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NONAUTONOMOUS DIFFERENTIAL SYSTEMS OF ALTERNATELY RETARDED AND ADVANCED TYPE

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In this paper, a solution formula of nonautonomous systems of differential equation

$\dot{x}(t) + A(t)x(t) + B(t)\,x(g(t)) = f(t)$

is obtained. At the same time, we study its oscillation and asymptotically stability properties and obtain a sufficient condition of all solutions of the linear nonautonomous difference systems.

Key Words : Alternately Retarded and Advanced Type; Oscillation; Asymptotic Stability

1. INTRODUCTION AND PRELIMINARY

Equations with piecewise constant argument (EPCA) are interesting in their own rights, have some curious and unpredictable properties and are investigated in many papers (see [1]-[10]). The nonautonomous differential equations of alternately retarded and advanced type can be studied in [10].

In this paper, we investigate the global asymptotic behaviour as well as oscillation of the system with piecewise constant argument

$$\dot{x}(t) + A(t)x(t) + B(t)\,x(g(t)) = f(t) \text{ for } t > 0 \qquad \dots (1)$$

subject to the initial condition

$$x(0) = x_0 \qquad \dots (2)$$

where $A(t)$, $B(t)$ are $r \times r$, $x(t)$ is an r-vector, $f(t)$ are locally integrable function on $[0, \infty)$, and $g(t)$ is piecewise constant function defined by

$$g(t) = np \text{ for } t \in [np - l, (n+1)\,p - l)\; n \in N)$$

where p and l are positive constants satisfying $p > l$.

In section 2, we study the case where $A(t)$ is the $r \times r$ zero matrix. We provide a closed formula for the solution of (1)-(2) (see Theorem 1) and show that, under some restrictions on the matrix $B(t)$ and the function $f(t)$, every solution of (1) sends to zero as $t \to \infty$ (see Theorem 2). Before Theorem 3 is given, we study the oscillation of linear nonautonomous deference system and obtain Theorem A. In Theorem 3, we give a sufficient condition for the oscillation of all solution of (1) when $f(t) = 0$. In section 3, we treat the case where $A(t)$ is not zero matrix. We provide a formula for the solution of (1)-(2) (see theorem 4), state conditions under which every solution of (1) converges to zero as $t \to \infty$ (see Theorem 5), and provide oscillatory results (see Theorem 6).

Definition 1 — A function $x : [0, \infty) \to R^r$ is a solution of (1)-(2) if the following conditions hold :-

(i) x is continuous on $[0, \infty)$.

(ii) x is differentiable in $[0, \infty)$, except possibly at the points $t = np - 1$, $n \in \{1, 2, ...\}$, where one-sided derivative exist.

(iii) $x(0) = x_0$ and x satisfies (1) in $(0, p - l)$ and in every interval of the form $[np - l, (n+1)p - l)$ for $n \in \{1, 2, ...\}$.

Definition 2 — A solution of (1)-(2) is oscillatory if every component of it has no last zero.

A solution of discrete (1)-(2) is oscillatory if every component of it has no last zero.

Let $[\cdot]$ denote the greatest integer function and let $|\cdot|$ denote both any vector norm in R^r and its induced matrix norm.

Considering matrix, we define $\prod_{i=n}^{1} A_i = A_n A_{n-1} \dots A_1$.

2 THE CASE : $A(t) \equiv 0$

2.1 *Stability* — In this case, (1) becomes

$$\dot{x}(t) + B(t)\,x(g(t)) = f(t) \text{ for } t > 0 \quad \dots (4)$$

To simplify the notation, define

$$B(a, b) = I - \int_a^b B(s)ds, \quad B(0, -l) = I,$$

$$x(np) = x_n \quad \dots (5)$$

and $I_n = [np - l, (n+1)p - l)$ for $n = 1, 2, \dots$.

Theorem 1 — *Let $B(t)$ and $f(t)$ be locally integrable on $[0, \infty)$. Then (4)-(2) has a unique solution on $[0, \infty)$ given by*

$$x(t) = B(g(t), t) \left(\prod_{j=g(t)/p}^{1} B^{-1}(jp, jp - l)\, B((j-1)p, jp - l) \right.$$

$$\left[x_0 + \sum_{j=1}^{g(t)/p} \left(\prod_{i=1}^{j} {}^{-1}((i-1), ip - l)\, B(ip, ip - l) \right) \right.$$

$$\left.\left(B^{-1}(jp, jp-l)\int_{(j-1)p}^{jp} f(s)\,ds\right)\right] + \int_{g(t)}^{t} f(s)ds \qquad \dots (6)$$

where $B(a, b)$ is defined in (5).

