1 F_1 dR}{\int_{R_0}^1 F_1 dR}, \quad \text{... (3.9)}$$

$$\sigma_\theta = \sigma_r + \frac{R^{-2n + C(n-1)} \exp f_1}{\int_{R_0}^1 F_1 \, dR} \quad \text{... (3.10)}$$

and

$$\sigma_z = \left(\frac{1-C}{2-C}\right)(\sigma_r + \sigma_\theta) + E_1 \, e_{zz} - E_1 \, \alpha \, \theta, \quad \text{... (3.11)}$$

where

$$E_1 \, e_{zz} = \frac{\dfrac{CR_0^2}{(2-C)} + E_1 \, \alpha \bar{\theta}_0 \left(R_0^2 \log(1/R_0) + \dfrac{R_0^2 - 1}{2} \right)}{1 - R_0^2}, \quad \text{... (3.12)}$$

$$F_1 = R^{-2n + C(n-1) - 1} \exp f_1 \text{ and } f_1 = \frac{\alpha \bar{\theta}_0 (3 - 2C)(bR)^n}{D^n}.$$

For incompressible material ($C \to 0$), the eqs. (3.9)-(3.12) become

$$\sigma_r = \frac{-\int_R^1 F_2 \, dR}{\int_{R_0}^1 F_2 \, dR}, \quad \text{... (3.13)}$$

$$\sigma_\theta = \sigma_r + \frac{R^{-2n} \exp f_2}{\int_{R_0}^1 F_2 \, dR}, \quad \text{... (3.14)}$$

and

$$\sigma_z = \frac{\sigma_r + \sigma_\theta}{2} + E_1 \, e_{zz} - E_1 \, \alpha \, \theta, \quad \text{... (3.15)}$$

where

$$e_{zz} = \frac{\alpha \bar{\theta}_0 \left(R_0^2 \log(1/R_0) + \dfrac{R_0^2 - 1}{2} \right)}{1 - R_0^2}, \quad \text{... (3.16)}$$

$$F_2 = R^{-2n-1} \exp f_2 \text{ and } f_2 = \frac{3\,\alpha\,\bar{\theta}_0\,(bR)^n}{D^n}.$$

As a particular case, creep stresses for thick-walled circular cylinder without thermal effect are obtained by putting $\theta_0 = 0$ in eqs. (3.9) to (3.11) as

$$\sigma_r = -\left[\frac{R^{-2n+C(n-1)} - 1}{R_0^{-2n+C(n-1)} - 1}\right], \quad \text{... (3.17)}$$

$$\sigma_\theta = \sigma_r - \frac{(-2n + C(n-1))\, R^{-2n+C(n-1)}}{(R_0^{-2n+C(n\ 1)} - 1)} \quad \text{... (3.18)}$$

and

$$\sigma_z = \left(\frac{1-C}{2-C}\right)(\sigma_r + \sigma_\theta) + E_1\, e_{zz}, \quad \text{... (3.19)}$$

where

$$E_1\, e_{zz} = \frac{C R_0^2}{(2-C)\,(1-R_0^2)}. \quad \text{... (3.20)}$$

These equations are the same as obtained by Gupta[10].

Eqs. (3.17)-(3.19) for incompressible material ($C \to 0$) become

$$\sigma_r = -\left[\frac{R^{-2n} - 1}{R_0^{-2n} - 1}\right], \quad \text{... (3.21)}$$

$$\sigma_\theta = \sigma_r + \frac{2nR^{-2n}}{(R_0^{-2n} - 1)} \quad \text{... (3.22)}$$

and

$$\sigma_z = \frac{\sigma_r + \sigma_\theta}{2}. \quad \text{... (3.23)}$$

These expressions are same as obtained by Finnie & Heller[1] and Odquist[14].

When the creep sets in, the strain should be replaced by strain rates. The stress strain rate relation can be written as

$$\dot{e}_{ij} = \frac{1+\nu}{E} T_{ij} - \frac{\nu}{E} \delta_{ij}\, \Theta + \alpha\, \theta, \quad \text{... (3.24)}$$

where $\dot{e}_{ij}$ is the strain rate tensor w.r.t. flow parameter t and $\Theta = T_{11} + T_{22} + T_{33}$ and

$\nu = (1-C)/(2-C)$ is the Poisson's ratio.

Differentiating eq. (2.2) w.r.t. t, we get

$$\dot{e}_{\theta\theta} = -\beta^{n-1}\,\dot{\beta}. \quad \text{... (3.25)}$$

For SWAINGER measure ($n = 1$) we have from eq. (3.25),

$$\dot{\varepsilon}_{\theta\theta} = -\dot{\beta}. \qquad \text{... (3.26)}$$

The transition value of eq. (3.1) at $P \to -1$ gives

$$\beta = \left[\frac{n(3-2C)}{E(2-C)}\right]^{\frac{1}{n}} (T_{rr} - T_{\theta\theta})^{\frac{1}{n}}. \qquad \text{... (3.27)}$$

