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EFFECTS OF STRATIFICATION, SURFACE TENSION AND RIGID PLANES ON RAYLEIGH-TAYLOR INSTABILITY

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The stability of a slab of incompressible fluid with exponentially-increasing density supported by a fluid of constant density in the presence of surface tension and rigid planes is discussed. A general dispersion relation is obtained and some special cases are recovered. It is found that surface tension and rigid planes have a stabilizing effect while stratification has a destabilizing effect.

Key Words : Stratification; Surface Tension; Rigid Planes; Rayleigh Taylor Instability

1. INTRODUCTION

The problem of Rayleigh-Taylor type of instability, which occurs when a dense fluid is supported against gravity by a fluid with less density, or from the equilibrium of an incompressible heavy fluid of variable density, has recently assumed importance because of its applications to such diverse problems such as controlled thermonuclear experiments, magnetohydrodynamic power generation, and rapid acceleration of a plasma by a magnetic field.

Lord Rayleigh[8] was the first who investigated the character of the equilibrium of an incompressible heavy fluid of variable density. He was followed by Geoffrey Taylor[7] who showed that, when two superposed fluid of different densities are accelerated in a direction perpendicular to their interface, this surface is stable or unstable according to whether the acceleration is directed from the heavier to the lighter fluid or vice versa.

Chandrasekhar generalized Taylor's problem by including surface tension. A1 Ansary[3] discussed the linear and nonlinear Rayleigh-Taylor Stability for a slab of incompressible fluid with exponentially-increasing density, supported by a semi-infinite homogeneous region and supporting a semi-finite region of exponentially-decreasing density using the nonlinear therapy developed by Callebaut. In the present paper we shall study the linear Rayleigh-Taylor Stability problems for a simple model of physical interest region with exponentially varying density supported by a region of constant density. The two fluids are confined between two rigid planes. The surface tension has been taken into account at the interface between the two fluids. The two fluids are assumed incompressible and inviscid and subjected to a constant downward gravitational field. The physial situation is represented approximately by a water layer superposed on an oil layer in a flat horizontal cavity in the crust of the earth. This paper is divided as follows. In Section (2), the basic equations and equilibrium are formulated. In Section (3), the perturbation analysis is followed. In Section (4), the dispersion relation is obtained. In Section (5), special cases are discussed. In the last section a summary and conclusions are given.

2. Basic Equations and Equilibrium

Under the circumstances mentioned in the introduction the relevant basic equations are the equation of state (the incompressibility requirement), the equation of motion of the interface betwee two fluids and the dynamic boundary condition (namely the pressure across the interface between two fluids is balanced by the surface tension) these equations take, respectively, the following forms :

$$\rho\,[\partial_t \bar{u} + (\bar{u}\cdot\bar{\nabla})\,\bar{u}] = -\bar{\nabla} u + \rho\,\bar{g}, \qquad \dots (1)$$

$$(\partial_t + \bar{u}\cdot\bar{\nabla})\,\rho = 0, \qquad \dots (2)$$

$$\bar{\nabla}\cdot\bar{u} = 0 \qquad \dots (3)$$

and

$$\langle P_s \rangle = -T\,(R_1^{-1} + R_2^{-1}) = -T\,\mathrm{div}\,\bar{\boldsymbol{n}}, \qquad \dots (4)$$

where $\rho, \bar{u}$ and P are the fluid density, velocity and pressure, respectively. We use a rectangular cartesian system of coordinates of Fig. 1 in which the z-axis chosen vertically one has $\bar{g} = (0, 0, -g)$.

Eq. (4) follows from fact that the time derivatives of the coordinates of points in the fluid surface must equal the velocities of these points. The displacement z is a function of x, y and t, the brackets $\langle\,\rangle$ represent the jump across the surface (which is here the interface). The unit vector $\bar{\boldsymbol{n}}$ is the outward normal to the surface (defined by $f = (Z - z\,(x, y, t, z_0)$ whose principal radii of curvature are R_1 and R_2).