In addition, if $B(t)$ and $f(t)$ are integrable on $(-\infty, 0]$, this solution can be continued backwards on $(-\infty, 0]$ and is given by

$$x(t) = B(g(t), t)\left(\prod_{j=-g(t)/p}^{1} B^{-1}(-jp, -jp-l)\,B((-j-1)p, -jp-l)\right)$$

$$\left[x_0 + \sum_{j=-g(t)/p}^{1}\left(\prod_{i=1}^{j} B^{-1}((-i-1)p, -ip-l)\,B(-ip, -ip-l)\right)\right.$$

$$\left.\left(B^{-1}(-jp, -jp-l)\int_{(-j-1)p}^{-jp} f(s)ds\right)\right] + \int_{g(t)}^{t} f(s)ds$$

PROOF : We use the notation given in (5)

In each interval of the type I_n, (4) becomes

$$\dot{x}(t) + B(t)\,x(np) = f(t)$$

which has a unique solution whenever a preassigned value for $x(np)$ is given. The solution of (4) with $x(np) = x_n$ is

$$x(t) = B(np, t)\,x_n + \int_{np}^{t} f(s)ds \text{ for } t \in I_n \qquad \dots (7)$$

and with $x((n+1)p) = x_{n+1}$ is

$$x(t) = B((n+1)p, t)\,x_{n+1} + \int_{(n+1)p}^{t} f(s)\,ds \text{ for } t \in I_{n+1}.$$

Continuity of the solution at $t = (n+1)p - l$ requires

$$B(np, (n+1)p-l)\,x_n + \int_{np}^{(n+1)p-l} f(s)ds$$

$$= B((n+1)p, (n+1)p-l)\,x_{n+1} + \int_{(n+1)p}^{(n+1)p-l} f(s)\,ds$$

so that

$$x_{n+1} = B^{-1}((n+1)p, (n+1)p - l)\, B(np, (n+1)p - l)\, x_n$$

$$+ B^{-1}((n+1)p, (n+1)p - l) \int_{np}^{(n+1)p} f(s)\, ds,$$

from which it follows that

$$x_n = \prod_{j=n}^{1} B^{-1}(jp, jp - l)\, B((j-1)p, jp - l) \left[x_0 + \sum_{j=1}^{n} \right.$$

$$\left(\prod_{i=1}^{j} B^{-1}((i-1)p, ip - l)\, B(ip, ip - l)) B^{-1}(jp, jp - l) \int_{(j-1)p}^{jp} f(s)\, ds \right]$$

Substituting the above formula into (7) yields (6). The continuation of (6) on $(-\infty, 0]$ is obtained in a similar way. The proof is complete.

Theorem 2 — *Let $B(t)$ be locally integrale on $[0, \infty)$. Assume that $|B(t)| < B_1$ $(B_1 > 0)$ for $t \in [0, \infty)$ and*

$$|B^{-1}(np, np - l)\, B((n-1)p, np - l)| < \alpha < 1 \text{ for } n \in \{1, 2, \ldots\}.$$

(a) If $f(t) \equiv 0$ then the trivial solution of (4) is globally asymptotically stable.

(b) If $\lim_{t \to \infty} f(t) = 0$ and $lB_1 < 1$, then every solution of (4) tends to zero as $t \to \infty$.

PROOF : (*a*) Note that $t \in I_n$

$$\left| B(g(t), t) \left(\prod_{j=g(t)/p}^{1} B^{-1}(jp, jp - l)\, B((j-1)p, jp - l)\, x_0 \right. \right|$$

$$< B_2\, \alpha^n\, |x_0|$$

where $B_2 = 1 + B_1 \max\{l, p - l\}$. Therefore (a) is proved.