Using eqs. (3.25), (3.26) and (3.27) in eq. (3.24), we get

$$\left.\begin{aligned}
\dot{\varepsilon}_{rr} &= \frac{1}{E_1}\left[\frac{n(\sigma_r - \sigma_\theta)(3-2C)}{E_1(2-C)}\right]^{\frac{1}{n}-1} [\sigma_r - \nu(\sigma_\theta + \sigma_z) + \alpha\,\theta\,E_1],\\
\dot{\varepsilon}_{\theta\theta} &= \frac{1}{E_1}\left[\frac{n(\sigma_r - \sigma_\theta)(3-2C)}{E_1(2-C)}\right]^{\frac{1}{n}-1} [\sigma_\theta - \nu(\sigma_z + \sigma_r) + \alpha\,\theta\,E_1],\\
\text{and } \dot{\varepsilon}_{zz} &= \frac{1}{E_1}\left[\frac{n(\sigma_r - \sigma_\theta)(3-2C)}{E_1(2-C)}\right]^{\frac{1}{n}-1} [\sigma_z - \nu(\sigma_r + \sigma_\theta) + \alpha\,\theta\,E_1].
\end{aligned}\right] \qquad \text{... (3.28)}$$

From eqs. (3.28) we get

$$\frac{\dot{\varepsilon}_{\theta\theta} - \dot{\varepsilon}_{rr}}{\sigma_{\theta\theta} - \sigma_{rr}} = \frac{\dot{\varepsilon}_{rr} - \dot{\varepsilon}_{zz}}{\sigma_{rr} - \sigma_{zz}} = \frac{\dot{\varepsilon}_{zz} - \dot{\varepsilon}_{\theta\theta}}{\sigma_{zz} - \sigma_{\theta\theta}}. \qquad \text{... (3.29)}$$

As a particular case, for incompressible material ($C \to 0$) without thermal effects, the creep strain rates (3.28) using eqn. (3.23) become

$$\left.\begin{aligned}
\dot{\varepsilon}_{rr} &= -\left(\frac{1}{E_1}\right)^{\frac{1}{n}} (-n\sqrt{3})^{\frac{1}{n}-1} \left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{n}+1} (\sigma_\theta - \sigma_r)^{\frac{1}{n}},\\
\dot{\varepsilon}_{\theta\theta} &= \left(\frac{1}{E_1}\right)^{\frac{1}{n}} (-n\sqrt{3})^{\frac{1}{n}-1} \left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{n}+1} (\sigma_\theta - \sigma_r)^{\frac{1}{n}}\\
\text{and } \dot{\varepsilon}_{zz} &= 0
\end{aligned}\right] \qquad \text{... (3.30)}$$

Expressions (3.30) are the same as obtained by Odquist[14] provided we put

$$E_1 = (-n\sqrt{3})^{n-1} = \sigma_c \text{ and } n = 1/N.$$

Thus it has been shown in eqs. (3.29) and (3.30) that the assumptions (2) and (3) stated above come out from the solution itself whereas these were assumed by the authors[1-5, 14] prior to the solution.

NUMERICLA ILLUSTRATION AND DISCUSSION

For calculating the stresses and strain rate distribution based on the above analysis, the following values have been taken.

$$n = 1/7,\ 1/3 \text{ and } 1 (N = 7, 3, 1),\ C = 0.75, 0.57, 0.25, 0.00,\ D = 1$$

$$\alpha = 5.0 \times 10^{-5} \text{ deg } F^{-1} \text{ (for Methyl Methacrylate) and } \theta_0 = 700^{\circ} \text{ F}.$$

Curves have been drawn in Figs. 1 and 2 between the stresses σ_r, σ_θ, σ_z and radius R for Methyl Methacrylate material with and without thermal effects. For $n = 1/7$, it can be seen that the