In the unperturbed state we consider the fluid at rest, i.e., $u_0 = 0$. In this state the interface separating the two fluids at $z_0 = 0$ is a horizontal plane the suffix zero will be used to characterize the equilibrium quantities. We suppose that the fluids is confined between two rigid planes at $z = -h_1$ and $z = h_2$ of Fig. (1) which are called regions I and II, one has

$$\textit{Region I} \;—\; (-h_1 \le z \le 0) : \rho_0\,(z) = \rho_1 \qquad \dots (5)$$

and

$$\textit{Region II} \;—\; (0 < z \le h_2) : \rho_0\,(z) = \rho_2\, e^{\beta z}, \qquad \dots (6)$$

where ρ_1 and ρ_2 are independent constants, β represents the inverse scale height in its corresponding region.

The hydrostatic equilibrium then reads :

$$\frac{d}{dz} P_0\,(z) = -\rho_0\,(z)\,g. \qquad \dots (7)$$

This means that the pressure at any level is determined by the total weight of the fluid above this level (and the weight of the wall, present, on top).

Integration of eq. (7) in each region yields :

Region I $(-h_1 \le z \le 0)$

$$P_0\,(z) = \int_0^z -\rho_1\, g z + \int_{h_2}^0 -\rho_2\, g\, e^{\beta z}\, dz$$

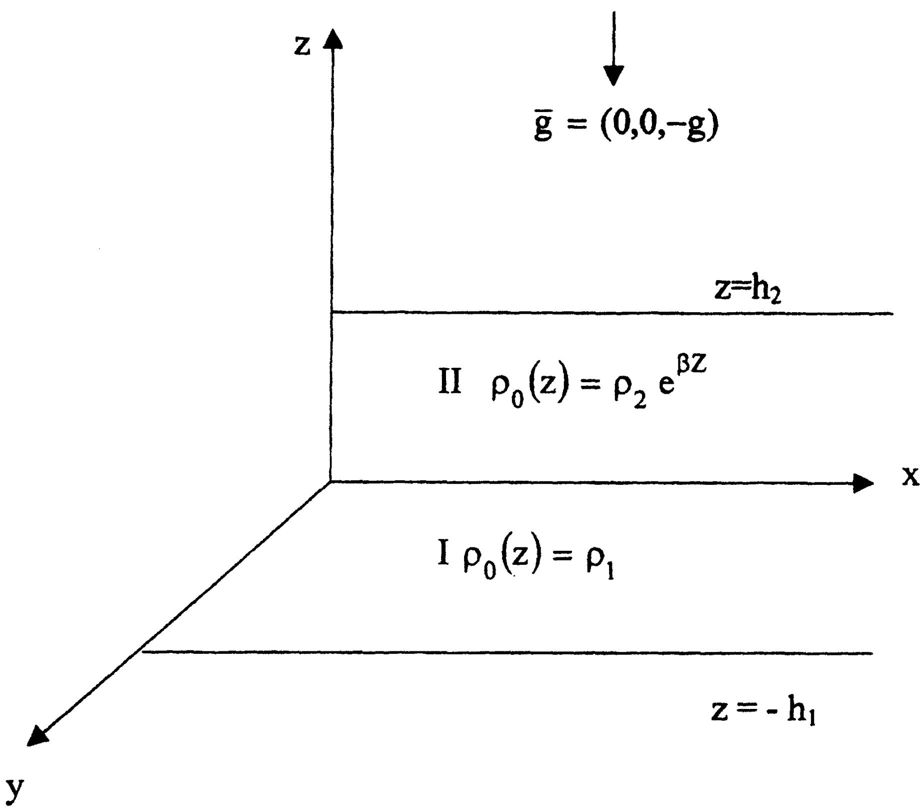

FIG. 1. Equilibrium and coordinate system.

$$= -\rho_1 gz - \frac{\rho_2 g}{\beta}(1 - e^{\beta h_2}) \qquad \dots (8)$$

Region II $(0 < Z \leq h_2)$

$$P_0(z) = \int_{h_2}^{z} -\rho_2 g e^{\beta z} dz = \frac{-\rho_2 g}{\beta}(e^{\beta z} - e^{\beta h_2}) \qquad \dots (9)$$

