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ITERATIVE PROCEDURES TO APPROXIMATE FIXED POINTS

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Let X be a locally convex topological vector space and $T: X \to X$ a linear mapping such that the family $\{T^n : n = 1, 2, \ldots\}$ is equi-continuous. Then the Mann-Krasnoselskii iteration process can be applied to approximate a fixed point of T. We can also apply this process to approximate a common fixed point of a family of mappings under appropriate conditions.

Key Words : Common Fixed Point; Commutative Family; Equicontinuous Linear Mappings; Locally Convex Convex Topological Vector Space; Iteration Method

The well-known Banach Contraction Principles states that if $f: (X, d) \to (X, d)$ is a contraction, then for any initial x in X, the sequence of iterates $\{f^n(x)\}$ converges to the unique fixed point of f. However, for more general mappings, this iteration process fails. Mann[6] and Krasnoselskii[5] have developed other iteration procedures which are applicable to more general situations. Also Rhoades has made some contributions in this area[8&9]. It is the main purpose of this paper to apply their iteration methods to approximate fixed points of a certain class of mappings. We obtain results that generalize some theorems of Dotson[2], Outlaw and Groetsch[7]. We can also successfully approximate a common fixed point of an equicontinuous semi-group of linear mappings that map a compact convex subset K of a locally convex topological vector space X into K.

Theorem 1 — *Let X be a locally convex topological vector space and $T: X \to X$ a linear mapping such that the family $\{T^n : n = 1, 2, \ldots\}$ is equicontinuous. If for some x in X, the sequence $\{S^n x\}$ contains weakly convergent subsequence, then $\{S^n x\}$ converges to a fixed point of T (where $S = \frac{I+T}{2}$).*

PROOF : First, we shall show that $S^n x - TS^n x \to 0$ as $n \to \infty$. Since $(I - T) S^n = S^n (I - T)$, hence

$$S^n x - TS^n x = (I - T) S^n x = S^n (I - T) x = S^n (x - Tx)$$

$$= ((I + T)/2)^n (x - Tx) = \sum_{k=0}^{n} 2^{-n} C(n, k) T^k (x - Tx)$$

$$= 2^{-n} \{C(n, 0)(x - Tx) + C(n, 1)(Tx - T^2 x) + \ldots$$

$$+ C(n, n)(T^n x - T^{n+1} x)\}$$

$$= 2^{-n}\{C(n, 0)\, x + C(n, 1) - C(n, 0))\, Tx + \dots$$

$$+ (C(n, n) - C(n, n-1))\, T^n x - C(n, n)\, T^{n+1} x\}.$$

Let $[n/2] = r$ ($[x]$ is the Gauss greatest integer function), $A_n = 2^{-1}\, C(n, r)$. Then $C(n, k) \le C(n, r)$ for $k = 0, 1, 2, \dots, n$, and $A_n = 2^{-n}\, C(n, r) \to 0$ as $n \to \infty$ by Stirling's formula.

$$S^n x - TS^n x = A_n\, (F_1(x) - F_2(x)),$$

where

$$F_1(x) = \frac{C(n, 0)\, x + (c(n, 1) - C(n, 0))\, Tx + \dots + (C(n, r) - C(m, r-1))\, T^r x}{C(n, r)}$$

$$F_2(x) = \frac{(C(n, r) - C(n, r+1))\, T^{r+1} x + \dots + (C(n, n-1) - C(n, n))\, T^n x + C(n, n)\, T^{n+1} x}{C(n, r)}.$$

Suppose a neighbourhood $U(0)$ is given, we may choose a balanced convex neighbourhood $V(0)$ such that $V + V \subseteq U$. For this $V(0)$, we may use the equicontinuity of $\{T^n : n = 1, 2, \dots\}$ to get balanced convex $W(0) \subseteq V$ such that $y \in W$ implies $T^n y \in V$ for $n = 1, 2, \dots$, and $W(0)$ is absorbing implies the existence of some $t > 0$ such that $x \in tW$ or $\frac{1}{t} x \in W$. On the other hand $A_n \to 0^+$ implies $\frac{1}{A_n} \to \infty$ which in turn implies the existence of some N such that $\frac{1}{A_n} > t$ whenever $n \ge N$. Thus $n \ge N$ implies $x \in \left(\frac{1}{A_n}\right) W$ or $A_n x \in W$ which implies $T^m (A_n x) = A_n T^m x \in V$ or $T^m x \in \left(\frac{1}{A_n}\right) V$ for all $m = 1, 2, \dots$, and hence $F_1(x), F_2(x) \in \frac{1}{A_n} V$. Consequently, $F_1(x) - F_2(x) \in \frac{1}{A_n}(V + V) \subseteq \frac{1}{A_n} U$ or $A_n\, (F_1(x) - F_2(x)) \in U$ whenever $n \ge N$. Thus $S^n x - TS^n x \to 0$. Since T is linear and continuous, we get

$$S^n x - T^2 S^n x = (S^n x - TS^n x) + T\, (S^n x - TS^n x) \to 0.$$

Inductively, we get $S^n x - T^m S^n x \to 0$ for each $m = 1, 2, \dots$.