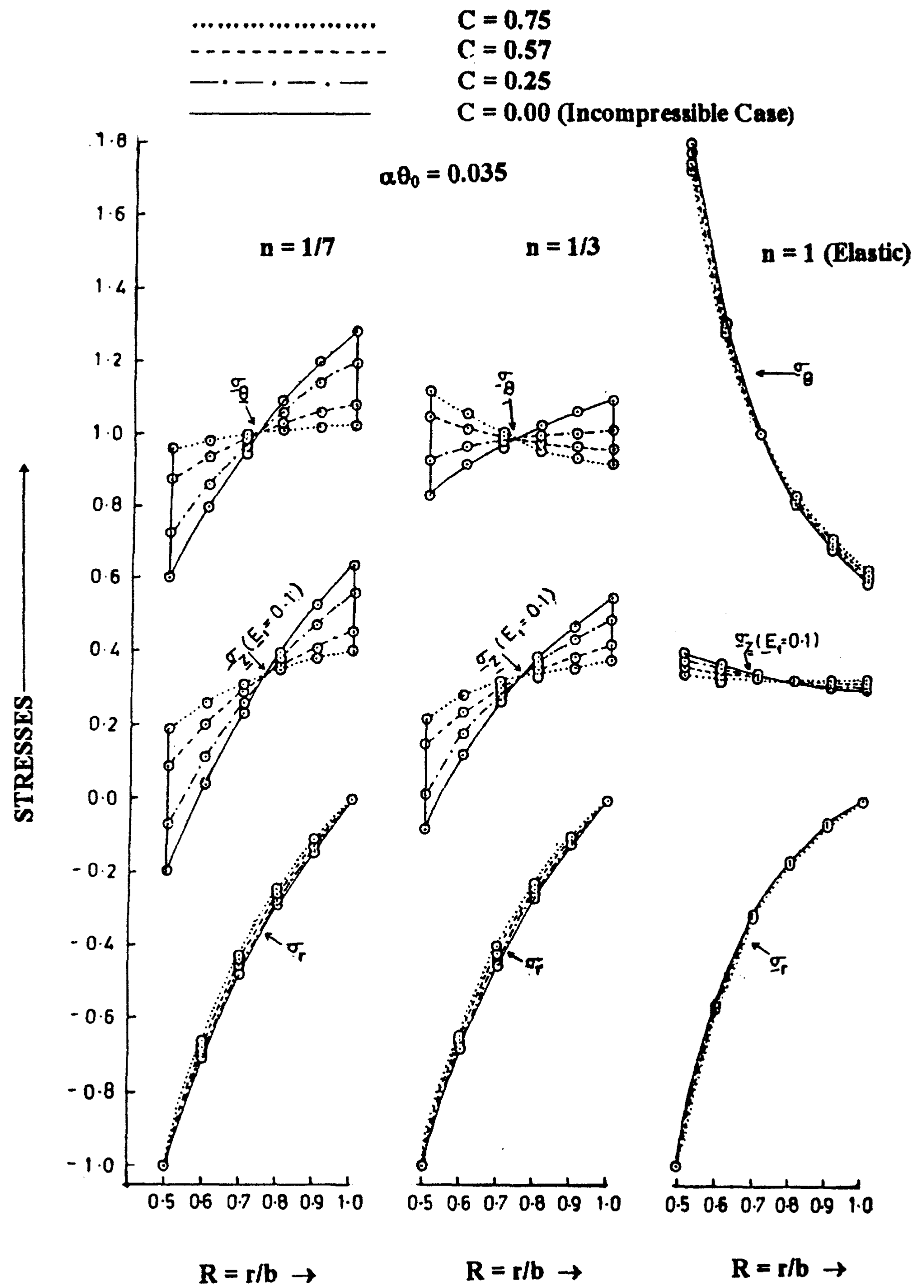

FIG. 1. Thermal creep stresses for a thick-walled circular cylinder under internal pressure along the radius

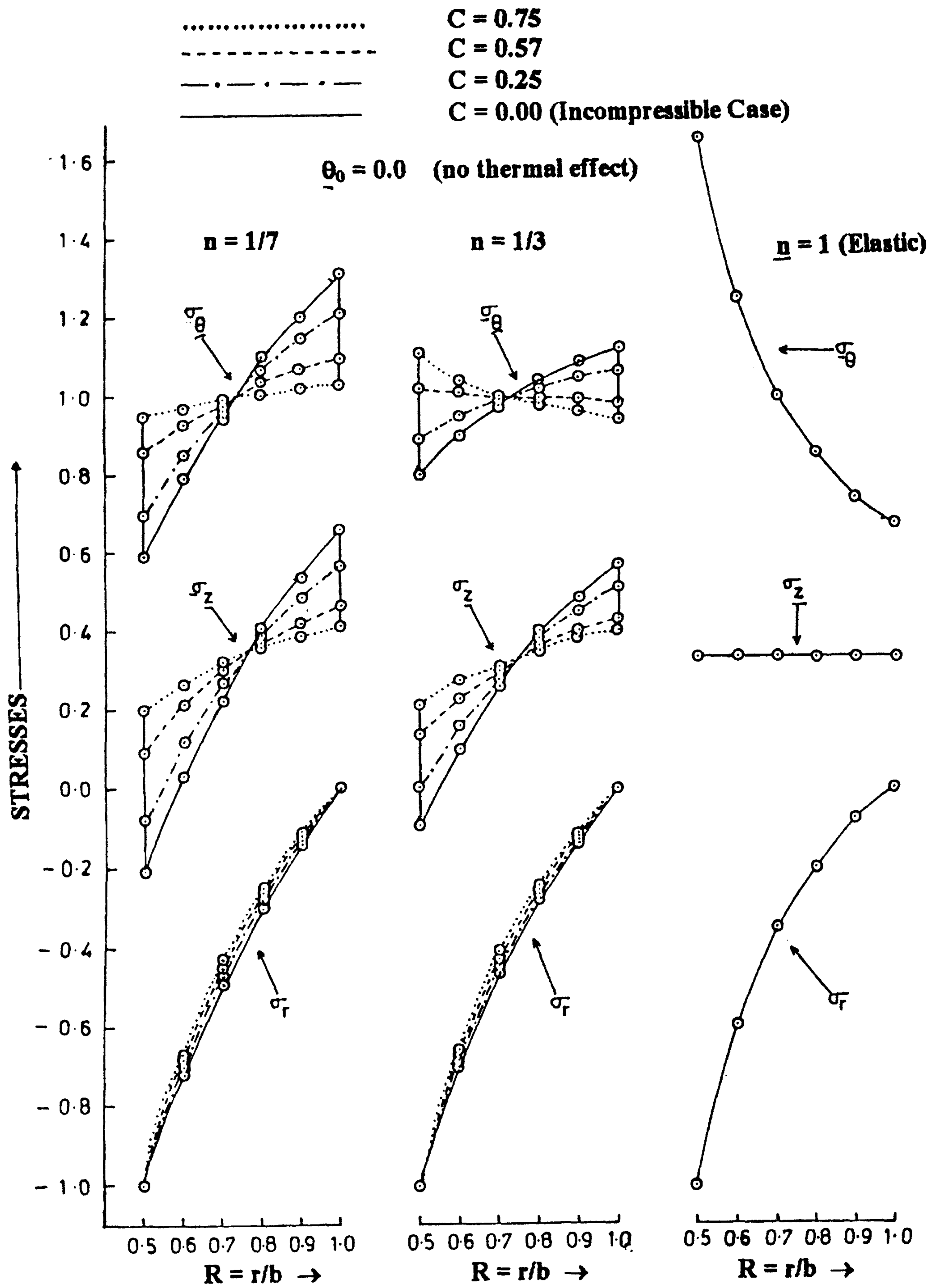

FIG. 2. Creep stresses for a thick-walled circular cylinder under internal pressure along the radius

