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COMPACT-COVERING CS-IMAGES OF METRIC SPACES

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In this paper, we establish the characterizations of metric spaces under compact-covering and cs-mappings by means of compact-finite-partition networks introduced by Lin and Yan [1].

Key Words : Compact-Covering Mapping; cs-Mapping; Compact-Countable Collection; Compact-Finite-Partition Network

Recently, Z. Qu and Z. Gao introduced the concept of cs-mapping (see [2]), that is, $f: X \to Y$ is called a *cs*-mapping, if for any compact subset C of Y, $f^{-1}(C)$ is separable in X. They proved that a space with a compact-countable closed k-network is a compact-covering cs-image of a metric space. But, this result gives only a sufficient condition. The present paper contributes to the problem of characterizing compact-covering cs-images of metric spaces.

All spaces in this paper are assumed to be T_2. Mappings are continuous and onto.

Definition 1 — Let X be a topological space, and let $\mathcal{P}$ be a collection of subsets of X.

(1) $\mathcal{P}$ is a CFP (i.e., compact finite partition) cover of compact subset K in X if there are a finite collection $\{K_n : n \leq k\}$ of closed subsets of K and a collection $\{P_n : n \leq k\} \subset \mathcal{P}$ such that $K = \bigcup\{K_n : n \leq k\}$ and each $K_n \subset P_n$ (see [3]).

(2) $\mathcal{P}$ is a cfp-network (i.e., compact-finite-partition network) if, whenever $K \subset V$ with K compact and V open in X, there exists a finite subcollection $\mathcal{P}^* \subset \mathcal{P}$ such that $\mathcal{P}^*$ is a CFP cover of K and $\bigcup \mathcal{P}^* \subset V$.

(3) $\mathcal{P}$ is called compact-countable if for any compact subset K of X, only countably many members of $\mathcal{P}$ intersect K.

(4) $\mathcal{P}$ is a k-network[7] if whenever $K \subset U$ with K compact and U open in X, then $K \subset U^* \mathcal{P}' \subset U$ for some $\mathcal{P}' \subset \mathcal{P}$.

(5) $\mathcal{P}$ is a *cs*-network[8] if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x\} \bigcup \{x_{n_i} : i \in N\} \subset P \subset U$ for some subsequence $\{x_{ni}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

Definition 2 — Let $f: X \to Y$ be a map.

(1) f is an *s*-map if each $f^{-1}(y)$ is separable.

(2) f is a compact-covering map[4] if each compact subset of Y is the image of some compact subset of X.

(3) f is an ss-map[5] if each $y \in Y$, there exists a neighbourhood U of y in Y such that $f^{-1}(U)$ is separable.

Lemma[1] — Suppose that $\mathcal{P}$ is a point-countable cover of X, and K is compact in X, then the collection of minimal CFP covers of K consisting of members of $\mathcal{P}$ is at most countable.

Theorem — *The following statements are equivalent for a space X :*

(1) X is a compact-covering cs-image of a metric space.

(2) X has a compact-countable cfp-network.

PROOF : (2) $\Rightarrow$ (1) Assume $\mathcal{P}$ is a compact-countable cfp-network for X. Denote $\mathcal{P} = \{P_\alpha : \alpha \in A\}$. For each $i \in N$, let A_i be a copy of A, and it is endowed with discrete topology. Put

$$M = \{\alpha = (\alpha_n) \in \prod_{i \in N} A_i : \{P_{\alpha_i} : i \in N\} \text{ is a network of some point } x_\alpha \text{ in } X\},$$

and give M the subspace topology induced from the product topology of the product space $\prod_{i \in N} A_i$. The point x_α is unique in X because X is T_2. We define $f : M \to X$ by $f(\alpha) = x_\alpha$. Obviously, M is a metric space, and f is a mapping.

(i) f is a *cs*-mapping.

For each compact subset C of X, by the compact countable property of $\mathcal{P}$, $\{\alpha \in A : C \cap P_\alpha \neq 0\}$ is countable. Put $L = \left(\prod_{n \in N} \left\{\alpha \in A_n : C \cap P_\alpha \neq 0\right\}\right) \cap M$, then L is a hereditarily separable subspace of M and $f^{-1}(C) \subset L$. Thus $f^{-1}(C)$ is a separable subspace of M. Hence f is a cs-mapping.

(ii) f is a compact-covering mapping.

For any compact subset K of X, let $\mathcal{T}$ be the collection of minimal CFP covers of K, by Lemma, we can suppose $\mathcal{T} = \{\mathcal{P}_i : \in N\}$, where each $\mathcal{P}_i$ is a minimal CFP cover of K. For each $i \in N$, assume $\mathcal{P}_i = \{P_\alpha : \alpha \in B_i\}$ is finite then there is a finite collection $\{F_\alpha : \alpha \in B_i\}$ of closed subsets of K such that $K = \bigcup_{\alpha \in B_i} F_\alpha$ and each $F_\alpha \subset P_\alpha$. Put $L = \left\{\alpha = (\alpha_i) \in \prod_{i \in N} B_i : \bigcap_{i \in N} F_{\alpha_i} \neq 0\right\}$, then

(a) L is closed in $\prod_{i \in N} B_i$, and so L is compact in $\prod_{i \in N} A_i$.