Next, we show that the family of linear mappings $\{S_n\}$ is equicontinuous. Let $U(0)$ be given and we may choose a balanced convex neighbourhood $V(0)$ such that $V \subseteq U$. For this $V(0)$, we may use the equicontinuity of $\{T^n : n = 1, 2, \dots\}$ to get $W(0) \subseteq V$ such that $T^n (W) \subseteq V$ for all $n = 1, 2, \dots$. Thus for $x \in W$, we have $T^n x \in V$ for all $n = 1, 2, \dots$, which implies $S^n x \in V$ since $S^n x \in Co\ \{x, Tx, \dots, T^n x\} \subseteq V \subseteq U$, showing that $\{S_n\}$ is equi-continuous. Also $\{S^n x\}$ contains a weakly convergent subsequence by hypothesis. Thus we may apply Eberlein's mean ergodic theorem to obtain $\lim_{n \to \infty} S^n x = y$ and $Ty = y$.

We may now deduce the following corollary which generalizes a result of Dotson[2] as well as a theorem of Outlaw and Groetsch[7].

Corollary 1 — Let X be a reflexive Banach space and $T : X \to X$ be a linear mapping such that $\{T^n : n = 1, 2, \ldots\}$ is uniformly bounded, i.e. $\| T^n \| \le M < \infty$ for $n = 1, 2, \ldots$. Then for any $x \in X$, $\{S^n x\}$ converges to a fixed point of T.

PROOF : $\{T^n\}$ is uniformly bounded implies $\{T^n\}$ is equi-continuous. Also

$$\| S^n x \| = \left\| \sum_{k=0}^{n} 2^{-n} C(n, k) T^k x \right\| \le \sum_{k=0}^{n} 2^{-n} C(n, k) \| T^k \| \| x \|$$

$$\le \sum_{k=0}^{n} 2^{-n} C(n, k) M \| x \| = M \| x \| \le r < \infty.$$

Hence the sequence $\{S^n x\}$ is contained in the closed ball $B\ [0; r]$ with center at the origin and radius r. By the reflexivity of X, $B\ [0; r]$ is weakly compact and thus weakly sequentially compact by Eberlein-Smulian theorem. Hence $\{S^n x\}$ contains a weakly convergent subsequence $\{S^{n_i} x\}$. It follows now from Theorem 1 that $\{S^n x\}$ converges to a fixed point of T.

PROOF : Observe that uniformly convex Banach spaces are certainly reflexive and T being nonexpansive implies $\| T^n \| \le 1$ for $n = 1, 2, \ldots$. Hence, the conclusion follows from Corollary 1.

Next, suppose K is a compact convex subset of a locally convex topological vector space and $\mathcal{S}$ is an equicontinuous group of linear mappings such that $\mathcal{S}(K) \subseteq K$. Then $\mathcal{S}$ has a common fixed point in K by Kakutani's fixed point theorem. Also Markov-Kakutani's Theorem states that if $\mathcal{F}$ is a commutative family of continuous linear mappings which map K into itself, then $\mathcal{F}$ has a common fixed point in K (cf. [3]). By combining equicontinuity and commutativity together, we obtain an iterative procedure to approximate a common fixed point in the following theorem.

Theorem 2 — *Let K be a compact convex subset of a locally convex topological linear space X. Suppose $\mathcal{S}$ is an equicontinuous commutative semigroup of linear mappings such that $\mathcal{S}(K) \subseteq K$. Assume furthermore, that $\mathcal{S}$ is finitely generated, say $\mathcal{S} = [T_1, T_2, \ldots, T_m]$. Then for any x in K,* $\left(\dfrac{mI + T_1 + T_2 + \ldots + T_m}{2m} \right)^n (x)$ *converges to a common fixed point of $\mathcal{S}$ as $n \to \infty$.*

PROOF: Let $T = \left(\dfrac{T_1 + T_2 .. + T_m}{m} \right)$. Since $\mathcal{S} = [T_1, T_2, \ldots, T_m] = \left\{ T_1^{n_1} \circ T_2^{n_2} \circ \ldots \circ T_m^{n_m} : n_i \ge 0 \right\}$ is equicontinuous and each T^n is a convex combination of elements of $\mathcal{S}$, we may use methods as in the proofs of previous theorems to show that the sequence $\{T^n\}$ is also equicontinuous. K is convex implies each T^n maps K into K. It follows from Theorem 1 that $\left(\dfrac{mI + T_1 + T_2 + \ldots + T_m}{2m} \right)^n (x) = \left(\dfrac{I + T}{2} \right)^n (x)$ converges to a fixed point p of $T = \dfrac{T_1 + T_2 + \ldots + T_m}{m}$. It follows then from a result of Anzai and Ishikawa1 that p is a common fixed point for each T_i, $i = 1, 2, \ldots$, m and consequently p is a common fixed point of $\mathcal{S}$.

Remark : We remark explicitly that Theorem 1 is still valid if the mapping $S = \frac{I+T}{2}$ is replaced by the more general $S_\alpha = \alpha I + (1-\alpha)T\,(0 < \alpha < 1)$. Theorem 2 is still valid if the mapping $\frac{mI + T_1 + T_2 + \ldots + T_m}{2m}$ is replaced by the more general $\alpha I + \alpha_1 T_1 + \ldots + \alpha_m T_m$ with $\alpha, \alpha_i > 0$ and $\alpha + \sum_{i=1}^{m} \alpha_i = 1$. These results are stated in their present form to avoid cumbersome notations.

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