(*b*) We observe that the remaining terms in (6) tends to zero as $t \to \infty$. For $t \in I_n$

$$\left| \int_{g(t)}^{t} f(s)\, ds \right| < \max\{l - p - l\} \max\{|f(t)| : t \in I_n\}.$$

Similarly, $F_j = \int_{(j-1)p}^{ip} f(s) ds \to 0$ as $j \to \infty$. Hence, given $\varepsilon > 0$, choose P_1 such that $|F_j| < K$

if $j < P_1$ and $|F_i| < \frac{\varepsilon B_s(1-\alpha)}{2B_2}$ for $j \geq P_1$, choose P_2 so that if $n > P_2$ then $\alpha^n < \frac{\varepsilon B_3}{2KB_2P_1}$, where $B_3 = \frac{1}{1 - l B_1}$. If $n > \max\{P_1, P_2\}$, then

$$\left| \prod_{j=g(t)/p}^{1} B^{-1}(jp, jp-l)\,(B((j-1)p, jp-l)\right.$$

$$\left.\left[\sum_{j=1}^{g(t)/p}\left(\prod_{i=1}^{j} B^{-1}((i-1)p, ip-l)\,B(ip, ip-l)\right)\left(B^{-1}(jp, jp-l)\int_{(j-1)p}^{jp} f(s)\,ds\right)\right]\right|$$

$$\leq \sum_{j=1}^{n}\left[\left(\prod_{i=n}^{j+1} |B^{-1}((i-1)p, ip-l)\,B(ip, ip-l)|\right)|B^{-1}(jp, jp-l)\,F_j|\right]$$

$$\leq \frac{1}{B_3}\sum_{j=1}^{P_1}\left[\left(\prod_{i=n}^{j+1}\alpha^i\right)|F_j|\right] + \frac{1}{b_3}\sum_{j=p_1+1}^{n}\left[\left(\prod_{i=n}^{j+1}\alpha^i\right)|F_j|\right] \leq \frac{\varepsilon}{B_2},$$

where we define $\prod_{i=n}^{n+1} B^{-1}(ip, ip-l)\,B((i-1)p, ip-l) = 1$. The proof is complete.

2.2 *Oscillation* — Our aim in section is to obtain sufficient conditions, in the sense of Definition 2, of all solutions of the linear non-autonomous deference system

$$x^i_{n+1} - x^i_n + \sum_{j=1}^{m} b_{ij}(n)\,x^j_{n-k} = 0 \quad (i = 1, 2, \ldots, m) \qquad \ldots (8)$$

where

$$b_{ij}(n) \in R \text{ for } i, j = 1, 2, \ldots, m \text{ and } n, k \in \{0, 1, 2, \ldots\} \qquad \ldots (9)$$

we also assume that the coefficient $b_{ij}(n)$ of (8) satisfy the following hypothesis.

(H) There exist a (componentwise) positive vector $u = [u_1, \ldots, u_m]^T$ and a function $p(n) \in R^+$ such that for all $j = 1, 2, \ldots, m$,

$$-u_j b_{ij}(n) + \sum_{i=1, i\neq j}^{m} u_i |b_{ij}(n)| \leq -p(n)\,u_j. \qquad \ldots (10)$$

Theorem A in this section is the following theorem which compares the oscillation of all solutions of the system (8) to the oscillation of all solutions of the scalar equation

$$y_{n+1} - y_n + (n)\,y_{n-k} = 0. \qquad \ldots (11)$$

At the end of the section, we use Theorem A to obtain an interesting sufficient condition for the oscillation of all solutions of the linear non-autonomous deference system (8), which parallels the oscillation theory of delay differential system (in [4]).

The following lemma A is extracted from [11]

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)\, x_{n-k_i} = 0 \qquad \text{... (12)}$$

and
$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n)\, x_{n-k_i} \le 0, \qquad \text{... (13)}$$

where for each $i = 1, 2, ..., m$

and
$$p_i(n) \in R^{+}, n, k \in \{0, 1, 2, ...\}. \qquad \text{... (14)}$$

Lemma A — Assume that (14) hold. Then the following statements are equivalent :

(*a*) Eq. (12) is oscillatory.

(*b*) Inequality (13) has no eventually positive solution.

Theorem A — *Suppose that (9) and hypothesis H are satisfied and assume that every solution of the scalar eq. (11) oscillates. Then every solution of (8) oscillates.*

PROOF : Assume, for the sake of contradiction, that (8) has a non-oscillatory solution $x_n = \left[\overset{1}{x}_n, \ldots, \overset{m}{x}_n \right]^m$. Then there exists a N such that for $i = 1, 2, ..., m$ and for $n > N$,

$$\varepsilon_i = sgn\, \overset{i}{x}_n,\ \ \varepsilon_i \overset{i}{x}_n = |\overset{i}{x}_n|$$

