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A CHARACTERISTIC CONDITION FOR CONVERGENCE OF GENERALIZED STEEPEST DESCENT APPROXIMATION TO ACCRETIVE OPERATOR EQUATIONS

HAIYUN ZHOU

Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang, 050003 *P.R. China*

AND

JINTI GUO

Department of Mathematics, Huabei Petroleum Education College, Renqiu, 062551 *P.R. China*

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In the present paper a sufficient and necessary condition for convergence of generalized steepest descent approximation to accretive operator equations is estalished, and for the sufficiency part a specific error estimation is also given.

Key Words and Phrases : Quasi-accretive Operator; Generalized Steepest Descent Approximation; Error Estimation.

1. INTRODUCTION

Let X be a real Banach space with norm $\|\cdot\|$ and dual X^*. The normalized duality mapping from X to the family of subsets of X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A mapping T with domain $D(T)$ in X is said to be accretive if for each $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0. \quad \text{... (1)}$$

Furthermore, T is called strongly accretive if there exists a constant $k > 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k \|x-y\|^2. \quad \text{... (2)}$$

T is said to be ϕ-strongly accretive if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that the inequality

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\| x-y \|) \| x-y \| \qquad \dots (3)$$

holds for all $x, y \in D(T)$. Let $N(T) = \{x \in X : Tx = 0\}$. If $N(T) \neq \phi$, and the inequalities (1), (2) and (3) hold for any $x \in D(T)$, but $y \in N(T)$, then the corresponding operator T is called quasi-accretive, strongly quasi-accretive and ϕ-strongly quasi-accretive, respectively. Such operators have been extensively studied and used by several researchers (see, e.g. [5, 10]).

Recall that a quasi-accretive operator A is said to be satisfy Condition (I) if, for any $x \in D(A), p \in N(A)$ and any $j(x-p) \in J(x-p)$ the equality $\langle Ax, j(x-p) \rangle = 0$ holds if and only if $Ax = Ap = 0$.

Recently, Xu and Roach studied the characteristic conditions for the convergence of the steepest descent approximation process

$$(@) \begin{cases} x_0 \in X \\ x_{n+1} = x_n - t_n A x_n \; n \geq 0, \end{cases}$$

where $t_n \in (0, \infty)$, $\sum_{n=0}^{\infty} t_n = \infty$, and $t_n \to 0$ as $n \to \infty$. They proved the following theorem.

***Theorem* XR** — *Let X be a uniformly smooth Banach space and let $A : D(A) = X \to X$ be a quasi-accretive, bounded operator which satisfies condition (I). Then, for any initial value $x_0 \in D(A)$, there is a positive real number $T(x_0)$ such that the steepest descent approximation method (@) with $t_n \leq T(x_0)$ for any n, converges strongly to a solution x^* of the equation $Ax = 0$ if and only if there is a strictly increasing function $\phi : R^+ \to R^+$, $\phi(0) = 0$, such that*

$$\langle Ax - Ax^*, j(x_n - x^*) \rangle \geq \phi(\| x_n - x^* \|) \| x_n - x^* \|. \qquad \dots (4)$$

We remark immediately that in the above Theorem XR, the choice of the initial value $x_0 \in D(A)$ should satisfy certain restrictions in order to guarantee that $\phi^{-1}(\| Ax_0 \|)$ is well-defined. Moreover, the choice of iteration parameter t_n depends heavily both on the modulus of smoothness $\rho_x(\tau)$ of X and on constants K and C. Clearly, it is rather hard for one to determine $t_n \leq T(x_0)$ for all $n \geq 0$, because of computational difficulties. On the other hand, in some applications, the operator A is, in general, not defined on the whole of X. The domain of A, $D(A)$, is generally a proper subset of X. In this case, the steepest descent approximation process (@) may not even be well-defined. Recently, author[11] has completely solved above all problems by using new approximating techniques and proving more general result.

It is our objective in this paper to extend the results of author[11] to the generalized steepest descent approximation setting. For this purpose, we need to introduce the following concepts and known fact.