circumferential stress is maximum at the external surface of a cylinder made of incompressible material as compared to that of compressible material. For measure $n = 1/3$ (or $N = 3$) even though the circumferential stress has maximum value at the external surface yet it has lesser values as compared to measure $n = 1/7$. It has been seen that introduction of thermal effects reduces the stresses at the outer surface. For measure $n = 1$, it gives elastic stress distribution.

In Figs. 3(*a*, 3(*b*) and 3(*c*), curves have been drawn for creep strain rates along the radius for measure $n = 1/3$ (or $N = 3$) and $E_1 = E/p = 0.1$, 1.0 and 1.5 respectively. It has been observed

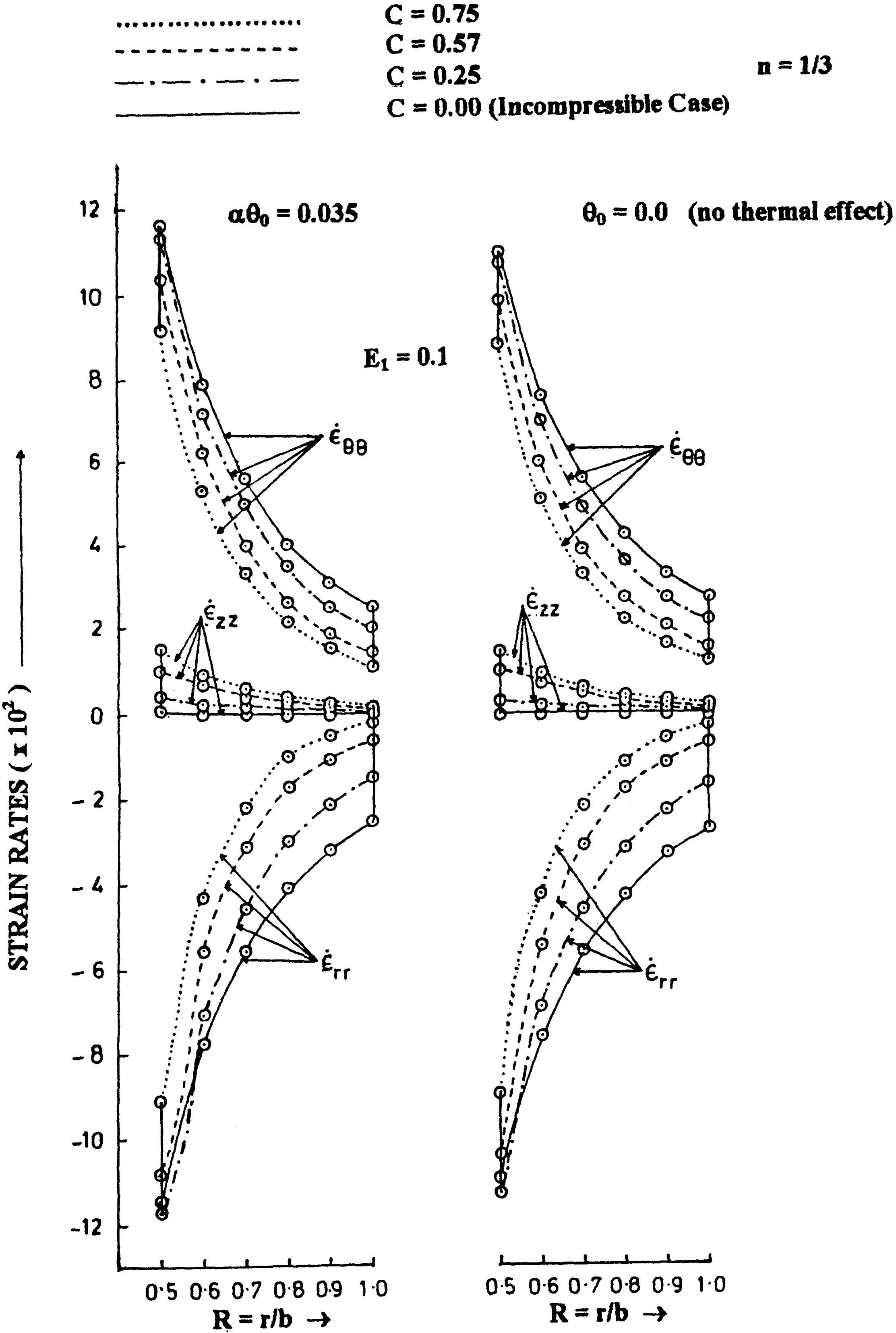

FIG. 3(a). Strain rate distribution for a thick-walled circular cylinder under internal pressure for n = 1/3 and E_1 = 0.1 along the radius

