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ON SOME NEW WEYL-TYPE DISCRETE INEQUALITIES

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In this paper, some new Weyl-type discrete inequalities are obtained, which generalise and improve some known results due to Pachpatte and Alic at el.

Key Words : Weyl-type; Discrete Inequality; Summation by Parts

1. INTRODUCTION

Weyl[8] established the following very important inequality:

$$\int_0^\infty f^2\,dx \le 2\left(\int_0^\infty x^2 f^2\,dx\right)^{1/2}\left(\int_0^\infty f'^2\,dx\right)^{1/2}, \quad \ldots (1)$$

where f is a real-valued continuously differentiable function on $(0, \infty)$ and the integrals on the right side of (1) exist. A more general version of this inequality was present in Hardy *et al.*[4]. Some other important generalizations of (1) were derived by Benson[2] and subsequently Bernis[3] and Pachpatte[6] proved some multivariable variants of it.

Pachpatte[7] obtained some discrete Weyl-type inequalities and Alic *et al.*[1] subsumed Pachpatte's results into a single and more general inequality. The derivation in [7] and [1] depends on the following inequality due to Pachpatte [7, pp. 713-714]:

$$(\alpha+1)\sum_{n=0}^{m-1} n^{\alpha} v_n^r \le (\alpha+2)\sum_{n=0}^{m-1}(n+1)^{\alpha+1}\frac{v_n^r}{m} + \sum_{n=0}^{m-1}(n+1)^{\alpha+1} \times\left(1-\frac{n+1}{m}\right)\cdot r\cdot v_n^{r-1}\,|\Delta u_n| \quad \ldots (2)$$

where $\alpha \ge 0, p \ge 0, q \ge 1$ are real constants, $r = p + q$, $\{u_n\}_0^m$ is a sequence of real numbers, $v_n = |u_n|$ and $\Delta u_n := u_{n+1} - u_n$.

By means of a more general and sharper version of (2), we will generalise and improve the main results established in [1, 7].

2. Results

Let $N = \{1, 2, ...\}$, $N_0 = \{0, 1, 2, ...\}$ and $N_{0,m} = \{0, 1, 2, ..., m\}$ for some fixed $m \in N$.

Throughout, we assume that all the sums involved exist on the respective domains of their definitions, and the usual convention that any empty sum is taken to be zero, is also used.

Lemma — Let $p \geq 0, q \geq 1$ be real constants, $\{u_n\}$, $\{v_n\}$ $(n \in N_{0,m})$ be sequences of real numbers with $v_n \geq 0$ for $n \in N_{0,m}$. Then

$$\sum_{n=0}^{m-1} v_n |u_n|^r \leq \sum_{n=0}^{m-1} \left(1 - \frac{n+1}{m}\right) V_{n+1} \cdot r \cdot |u_n|^{r-1} |\Delta u_n| + \sum_{n=0}^{m-1} (V_{n+1} + nv_n) |u_n|^r \qquad \dots (3)$$

where $r = p + q$ and $V_n = \sum_{i=0}^{n-1} v_i$.

Proof : Constructing the following finite sum and taking summation by parts, we have

$$\sum_{n=0}^{m-1} \left[v_n - \frac{1}{m}(V_{n+1} + nv_n) \right] |u_n|^r$$

$$= \sum_{n=0}^{m-1} |u_n|^r \Delta\left[\left(1 - \frac{n}{m}\right) V_n\right]$$

$$= -\sum_{n=0}^{m-1} \left(1 - \frac{n+1}{m}\right) V_{n+1} \Delta(|u_n|^r)$$

$$\leq \sum_{n=0}^{m-1} \left(1 - \frac{n+1}{m}\right) V_{n+1} \cdot r |\Delta u_n|^{r-1} (|u_n| - |u_{n+1}|), \qquad \dots (4)$$

here we used the elementary inequality (see, [4], p.39)

$$xy^{r-1}(x - y) \leq x^r - y^r \, rx^{r-1} \leq (x - y), \text{ for } x, y \geq 0 \text{ and } r \geq 1.$$

By triangle inequality, inequality (4) can be rewritten as inequality (3).

Remark 1 : When $v_n = n^\alpha$ (where $\alpha \geq 0$ is real constant), we observe from (3) that

$$\sum_{n=0}^{m-1} n^\alpha |u_n|^r$$

$$\leq \sum_{n=0}^{m-1} \left(1-\frac{n+1}{m}\right)\left(\sum_{i=0}^{n} i^{\alpha}\right) \cdot r \cdot |u_n|^{r-1} |\Delta u_n|$$

$$+ \sum_{n=0}^{m-1} \frac{1}{m}\left(\sum_{i=0}^{n} i^{\alpha} + n^{\alpha+1}\right) |u_n|^r \qquad \text{... (5)}$$

Since $$\sum_{i=0}^{n-1} i^{\alpha} \leq \frac{(n+1)^{\alpha+1}}{\alpha+1}$$

and $$\sum_{i=0}^{n-1} i^{\alpha} + n^{\alpha+1} \leq \frac{(n+1)^{\alpha+1}}{\alpha+1} + n^{\alpha+1} < \frac{\alpha+2}{\alpha+1}(n+1)^{\alpha+1}.$$

hold, thus (5) is sharper than (2), and (3) is a generalization of (2).