Let $A : D(A) \subset X \to X$ be an operator and let $\{c_n\}$ and $\{d_n\}$ be two real sequences in [0, 1]. A scheme (GSDA) is called a generalized steepest descent approximation if a sequence $\{x_n\}_{n \geq 0}$ is defined by

$$(\text{GSDA}) \begin{cases} x_0 \in D(A) \\ y_n = x_n - d_n A x_n, \; n \geq 0, \\ x_{n+1} = x_n - c_n A y_n, \; n \geq 0, \end{cases}$$

An Operator A — $A : D(A) \subset X \to X$ is said to be locally bounded at some $x_0 \in D(A)$ if there exists a positive constant $r > 0$ such that the closed ball $B_r(x_0)$ is contained in $D(A)$, and $A(B_r(x_0))$ is bounded. Accordingly, A is said to be locally bounded on $D(A)$ if A is locally bounded at every $x \in D(A)$. It is well known that if X is a uniformly smooth Banach space and $A : D(A) \subset X \to X$ is an accretive operator with an open domain $D(A)$, then A is locally bounded on $D(A)$.

The following lemma, due to Petryshyn[3], is very useful in this paper for the proof of our main results.

Lemma 1.1 — Let X be a real Banach space. Then the following inequality

$$\| x + y \|^2 \le \| x \|^2 + 2 \langle y, j(x+y) \rangle$$

holds for all $x, y \in X$ and all $j(x+y) \in J(x+y)$.

2. Main Result

Theorem 2.1 — *Let X be a real uniformly smooth Banach space and let $A : D(A) \subset X \to X$ be a quasi-accretive locally bounded operator which satisfies condition (I). Then there exist a closed ball $B_r(x^*)$ and positive real numbers $T(x_0)$ such that the generalized steepest descent approximation scheme (GSDA) defined by*

$$\text{(GSDA)} \begin{cases} x_0 \in B_r(x^*) \\ y_n = x_n - d_n A x_n, \; n \ge 0, \\ x_{n+1} = x_n - c_n A y_n, \; n \ge 0, \end{cases}$$

with

(i) $0 < (c_n + d_n) \le T(x_0) = \min\left\{ \frac{\delta}{M}, \frac{r}{4M} \right\}$ for each $n \ge 0$;

(ii) $\sum_{n=0}^{\infty} c_n = \infty$; and

(iii) $\lim_{n \to \infty} c_n = \lim_{n \to \infty} d_n = 0$;

converges strongly to the solution x^* of the equation $Ax = 0$ if and only if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\text{(C)} \begin{cases} \langle Ax_n, j(x_n - x^*) \rangle \ge \phi(\| x_n - x^* \|) \| x_n - x^* \|, \; n \ge 0, \\ \langle Ay_n, j(y_n - x^*) \rangle \ge \phi(\| y_n - x^* \|) \| y_n - x^* \|, \; n \ge 0. \end{cases}$$

Moreover, for the sufficiency part, if $\inf_{t>0} \frac{\phi(t)}{t} > 0$, then we have the error estimation :

$$\| x_n - x^* \|^2 \le r_1 \, \theta_n, \; n \ge 0,$$

where $r_1 = \max\{r, 1\}$ and $\theta_n \to 0$ as $n \to \infty$.

PROOF : NECESSITY : Suppose that $x_n \to x^*$ as $n \to \infty$. Then $y_n \to x^*$ as $n \to \infty$. Define a new sequence $\{z_n\}$ as follows: $\{z_n\} = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots\}$. Then $z_n \to x^*$ as $n \to \infty$. Let $M = \sup \{\| z_n - x^* \| : n \geq 0\}$. Then $M \geq 0$. If $M = 0$, then $z_n = x^*$ for all $n \geq 0$ and hence the condition (C) follows trivially. Suppose $M > 0$ and for $t \in (0, M)$ define

$$C_t = \{n \in N : \| z_n - x^* \| \geq t\},$$

then C_t is nonempty and finite. Indeed, assume $C_t = \phi$, then $\| z_n - x^* \| < t$ for all $n \geq 0$, it follows that $M \leq t < M$, which is a contradiction. For every $t \in (0, M)$, there exists $n(t) \in N$ such that $\| z_n - x^* \| < t$ for all $n \geq n(t)$. Hence $C_t \subset \{0, 1, 2, \ldots, n(t) - 1\}$. Define

$$f(t) = \min \left\{ \frac{\langle Az_n, j(z_n - x^*) \rangle}{\| z_n - x^* \|} : n \in C_t \right\}.$$

Then $f(t)$ is nonnegative and nondecreasing. If $f(t) = 0$ for some $t \in (0, M)$, then there exists some fixed $n_0 \in C_t$ such that $\| z_{n_0} - x^* \| \geq t$ and $\langle Az_{n_0}, j(z_{n_0} - x^*) \rangle = 0$. By condition (I), we have $Az_{n_0} = Ax^* = 0$. In view of (GSDA), we have $z_n = z_{n_0}$ for all $n \geq n_0$. This implies that $z_{n_0} = x^*$, which contradicts $\| z_{n_0} - x^* \| \geq t$. This contradiction shows that $f(t) > 0$ for every $t \in (0, M)$.