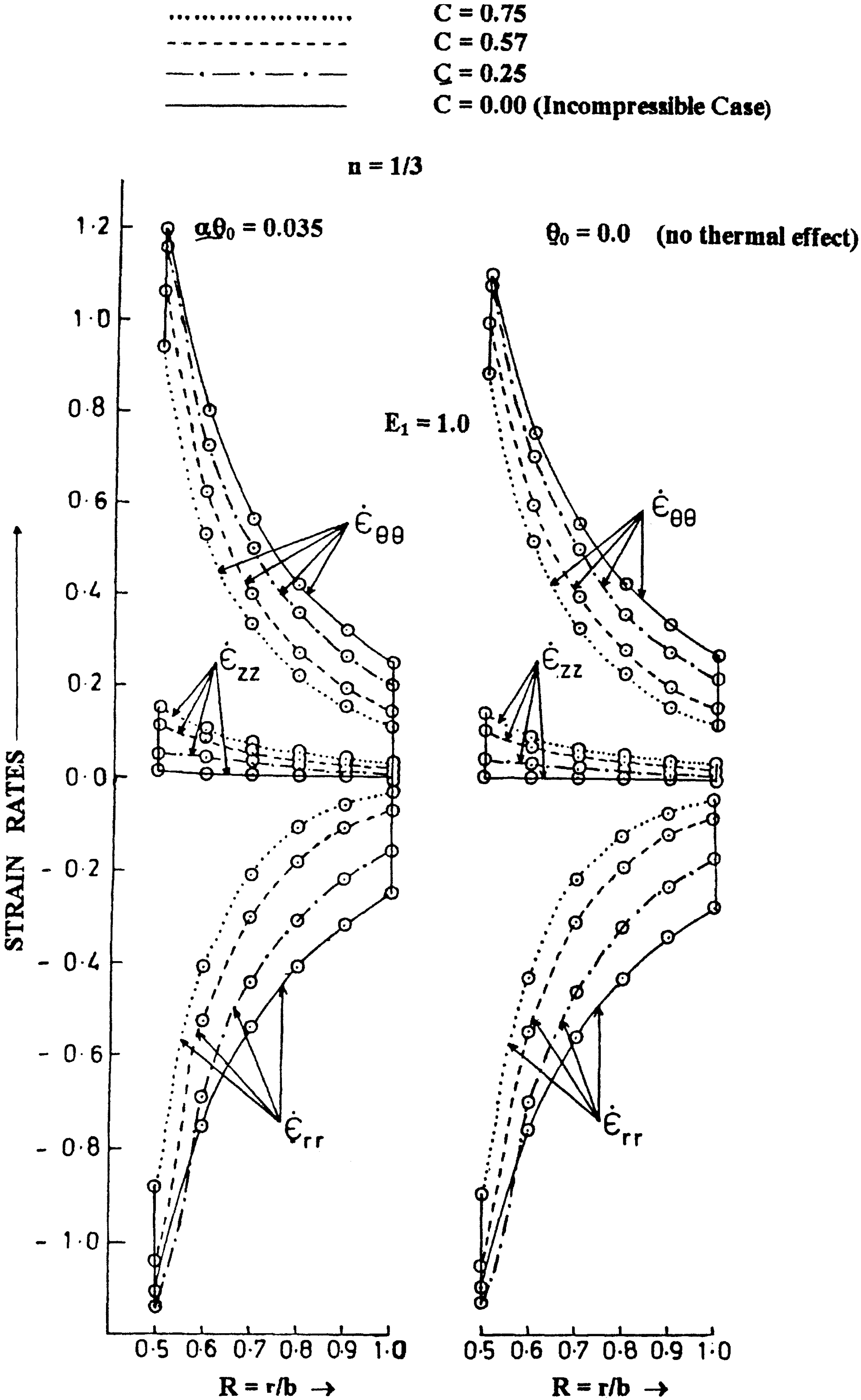

FIG. 3(b). Strain rate distribution for a thick-walled circular cylinder under internal pressure for $n = 1/3$ and $E_1 = 1.0$ along the radius