and
$$\sum_{i=1}^{m} |\overset{i}{x}_n| > 0$$

Set
$$z_n = \sum_{i=1}^{m} u_i\, \varepsilon_i\, \overset{i}{x}_n,\ n \ge N$$

Then $z_n > 0$ and for n sufficiently large,

$$z_{n+1} - z_n = \sum_{i=1}^{m} u_i\, \varepsilon_i \left(\overset{i}{x}_{n+1} - \overset{i}{x}_n \right) = - \sum_{i=1}^{m} u_i\, \varepsilon_i \sum_{j=1}^{m} b_{ij}(n)\, \overset{j}{x}_{n-k}$$

$$\le \sum_{j=1}^{m} \left[- u_j\, b_{jj}(n)\, \overset{j}{x}_{n-k}\, sgn\, \overset{j}{x}_{n-k} + \sum_{i=1, i \ne j}^{m} u_i\, |b_{ij}(n)|\, |\overset{j}{x}_{n-k}| \right]$$

$$\le \sum_{j=1}^{m} (-p(n)\, u_j \,|\, x^j_{n-k} \,|) = -p(n)\, z_{n-k}.$$

That is, the deference inequality

$$z_{n+1} - z_n + p(n)\, z_{n-k} \le 0 \qquad \text{... (15)}$$

has an eventually positive solution. Hence, by Lemma A, (11) is not oscillatory.

This contradicts hypothesis and the proof is complete.

We now apply Theorem A to obtain the following Theorem 3.

***Theorem* 3** — *Let $B(t)$ be locally integrable on $[0, \infty)$. Assume there exists a positive vector $u = [u_1, u_2, \ldots, u_m]^T$ and a function $p(n) \in R^+$ such that for all $j = 1, 2, \ldots, m$,*

$$-u_j\, b_{jj}(n) + \sum_{i=1,\, i \ne j}^{m} u_i \,|\, b_{ij}(n) \,| \le -p(n)\, u_j, \qquad \text{... (16)}$$

and

$$1 - p(n) \text{ is not eventually positive,} \qquad \text{... (17)}$$

where $(b_{ij}(n)) = I - B(np, (n+1)p - l)\, B^{-1}(np, np - l)$ $(i, j = 1, 2, \ldots, m)$.

Then every solution of equation

$$\dot{x}(t) + B(t)\, x(g(t)) = 0,\; x(0) = x_0 \qquad \text{... (18)}$$

oscillates.

PROOF : Let $x(t)$ be a solution of (18). The continuity of $x(t)$ at $t = (n+1)p - l$ in (7) gives

$$x((n+1)p - l) = B(np, (n+1)p - l)\, x_n$$

Again using (7) with $t = np - l$ we have that

$$x(np - l) = B(np, np - l)\, x_n$$

From the two above forms we obtain that

$$x((n+1)p - l) = B(np, (n+1)p - l)\, B^{-1}(np, np - l)\, x(np - l).$$

That is $x((n+1)p - l) - x(np - l) + [I - B(np, (n+1)p - l)\, B^{-1}(np, np - l)]\; x(np - l) = 0$

It is easy to see that $x(t)$ oscillates if $\{x(np - l)\}$ oscillates.

By (16) and (17), the conditions of Theorem A are satisfied. So the sequence $\{x(np - l)\}$ oscillates. The proof is complete.

The folllwing example is to illustrate Theorem 1 and Theorem 3.

Example — Consider equation

$$\dot{x}(t) + B(t)\, x\left(\left[\frac{t+1}{2}\right]\right) = 0,\ t > 0 \qquad \text{... (19)}$$

with

$$B(t) = \begin{pmatrix} -5 & 3 \\ 0 & -1-4t \end{pmatrix},\ x_0 = \begin{pmatrix} h_0 \\ v_0 \end{pmatrix}.$$

Theorem 1 asserts that (19) subject $x(0)$ has a unique solution on $[0, \infty)$. The solution of (19) is given by

$$x(t) = \begin{pmatrix} 1-10n+5t & 6n-3t \\ 0 & 1-2n-8n^2+t+2t^2 \end{pmatrix} \prod_{j=1}^{n} \begin{pmatrix} -\frac{3}{2} & \frac{3}{4(4j-1)} \\ 0 & \frac{1}{4j-1}-1 \end{pmatrix} x_0$$

for $t \in [2n-1, 2n+1)$.