Now we extend the domain of f to R^+ by defining $f(0) = 0$ and $f(t) = \sup \{f(s) : s < M\}$ for all $t \geq M$. Define $\phi(t) = \frac{tf(t)}{1+t}$ for $t \in R^+$. Then $\phi : R^+ \to R^+$ is a strictly increasing function with $\phi(0) = 0$ and satisfies condition (C). The proof of the necessity is complete.

SUFFICIENCY — Suppose that inequality (C) holds. Since A is locally bounded, for $x^* \in N(A)$, there exists a positive constant $r > 0$ such that $B_r(x^*) \subset D(A)$ and $A(B_r(x^*))$ is bounded.

Let $M = \sup \{\| Au \| : \| u - x^* \| \leq r\}$. Since X is uniformly smooth, j is uniformly continuous on the ball $B(0, 2r)$. At this point we can choose some $\delta > 0$ such that

$$\| jx - jy \| \leq \frac{r\phi\left(\frac{r}{4}\right)}{8M}$$

whenever $x, y \in B(0, 2r)$ and $\| x - y \| \leq \delta$.

Now set

$$T(x_0) = \min \left\{ \frac{\delta}{M}, \frac{r}{4M} \right\}.$$

Now we prove that the sequence $\{x_n\}$ defined by (GSDA) with $(c_n + d_n) \leq T(x_0)$ is bounded. We finish the proof by using mathematical induction.

We first prove that $\| y_n - x^* \| \leq r$ whenever $\| x_n - x^* \| \leq r$. Assume that $\| x_n - x^* \| \leq r$. Then $\| Ax_n \| \leq M$.

Observe that

$$\| y_n - x^* \| \leq \| x_n - x^* \| + d_n \| Ax_n \| \leq (r + r) = 2r,$$

and

$$\| y_n - x_n \| \leq d_n \| Ax_n \| \leq \delta,$$

so that

$$\| j(y_n - x^*) - j(x_n - x^*) \| \leq \frac{r\phi\left(\frac{r}{4}\right)}{8M}.$$

Now we want to show that $\| y_n - x^* \| \leq r$ for every $n \geq 0$. If it is not the case, then there exists an n for which

$$\| y_n - x^* \| > r.$$

By (GSDA) we have

$$\| x_n - x^* \| \geq \| y_n - x^* \| - d_n \| Ax_n \|$$

$$\geq r - T(x_0) M$$

$$\geq \frac{r}{2}, \qquad \ldots (5)$$

and hence $$\phi(\| x_n - x^* \|) \geq \phi\left(\frac{r}{2}\right). \qquad \ldots (6)$$

By using Lemma 1.1, (GSDA), (5) and (6), we have

$$\| y_n - x^* \|^2 = \| x_n - x^* - d_n Ax_n \|^2$$

$$\leq \| x_n - x^* \|^2 - 2d_n \langle Ax_n, j(y_n - x^*) \rangle$$

$$\leq \| x_n - x^* \|^2 - 2d_n \phi(\| x_n - x^* \|) \| x_n - x^* \|$$

$$+ 2d_n M \| j(y_n - x^*) - j(x_n - x^*) \|$$

$$\leq \| x_n - x^* \|^2 - 2d_n \frac{r}{2} \phi\left(\frac{r}{2}\right) + d_n r\phi\left(\frac{r}{2}\right)$$

$$\leq \| x_n - x^* \|^2, \qquad \ldots (7)$$

which implies that $\| y_n - x^* \| \leq \| x_n - x^* \| \leq r$, which contradicts the assumption $\| y_n - x^* \| > r$.

Now we prove that $\| x_n - x^* \| \le r$ for all $n \ge 0$. Let x_0 be some point inside $B_r(x^*)$. Then we have that $\| x_0 - x^* \| \le r$. Assume that $\| x_n - x^* \| \le r$. We want to show that $\| x_{n+1} - x^* \| \le r$.

Observe that

$$\| x_{n+1} - x^* \| \le r + c_n \| Ax_n \| \le 2r,$$

and

$$\| x_{n+1} - y_n \| \le (c_n \| Ay_n \| + d_n \| Ax_n \|) \le (c_n + d_n) M \le \delta,$$

so that

$$p_n := \| j(x_{n+1} - x^*) - j(y_n - x^*) \| \le \frac{r\phi\left(\frac{r}{4}\right)}{8M}.$$

Suppose that $\| x_{n+1} - x^* \| \ge r$. Then $\| x_n - x^* \| \ge (r - c_n M) \ge \frac{r}{2}$,

and $\| y_n - x^* \| \ge \left(\frac{r}{2} - d_n M\right) \ge \frac{r}{4}$. Consequently, $\phi(\| y_n - x^* \|) \ge \phi\left(\frac{r}{4}\right)$.