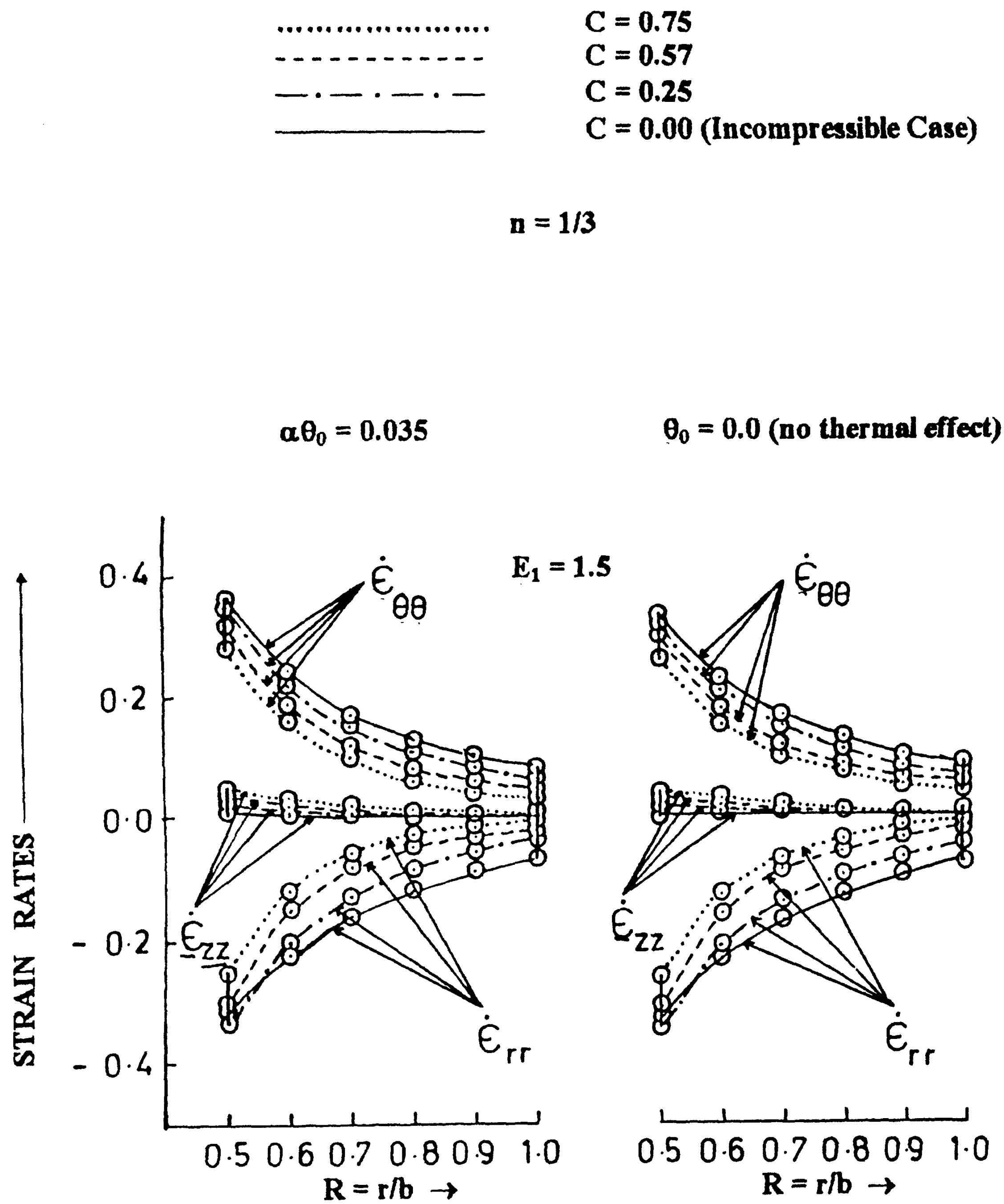

FIG. 3(c). Strain rate distribution for a thick-walled circular cylinder under internal pressure for $n = 1/3$ and $E_1 = 1.5$ along the radius

that for a thick-walled cylinder made of compressible material and $E_1 < 1.0$ (i.e. Young's Modulus of the material is less than the pressure applied) the creep rates have larger values at the internal surface as compared to $E_1 \geq 1.0$. These values further increase at the internal surface as n decreases ($n = 1/7$) or N increases ($N = 7$) and $E_1 < 1.0$ (see Fig. 4(a)). With the introduction of thermal effects the creep rates at the internal surface have much higher values for $n = 1/7$ as compared to