Theorem 1 — *Suppose that p, q, $\{u_n\}$ and $\{v_n\}$ are as defined in above Lemma and $\{w_n\}$ $(n \in N_{0,m})$ is a sequence of positive real numbers. Then*

$$\sum_{n=0}^{m-1} v_n |u_n|^{p+q} \leq \left\{ \sum_{n=0}^{m-1} V_{n+1} \; |u_n|^p w_n^q \left(g_n |\Delta u_n| + h_n \frac{|u_n|}{m} \right)^q \right\}^{1/q}$$

$$\left\{ \sum_{n=0}^{m-1} V_{n+1} \frac{1}{w_n^{q'}} |u_n|^{p+q} \right\}^{1/q'}, \qquad \text{... (6)}$$

where $$V_n = \sum_{i=0}^{n-1} v_i,\; g_n = (p+q)\left(1-\frac{n+1}{m}\right),\; h_n = 1 + \frac{nv_n}{V_{n+1}} \text{ and } q' = \frac{q}{q-1}.$$

PROOF : Letting $g_n = (p+q)\left(1-\frac{n+1}{m}\right), h_n = 1 + \frac{nv_n}{V_{n+1}}$ we derive from (3) that

$$\sum_{n=0}^{m-1} v_n |u_n|^{p+q} \leq \sum_{n=0}^{m-1} V_{n+1} \; |u_n|^{p+q-1} \left[g_n |\Delta u_n| + h_n \frac{|u_n|}{m} \right]$$

$$= \sum_{n=0}^{m-1} \left[V_{n+1}^{1/q} \; |u_n|^{p/q} w_n \left[g_n |\Delta u_n| + h_n \frac{|u_n|}{m} \right] \times V_{n+1}^{(q-1)/q} |u_n|^{(q-1)(p+q)/q} \cdot \frac{1}{w_n} \right]$$

$$\leq \left\{ \sum_{n=0}^{m-1} V_{n+1} |u_n|^p w_n^q \left[g_n |\Delta u_n| + h_n \frac{|u_n|}{m} \right]^q \right\}^{1/q}$$

$$\times\left\{\sum_{n=0}^{m-1} V_{n+1}\,|\,u_n\,|^{p+q}\frac{1}{w_n^{q'}}\right\}^{1/q'},$$

where the Hölder inequality with indices q, q' is used.

Remark 2 : When $v_n = n^{\alpha}$ (where $\alpha \geq 0$ is a real constant), since

$$\frac{n^{\alpha+1}}{\alpha+1} < V_{n+1} = \sum_{i=0}^{n} i^{\alpha} \leq \frac{(n+1)^{\alpha+1}}{\alpha+1},$$

$$g_n = (p+q)\left(1-\frac{n+1}{m}\right) < (p+q)\left(1-\frac{1}{m}\right)$$

and
$$h_n = 1+\frac{nv_n}{V_{n+1}} = 1+\frac{nv_n}{\sum_{i=0}^{n} i^{\alpha}} \leq 1+n^{\alpha+1}\times\frac{\alpha+1}{n^{\alpha+1}} = \alpha+2$$

hold, by setting $M_m' = \max\left\{\alpha+2,\,(p+q)\left(\frac{m-1}{m}\right)\right\}$ we derive from (6) that

$$\sum_{n=0}^{m-1} n^{\alpha}\,|\,u_n\,|^{p+q} \leq \left\{\sum_{n=0}^{m-1}\left(\sum_{i=0}^{n} i^{\alpha}\right)|\,u_n\,|^{p}\,w_n^{q}\left[g_n\,|\,\Delta u_n\,|+h_n\frac{|\,u_n\,|}{m}\right]^{q}\right\}^{1/q}$$

$$\times\left\{\sum_{n=0}^{m-1}\left(\sum_{i=0}^{n} i^{\alpha}\right)|\,u_n\,|^{p+q}\frac{1}{w_n^{q'}}\right\}^{1/q'} \qquad \ldots (7)$$

$$\leq M_m\left\{\sum_{n=0}^{m-1}(n+1)^{\alpha+1}\,w_n^{q}\,|\,u_n\,|^{p}\left(\frac{|\,u_n\,|}{m}+|\,\Delta u_n\,|\right)^{q}\right\}^{1/q}$$

$$\times\left\{\sum_{n=0}^{m-1}(n+1)^{\alpha+1}\frac{1}{w_n^{q'}}\,|\,u_n\,|^{p+q}\right\}^{1/q'},$$

where
$$M_m = \frac{1}{\alpha+1}\times M_m' = \max\left\{\frac{\alpha+2}{\alpha+1},\,\frac{p+q}{\alpha+1}\left(\frac{m-1}{m}\right)\right\}.$$

Inequality (7)' is the main result of Alic *et al.* [1, Th. 1], which contains the main results of Pachpatte [7, Theorem] as special case when $w_n = (n+1)^{1/q}, 1, (n+1)^{(\alpha+1)/q'}$, and $(n+1)^{\alpha-(2\alpha+1)/q}$, respectively. This fact shows that inequalities (6) and (7) extend and improve the above-mentioned main results of Pachpatte and Alic et al, respectively.