It follows from Lemma 1.1 and the above argument that

$$\begin{aligned}\| x_{n+1} - x^* \|^2 &\le \| x_n - x^* \|^2 - 2c_n \langle Ay_n, j(x_{n+1} - x^*) \rangle \\ &\le \| x_n - x^* \|^2 + 2Mc_n p_n - 2c_n \phi(\| y_n - x^* \|) \| y_n - x^* \| \\ &\le \| x_n - x^* \|^2 + 2Mc_n \frac{r\phi\left(\frac{r}{4}\right)}{8M} - 2c_n \phi\left(\frac{r}{4}\right)\frac{r}{4} \\ &\le \| x_n - x^* \|^2, \qquad \ldots (8)\end{aligned}$$

which implies that $\| x_{n+1} - x^* \| \le \| x_n - x^* \| \le r$, a contradiction. This contradiction shows that $\| x_{n+1} - x^* \| \le r$. By mathematica induction, we have shown that $\| x_n - x^* \| \le r$ for all $n \ge 0$. By the definition of M, we know that $\| Ax_n \| \le M$ and $\| Ay_n \| \le M$ for all $n \ge 0$.

Set $M = \max\left\{ \sup_{n \ge 0} \{ \| Ax_n \| \}, \sup_{n \ge 0} \{ \| Ay_n \| \} \right\}$.

Using (GSDA) and (iii),

$$\begin{aligned} p_n &= \| j(x_{n+1} - x^*) - j(y_n - x^*) \| = \| j(x_{n+1} - y_n) \| = \| j(-c_n Ay_n + d_n Ax_n) \| \\ &\le (c_n + d_n) M, \qquad \ldots (9)\end{aligned}$$

and $p_n \to 0$ as $n \to \infty$, since j is uniformly continuous on bounded subsets of X. Again using Lemma

1.1, we have

$$\| x_{n+1} - x^* \|^2 \le \| x_n - x^* \|^2 - 2c_n \phi(\| y_n - x^* \|) \| y_n - x^* \| + 2Mc_n p_n. \quad \dots (10)$$

Set $\limsup_{n \to \infty} \| y_n - x^* \| = a$. Then $a \ge 0$. Suppose that $a > 0$. Then there exists a fixed integer n_0 such that $\| y_n - x^* \| \ge \frac{a}{4}$, and hence $\phi(\| y_n - x^* \|) \ge \phi\left(\frac{a}{4}\right)$ for all $n \ge n_0$. Since $p_n \to 0$ as $n \to \infty$, we can choose $n_1 \ge n_0$ so large that

$$4Mp_n \le a\phi\left(\frac{a}{4}\right), \quad \dots (11)$$

for all $n \ge n_1$. It follows from (9) and (10) that

$$\frac{a}{4}\phi\left(\frac{a}{4}\right) \sum_{n > n_1}^{\infty} c_n \le \| x_{n_1} - x^* \|^2 < \infty,$$

which contradicts assumption (ii). Consequently,

$$0 \le \liminf_{n \to \infty} \| y_n - x^* \| \le \limsup_{n \to \infty} \| y_n - x^* \| = 0.$$

This shows that $y_n \to x^*$ and hence $x_n \to x^*$ as $n \to \infty$.

Now we consider an error estimation. For this purpose, assume that $\inf_{n \ge 0} \frac{\phi(\| y_n - x^* \|)}{\| y_n - x^* \|} = \sigma > 0$. Without loss of generality, we assume that $p_n \le \frac{\sigma}{M}$ for all $n \ge 0$. Define iteratively a real sequence $\{\theta_n\}_{n \ge 0}$ as follows:

$$\begin{cases} \theta_0 = 1 \\ \theta_{n+1} = (1 - 2\sigma c_n)\,\theta_n + 2M\,c_n p_n,\ n \ge 0. \end{cases}$$

Then we have that $\theta_n \le 1$, $\lim_{n \to \infty} \theta_n = 0$ and $\| x_n - x^* \|^2 \le r_1\,\theta_n$ for all $n \ge 0$, where r_1 = max {r, 1}. We are done. □

Remark : We observe that if $A : D(A) \subset X \to X$ is a ϕ-strongly quasi-accretive operator, then A satisfies the condition (C).

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