Suppose $\alpha = (\alpha_i) \in \prod_{i \in N} B_i - L$, by the construction of L, we have $\bigcap_{i \in N} F_{\alpha_i} = 0$.

Thus there is $i_0 \in N$ such that $\bigcap_{i \le i_0} F_{\alpha_i} = 0$. Denote

$W = \left\{(\beta_i) \in \prod_{i \in N} B_i : \text{if } i \le i_0, \text{ then } \beta_i = \alpha_i\right\}$, then W is an neighbourhood of α in $\prod_{i \in N} B_i$ and $W \cap L = 0$.

(b) $L \subset M$ and $f(L) \subset K$.

Suppose $\alpha = (\alpha_i) \in L$, take $x \in \bigcap_{i \in N} F_{\alpha_i}$, then we shall show that $\{P_i : i \in N\}$ is a network of x in X. Suppose V is an open neighbourhood of x in X then there is an open neighbourhood W of x in K such that $cl_K(W) \subset V$ because K is a regular subspace. Since $\mathcal{P}$ is a cfp-network, there is a finite collection $\mathcal{P}' \subset \mathcal{P}$ such that $\mathcal{P}$ is a CFP cover of $cl_k(W)$ and $\bigcup \mathcal{P} \subset V$. In view of the compact subset $K - W \subset X - \{x\}$, $\mathcal{P}'$ is a CFP cover of $K - W$ and $\bigcup \mathcal{P}' \subset X - \{x\}$ for some finite collection $\mathcal{P}' \subset \mathcal{P}$. Let $\mathcal{P}^* = \mathcal{P} \bigcup \mathcal{P}'$, then $\mathcal{P}^*$ is a CFP cover of K, and thus $\mathcal{P}_k \subset \mathcal{P}^*$ for some $k \in N$. Since $x \in F_{\alpha_k} \subset P_{\alpha_k} \in \mathcal{P}_k$, we have $P_{\alpha_k} \in \mathcal{P}$. Thus $x \in P_{\alpha_k} \subset V$. This shows $\{P_{\alpha_i} : i \in N\}$ is a network of x in X. Hence, $\alpha \in M$ and $f(\alpha) = x \in K$, that is, $L \subset M$ and $f(L) \subset K$.

(c) $K \subset f(L)$.

Suppose $x \in K$. For each $i \in N$, there is $\alpha_i \in B_i$ such that $x \in F_{\alpha_i}$. Let $\alpha = (\alpha_i)$, by virtue of the proof of (b), we have that $\alpha \in L$, and $f(\alpha) = x$. Thus $K \subset f(L)$. Hence, $f(L) = K$. Therefore, f is a compact-covering and *cs*-mapping.

(1) $\Rightarrow$ (2) — Suppose $f : M \to X$ is a compact-covering and *cs*-mapping, where M is a metric space. Let $\mathcal{B}$ be a σ-locally-finite base of M. Since f is a cs-mapping, $f(\mathcal{B})$ is compact-countable in X. It is easy to show that the property having of cfp-network is preserved under compact-covering mappings. Thus $f(\mathcal{B})$ is a compact-countable cfp-network of X.

By definition[1], every closed k-network is a cfp-network, so we have

Corollary 1[2] — If X is a space with a compact-countable closed *k*-network, then X is a compact-covering *cs*-image of a metric space.

Corollary 2[2] — If X is a *k*-space with a compact-countable closed *k*-network, then X is a compact-covering and quotient *cs*-image of a metric space.

Example 1 — A compact-covering and *s*-image of a metric space which is not a compact-covering and *cs*-image of a metric space.

Let Z be the topological sum of the unite interval [0, 1] and the collection $\{S(x) : x \in [0, 1]\}$ of 2^{ω} convergent sequences $S(x)$. Let X be the space obtained from Z by identifying the limit point of $S(x)$ with $x \in [0, 1]$, for each $x \in [0, 1]$. X is a compact-covering and *s*-image of a metric space (indeed, X is a compact-covering and quotient compact image of a locally compact metric space), but X is not a compact-covering and *cs*-image of a metric space, because X has no compact-countable cs^*-network (cf. [6]).

Example 2 — A compact-covering and cs-image of a metric space is not a compact covering and ss-image of a metric space.

Let X be a paracompact space with a point-countable base, and not metrizable. Then X has a compact-countable base, and so X has a compact-countable cfp-network, By Theorem above, X is a compact-covering and *cs*-image of a metric space. But, X is not a compact-covering and ss-image of a metric space, because X has no locally countable *k*-network by Theorem in [5].

In view of Example 1 and Example 2, we have the following question relating to Michael-Nagami's problem[9].

Question Suppose X is a quotient *cs*-image of a metric space. Is X a compact-covering and *cs*-image of a metric space?

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