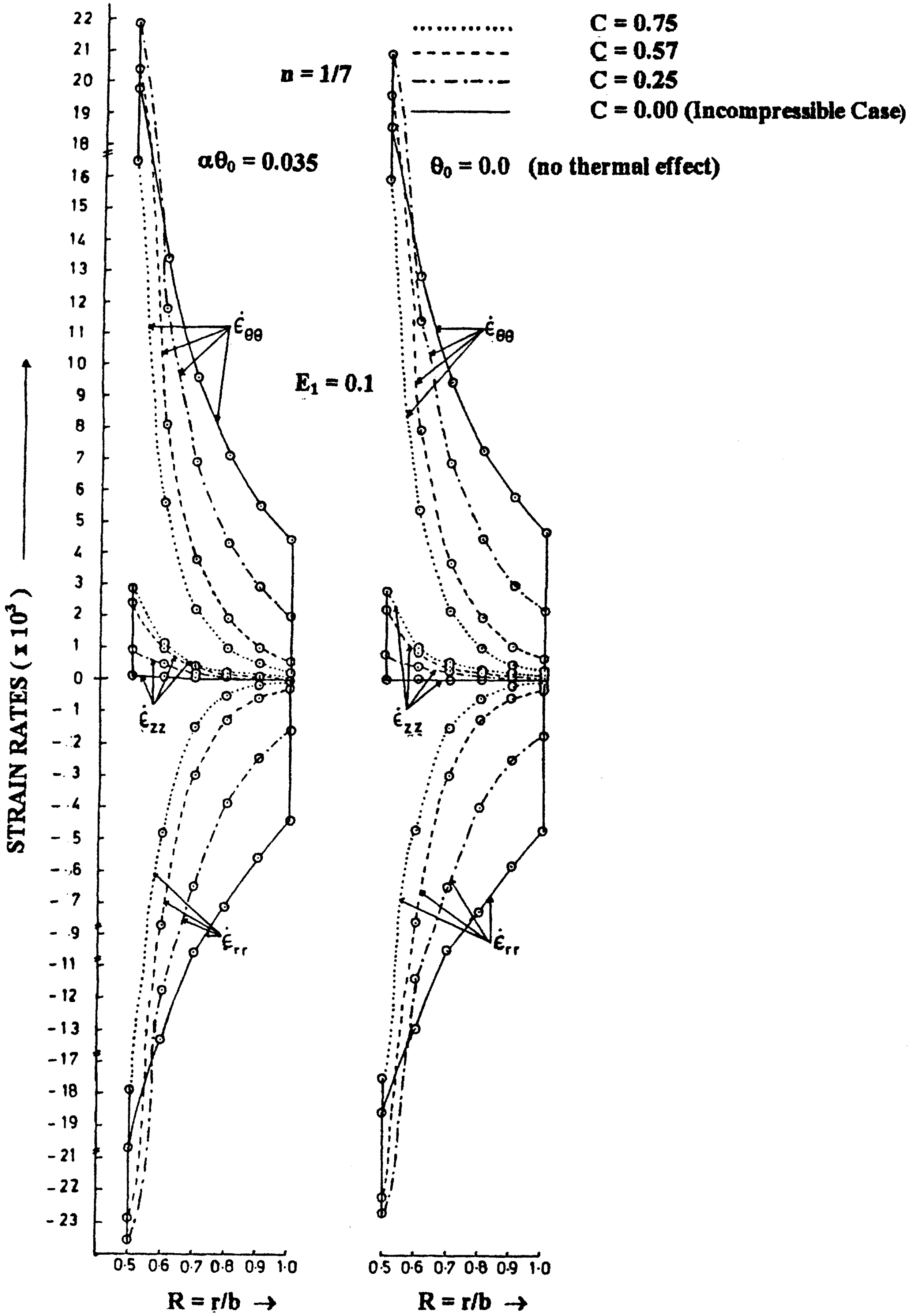

FIG. 4(a). Strain rate distribution for a thick-walled circular cylinder under internal pressure for $n = 1/7$ and $E_1 = 0.1$ along the radius