3. PERTURBATION ANALYSIS

The physical quantities (Say X) can be developed in a series ε as[3]

$$X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots, \qquad \dots (10)$$

where ε is the amplitude of the first-order term at all tiems, i.e.,

$$\varepsilon = \varepsilon_0 \exp(\sigma t), \qquad \dots (11)$$

ε_0 being its amplitude at $t = 0$, the growth rate is σ (for oscillations one should replace it by $i\omega$). It follows that the system of basic equations is convicted into a hierarchy of systems of equations which are independent of time. The zeroth-order system is of course the equilibrium. The first-order system is simply the customary linearized perturbation system, which by inserting the expressions (10) and (11) in eqs. (1)-(3) takes the form :

$$\sigma \rho_0 u_{1x} = -\partial_x P_1, \quad \text{... (12)}$$

$$\sigma \rho_0 u_{1y} = -\partial_y P_1, \quad \text{... (13)}$$

$$\sigma \rho_0 u_{1z} = -\partial_z P_1 - g\rho_1, \quad \text{... (14)}$$

$$\nabla \cdot u_1 = 0, \quad \text{... (15)}$$

$$\sigma \rho_1 = -u_{1z} \partial_z \rho_0, \quad \text{... (16)}$$

$$\sigma z_1 = u_{1z} |_{z_0}, \quad \text{... (17)}$$

$$\langle P_{1s} \rangle = T\left(\partial_x^2 + \partial_y^2 \right) \quad \text{... (18)}$$

and

$$\partial_z u_{1z} = -\partial_x u_{1x} - \partial_y u_{1y}. \quad \text{... (19)}$$

Eliminating u_{1x} and u_{1y} among eqs. (12), (13) and (19) we obtain

$$\rho_0 (\sigma^2 0\, D u_{1z} = -k^2 \sigma P_1, \quad \text{... (20)}$$

where

$$k^2 = k_x^2 + k_y^2. \quad \text{... (21)}$$

In what follows k will called the wave number of the perturbation, and a dash denotes, the derivative with respect z. If we eliminate P_1 and ρ_1 between eqs. (20), (14) and (16) yield.

$$\rho_0 D^2 u_{1z} + (D\rho_0)(Du_{1z}) - k^2 \left(\rho_0 - \frac{gD\rho_0}{\sigma^2} \right) u_{1z} = 0. \quad \text{... (22)}$$

This is an ordinary linear differential equation of second order in one function only u_{1z}.

For the other perturbed quantities we obtain

$$P_1 = \frac{-\rho_0}{k^2} \sigma D u_{1z}, \quad \text{... (23)}$$

$$\sigma \rho_1 = -D \rho_0 u_{1z}, \quad \text{... (24)}$$

$$u_{1x} = \frac{ik_x}{k^2} D u_{1z} \quad \text{... (25)}$$

and

$$u_{1y} = \frac{ik_y}{k^2} D u_{1z}. \quad \text{... (26)}$$

4. The Dispersion Relation

4.1. *Region (I)* — In this region the density constant and the eq. (22), becomes

$$(D^2 - k^2)\, u_{1z} = 0. \qquad \text{... (27)}$$

The solution of the differential eq. (27) is

$$u_{1z} = A_1\, e^{kz} + A_2\, e^{-kz}, \qquad \text{... (28)}$$

where A_1 is an arbitrary constant and A_2 is an integration constant, to be determined by the boundary conditions.

From eq. (23) we have in the region (I)

$$P_1 = \frac{-\rho_1\, \sigma}{k}\left(A_1\, e^{kz} - A_2\, e^{-kz}\right) \qquad \text{... (29)}$$

4.2. *Region (II)* — In the region (II) in which the density increases exponentially in the vertical direction eq. (22) becimes

$$D^2\, u_{1z} + \beta\, Du_{1z} - k^2\left(1 - \frac{g\beta}{\sigma^2}\right) u_{1z} = 0. \qquad \text{... (30)}$$

Its solution is

$$u_{1z} = A_3\, e^{b_1 z} + A_4\, e^{b_2}, \qquad \text{... (31)}$$

where A_3 and A_4 are two integration constants to be determined by the boundary conditions and where

$$b_{1,2} = \frac{1}{2}\left(-\beta \pm \sqrt{\beta^2 + 4k^2\left(\frac{9\beta}{\sigma_2}\right)}\right) \qquad \text{... (32)}$$

and

$$P_1 = \frac{-\rho_2\, \sigma\, e^{\beta z}}{k^2}\,(A_3\, b_1\, e^{b_1 z} + A_4\, b_2\, e^{b_2 z}) \qquad \text{... (33)}$$

4.3. *Boundary conditions*

(*i*) Continuity of the z-component of velocity perturbation on the surface $z = 0$.