***Theorem* 2** — *Let $p \geq 0, q \geq 1$ be real constants, $\{u_n\}$, $\{v_n\}$ $\{w_n\}$ $(n \in N_0)$ be sequences with $v_n \geq 0$ and $w_n > 0$ for $n \in N_0$. Then*

$$\sum_{n=0}^{\infty} v_n |u_n|^{p+q} \leq (p+q)\left[\sum_{n=0}^{\infty} V_{n+1} w_n^q |u_n|^p |\Delta u_n|^q\right]^{1/q}$$

$$\times\left[\sum_{n=0}^{\infty} V_{n+1} \frac{1}{w_n^{q'}} |u_n|^{p+q}\right]^{1/q'}, \qquad \dots (8)$$

here all relevent sums are assumed exist.

PROOF : By Minkowski's inequality, we derive from (6) that

$$\sum_{n=0}^{m-1} v_n |u_n|^{p+q} \leq \left\{\left[\sum_{n=0}^{m-1} V_{n+1} |u_n|^p w_n^q g_n^q |\Delta u_n|^q\right]^{1/q}\right.$$

$$\left. + \left[\sum_{n=0}^{m-1} V_{n+1} w_n^q h_n^q \frac{|u_n|^{p+q}}{m^q}\right]^{1/q}\right\} \left\{\sum_{n=0}^{m-1} V_{n+1} \frac{1}{w_n^{q'}} |u_n|^{p+q}\right\}^{1/q'}$$

$$\leq \frac{1}{m}\left[\sum_{n=0}^{\infty} V_{n+1} w_n^q h_n^q |u_n|^{p+q}\right]^{1/q}\left[\sum_{n=0}^{\infty} V_{n+1} \frac{1}{w_n^{q'}} |u_n|^{p+q}\right]^{1/q'}$$

$$+ \left[\sum_{n=0}^{\infty} V_{n+1} w_n^q g_n^q |u_n|^p |\Delta u_n|^q\right]^{1/q}\left[\sum_{n=0}^{\infty} V_{n+1} \frac{1}{w_n^{q'}} |u_n|^{p+q}\right]^{1/q'}.$$

In the last inequality by letting $m \to \infty$, we obtain (8).

Remark 3 : (*i*) When $v_n = n^{\alpha}$ (where $\alpha \geq 0$ is a real constant), from (8) we have

$$\sum_{n=0}^{\infty} n^{\alpha} |u_n|^{p+q} \leq (p+q)\left[\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} i^{\alpha}\right) w_n^q |u_n|^p |\Delta u_n|^q\right]^{1/q}$$

$$\times\left[\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} i^{\alpha}\right) \frac{1}{w_n^{q'}} |u_n|^{p+q}\right]^{1/q'}$$

$$\leq \frac{p+q}{\alpha+1}\left[\sum_{n=0}^{\infty} (n+1)^{\alpha+1} w_n^q |u_n|^p |\Delta u_n|^q\right]^{1/q}$$

$$\times\left[\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\frac{1}{w_n^q}|u_n|^{p+1}\right]^{1/q'} \qquad \dots (9)$$

Since $\frac{p+q}{\alpha+1}\leq M = \max\left\{\frac{\alpha+2}{\alpha+1},\frac{p+q}{\alpha+1}\right\}$, (9) is a tighter ineqality than Theorem 2 of Alic *et al.* [1, p. 578].

(*ii*) In the special case when $v_n = 1$, $\alpha = p = 0$, $q = 2$ and $w_n = 1/\sqrt{n+1}$, we derive from (8) that

$$\sum_{n=0}^{\infty}|u_n|^2\leq 2\left[\sum_{n=0}^{\infty}|\Delta u_n|^2\right]^{1/2}\left[\sum_{n=0}^{\infty}(n+1)^2|u_n|^2\right]^{1/2},$$

which is a closer discrete analogue to Weyl'e inequality (1).

(*iii*) If we let $p = 0$, $w_n(\bar{v}_{n+1})^{-1/q}$ in (8), then we shall obtain the following useful inequality

$$\sum_{n=0}^{\infty}v_n(u_n)^q\leq q\left(\sum_{n=0}^{\infty}|DU_n|^q\right)^{1/a}\left[\sum_{n=0}^{\infty}\bar{V}_{n+1}^{q'}|U_n|^q\right]^{1/q}.$$

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