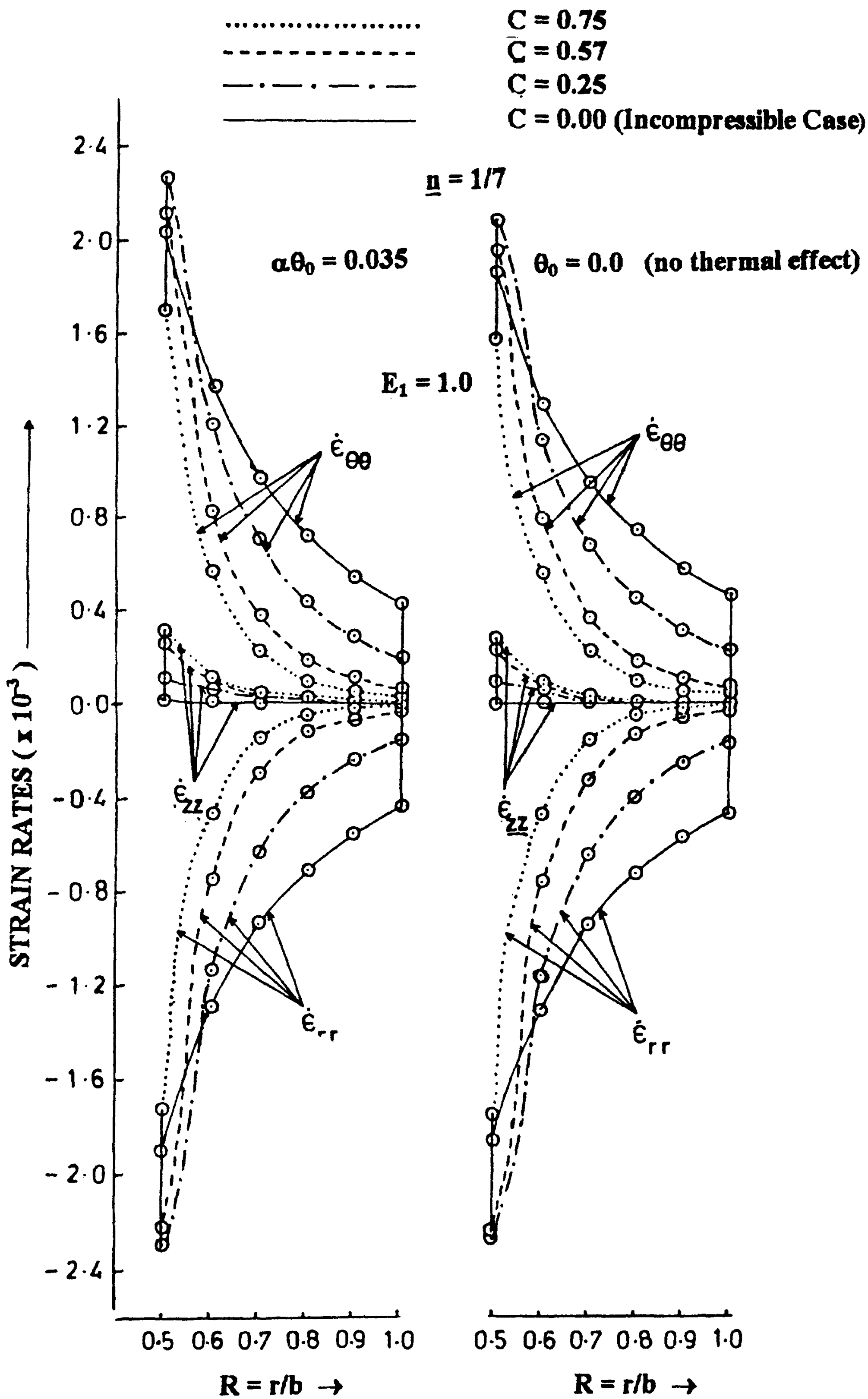

FIG. 4(b). Strain rate distribution for a thick-walled circular cylinder under internal pressure for $n = 1/7$ and $E_1 = 1.0$ along the radius

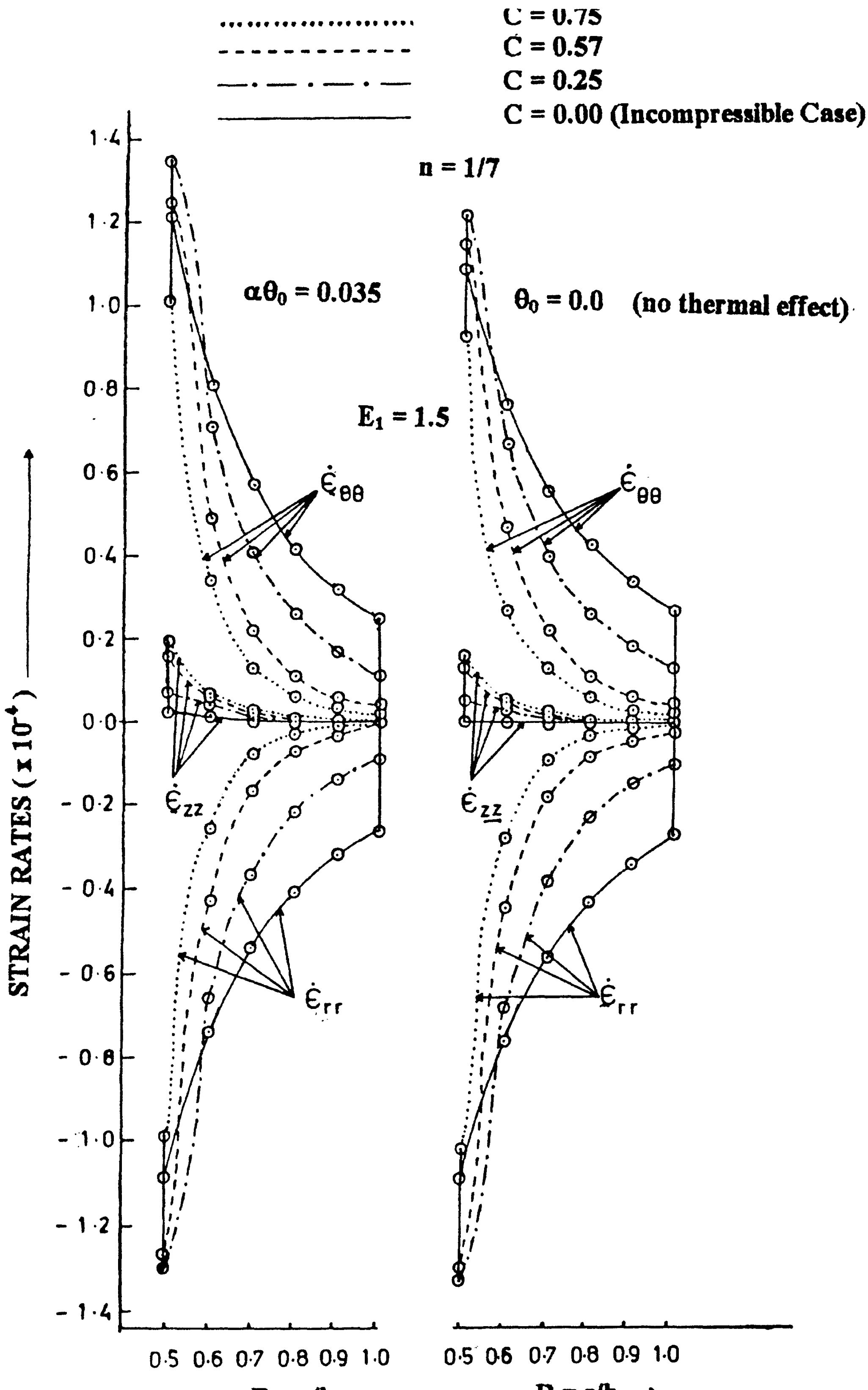

FIG. 4(c). Strain rate distribution for a thick-walled circular cylinder under internal pressure for $n = 1/7$ and $E_1 = 1.5$ along the radius

$n = 1/3$. It means that a thick-walled cylinder made of compressible material subjected to both pressure and temperature have large creep rates at the internal surface for measure $n = 1/7$ (or N - 7) and $E_1 < 1.0$ as compared to $n = 1/3$ (or $N = 3$), $E_1 \geq 1.0$ and cylinder made of incompressible material.

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