(*ii*) The difference pressure perturbations P_1 on the surface $z = 0$, equal to the coefficient of surface tension.

(*iii*) The z-component of velocity vanishes $z = -h_1$ and $= h_2$.

The boundary condition (18) can be rewritten in the form

$$[P_1]_{II} - [P_1]_I = T\left(\partial_x^2 + \partial_y^2\right) z_1 \ \text{ at } z_0 = 0, \qquad \text{... (34)}$$

where $$[P_1] = (P_1 + z_z P_0)\,|_{z_0=0} \qquad \ldots (35)$$

These boundary conditions allow us to determine the integration constants, e.g., A_2, A_3 and A_4 in terms of A_1, moreover, we obtain the dispersion relation

$$\rho_1 (\sigma^2 + gk) - \rho_1 (gk - \sigma^2)\, e^{-2kh_1}$$

$$- \frac{(1 - e^{-2kh_1})}{[1 - e^{(b_1 - b_2) h_2}]} [m_1 - m_2 e^{(b_2 - b_1) h_2}] = - k^3 T (1 - e^{-2kh_1}), \qquad \ldots (36)$$

where $$m_1 = \frac{\rho_2 (b_1 \sigma^2 + gk^2)}{k^2} \qquad \ldots (37)$$

and $$m_2 = \frac{\rho_2 (b_2 \sigma^2 + gk^2)}{k^2}, \qquad \ldots (38)$$

where the arbitrary constant A_1 is chosen as σ to have ε as the actual amplitude of the deformed interface.

Relation (36) represent the dispersion relation for the configuration which is shown in Fig. (1). Now we shall discuss some particular cases.

5. Special Cases

5.1. *Case* $h_1 \to \infty, h_2 \to \infty$ and $\beta = 0$:

The dispersion relation (36) can be written in the form :

$$\sigma^2 (\rho_1 + \rho_2) = gk (\rho_2 - \rho_1) - k^3 T. \qquad \ldots (39)$$

This is the dispersion relation given by Chandrasekar[2].

In the absence of surface tension ($T = 0$) eq. (39) is reduced to

$$\sigma^2 (\rho_1 + \rho_2) = gk(\rho_2 - \rho_1). \qquad \ldots (40)$$

Eqs. (39) and (40) can be written in a dimensionless form as the following

$$X^2 = K\alpha - K^3 \tau, \qquad \ldots (41)$$

where $$X^2 = \frac{\sigma^2 h}{g},\ K = kh,\ \tau = \frac{T}{gh^2 (\rho_1 + \rho_2)},\ \alpha = \frac{(\rho_2 - \rho_1)}{(\rho_1 + \rho_2)} \qquad \ldots (42)$$

and $$X^2 = K\, \alpha, \qquad \ldots (43)$$

5.2. *The case* $h_1 = h_2 = h, \beta = 0$

The dispersion relation (36) can be written in the form

$$\sigma^2 = \frac{[gk(\rho_2 - \rho_1) - k^3 T] \tanh(kh)}{(\rho_1 + \rho_2)} \qquad \text{... (44)}$$

In the absence of surface tension ($T = 0$), eq. (44) is reduced to

$$\sigma^2 = \frac{gk(\rho_2 - \rho_1)}{(\rho_1 + \rho_2)} \tanh(kh). \qquad \text{... (45)}$$

According to eq. (44), if $\rho_2 < \rho_1$, the configuration is stable, while when $\rho_2 < \rho_1$ the configuration is unstable, in the range $0 < k < k_c$, where

$$k_c = [g(\rho_2 - \rho_1)/T]^{1/2}. \qquad \text{... (46)}$$

However, in the latter case, the arrangement is stable for all disturbances with $k > k_c$. Thus, surface tension succeeds in stabilizing a potentially unstable arrangement for all sufficiently short wavelenghts, but the arrangement remains unstable for all sufficiently long wavelengths. Eqs. (44) and (45) can be written in a dimensionless from as the following.