From Theorem 3 all solutions of (19) are oscillatory for let

$$u = (1, 1)^T, p(n) = \frac{3}{2}$$

with

$$I - B(2n, 2n+1)\, B^{-1}(2n, 2n-1) = \begin{pmatrix} \frac{5}{2} & \frac{3}{4(4n-1)} \\ 0 & 2+\frac{3}{4n-1} \end{pmatrix}$$

3. The Case — $A(t) \neq 0$

To simplity the notation, define

$$B(a, b) = I - \int_a^b B(s)\, e^{\int_a^b A(u)\,du}\, ds,$$

$$\int_{-p}^{0} A(s)\, ds = 0,\ x(np) = x_n \text{ and} \qquad \text{... (20)}$$

$$I_n = [np - l, (n+1)\,p - l) \text{ for } n = 1, 2, \ldots .$$

We state some theorems for (1). The proofs of Theorem 4, 5 and 6 can be obtained by the techniques used in the proofs of Theorem 1, 2 and 3 of the previous section.

***Theorem* 4** — *Let A(t), B (t) and f(t) be locally integrable on* $[0, \infty)$. *Then (1)-(2) has a unique solution on* $[0, \infty)$ *given by*

$$x(t) = e^{-\int_{g(t)}^{1} A(s)\,ds} B(g(t), t) \left(\prod_{j=g(t)/p}^{1} B^{-1}(jp, jp-l)\, e^{-\int_{(j-1)p}^{jp} A(s)\,ds} B((j-1)p, jp-1) \right)$$

$$\left[x_0 - \sum_{j=1}^{g(t)/p} \left(\prod_{i=1}^{j} B^{-1}((i-1)p, ip-l \right) e^{\int_{(i-1)p}^{p} A(s)ds} B(ip, ip-l)) \right.$$

$$\left. (B^{-1}(jp, jp-l)\, F(jp, jp-l)) \right]$$

$$+\ e^{-\int_{g(s)}^{t} A(s)ds} F(g(t), t)$$

where $B(a, b)$ and $F(a, b)$ are defined in (20).

In addition, if $a(t)$, $b(t)$ and $f(t)$ are integrable on $(-\infty, 0]$, This solution can be continued backwards on $(-\infty, 0]$ and is given by

$$x(t) = e^{-\int_{g(s)}^{t} A(s)\,ds} B(g(t), t) \left(\prod_{j=-g(t)/p}^{1} B^{-1}(-jp, -jp-l)\, e^{-\int_{(-j-1)p}^{-jp} A(s)ds} \right.$$

$$\left. B((-j-1)p, -jp-l)) \left[x_0 - \sum_{j=1}^{-g(t)/p} \left(\prod_{i=1}^{j} B^{-1}((-i-1)p, -ip-l)\, e^{\int_{(-i-1)p}^{-ip} A(s)ds} B(-ip, -ip-l) \right) \right. \right.$$

$$(B^{-1}(-jp, -jp-l)\, F(-jp, -jp-l))] + e^{-\int_{g(t)}^{t} A(s)\,ds} F(g(t), t)$$

***Theorem* 5** — *Let* $A(t)$ *and* $B(t)$ *be locally integrable on* $[0, \infty)$. *Assume that* $|A(t)| < A$, $|B(t)| < B_1$ $(A, B_1 > 0)$ *for* $t \in [0, \infty)$ *and*

$$\left| B^{-1}(np, np-l)\, e^{-\int_{(n-1)p}^{np} A(s)ds} B((n-1)p, np-l) \right| < \alpha < 1 \text{ for } n \in \{1, 2, \ldots\}.$$

(a) If $f(t) \equiv 0$ then the trivial solution of (1) is globally asymptotically stable.

(b) If $\lim_{t \to \infty} f(t) = 0$ then every solution of (1) tends to zero as $t \to \infty$.

***Theorem* 6** — *Let A(t) and B(t) be locally integrale on* $[0, \infty)$. *Assume there exists a positive vector* $u = [u_1, u_2, \ldots, u_m]^T$ *and a function* $p(n) \in R^+$ *such that for all j = 1, 2, ..., m,*

$$-u_j b_{jj}(n) + \sum_{i=1, i \neq j}^{m} u_i \mid b_{ij}(n) \mid \leq -p(n) u_j,$$

and $\quad 1 - (n)$ is not eventually positive,

where $$b_{ij}(n) = I - e^{-\int_{np}^{(n+1)p} A(s)\, ds} B(np, (n+1)p - l) B^{-1}(np, np - l) e^{\int_{np}^{np-l} A(s) ds}$$

(i, j = 1, 2, ..., m).

Then every solution of equation

$$\dot{x}(t) + A(t)x(t) + B(t)x(g(t)) = 0, x(0) = x_0$$

oscillates.

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