$$X^2 = \alpha K \tanh(K) - K^3 \tau \tanh(K) \qquad \text{... (47)}$$

and

$$X^2 = \alpha K \tanh(K). \qquad \text{... (48)}$$

5.3. *Case* $h_1 \to \infty, h_2 = h, \beta = 0$

The dispersion relation (36) can be written in the form

$$\sigma^2 = \frac{(\rho_1 + \rho_2) gk - k^3 T}{(2\rho_1 + \rho_2 \coth(kh))}. \qquad \text{... (49)}$$

According to eq. (49), the arrangement is stable for all disturbances with $k > k_c$, where

$$k_c = [g(\rho_1 + \rho_2)/T]^{1/2} \qquad \text{... (50)}$$

The configuration is unstable for all wave number in the range $0 < k < k_c$. In the absence of surface tension ($T = 0$), eq. (50) is reduced to

$$\sigma^2 = \frac{(\rho_1 + \rho_2) gk}{(2\rho_1 + \rho_2 \coth(kh))}. \qquad \text{... (51)}$$

Eqs. (50) and (51) can be written in a dimensionless form as the following :

$$X^2 = \frac{K - K^3 \tau}{\alpha_1 + \alpha_2 \coth(K)} \qquad \text{... (52)}$$

and

$$X^2 = \frac{K}{\alpha_1 + \alpha_2 \coth(K)}, \qquad \text{... (53)}$$

where

$$\alpha_1 = \frac{2\rho_1}{\rho_1 + \rho_2}, \ \alpha_2 = \frac{\rho_2}{\rho_1 + \rho_2}. \qquad \text{... (54)}$$

5.4. *Case* $h_1 = h, h_2 \to \infty, \beta = 0$:

Eq. (36) can be written in the form

$$\sigma^2 = \frac{-k\,[\rho_1 g + k^2 T] \tanh(kh)}{\rho_1 + 2\rho_2 \tanh(kh)}. \qquad \text{...(55)}$$

According to eq. (55), the arrangement is stable for all disturbances k.
In the absence of surface tension ($T = 0$) eq. (55) is reduced to

$$\sigma^2 = \frac{-k\,\rho_1 g \tanh(kh)}{\rho_1 + 2\rho_2 \tanh(kh)}. \qquad \text{... (56)}$$

Eqs. (55) and (56) can be written in a dimensionless form as the following

$$X^2 = \frac{-[K + K^3 \tau_1] \tanh(K)}{1 + \alpha_3 \tanh(K)} \qquad \text{... (57)}$$

and

$$X^2 = \frac{K}{1 + \alpha_3 \tanh(K)}, \qquad \text{... (58)}$$

where

$$\alpha_3 = \frac{2\rho_2}{\rho_1}, \ \tau_1 = \frac{T}{gh^2 \rho_1}. \qquad \text{... (59)}$$

5.5. *The Case* $h_1 \to \infty, h_2 \to \infty, \beta \neq 0$:

The dispersion relation (36) can be written in the form.

$$\sigma^2 = gk\left[(\rho_2 - \rho_1) - \frac{k^2 T}{g} \right] \Big/ \left(\rho_1 - \frac{b\rho_2}{k} \right) \qquad \text{... (60)}$$

where

$$b = -\frac{1}{2}\left[\beta + \sqrt{\beta^2 - 4k^2\left(1 - \frac{g\beta}{\sigma^2}\right)} \right] \qquad \text{... (61)}$$

The dispersion relation (60) can be written in the quadratic form

$$AX^2 - BX + C = 0. \qquad \text{... (62)}$$

The solution

$$X_{1,2} = [B \pm \sqrt{B^2 - 4AC}]/2A, \quad \text{... (63)}$$

where
$$A = \lambda^2 + \lambda/K_1 - 1, \quad \text{... (64)}$$

$$B = 2\lambda K_1 - (1 + 2\lambda K_1)(\lambda + \tau_2 K_1^2), \quad \text{... (65)}$$

$$C = K_1^2 (1 - \lambda - \tau_2 K_1^2), \quad \text{... (66)}$$

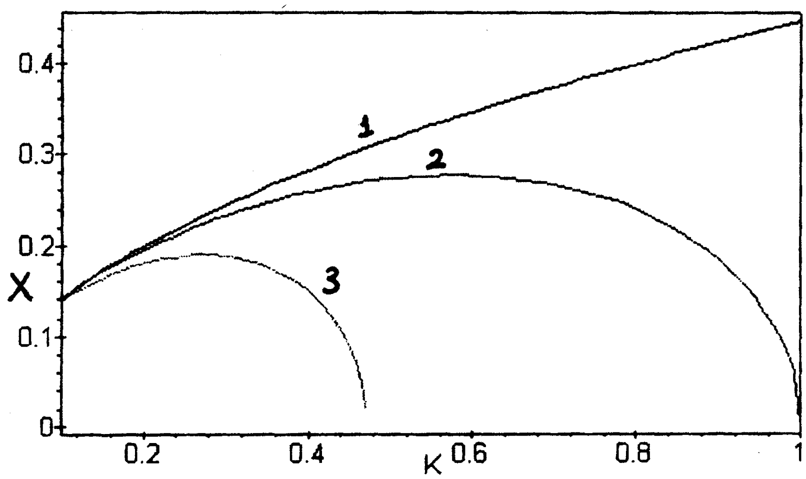

FIG. 2. The dependence of X (dimensionless growth rate) on K (dimensionless wave number) when $\alpha = 0.2$, $0.1 \leq K \leq 1$. (The curves labelled 1, 2, and 3 are for values of $\tau = 0$, 0.2 and 0.9).

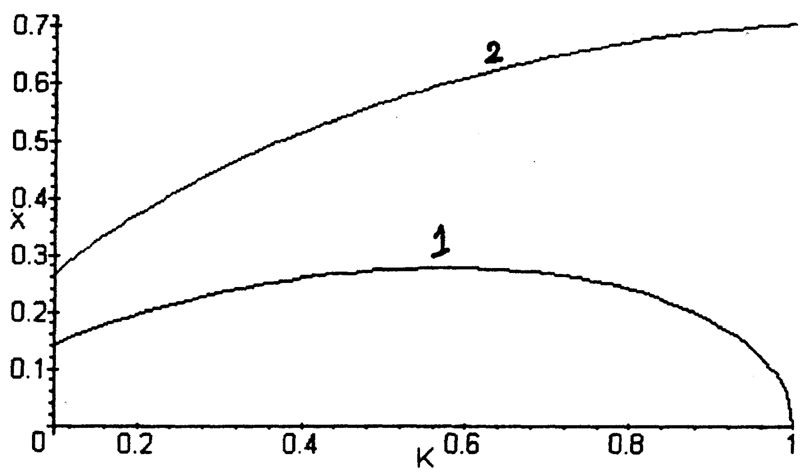

FIG. 3. The dependence of X (dimensionless growth rate) on K (dimensionless wave number) when $\tau = 0.2$, $0.1 \leq K \leq 1$. (The curves labelled 1 and 2 are for values of $\alpha = 0.2$ and 0.7 respectively).

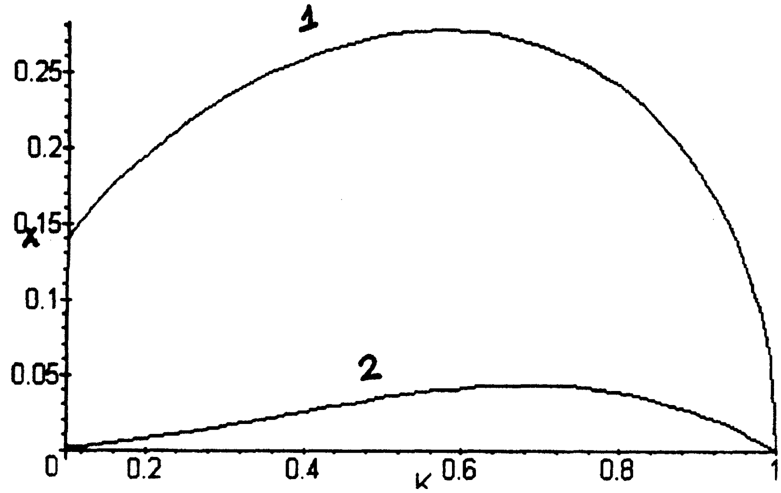

FIG. 4. The dependence of X (dimensionless growth rate) on K (dimensionless wave number) when τ = 0.2, α = 0.2 and 0.1 $\leq K \leq$ 1. (The curves labelled 1, 2 are for the absence and for the present of one rigid plane respectively).

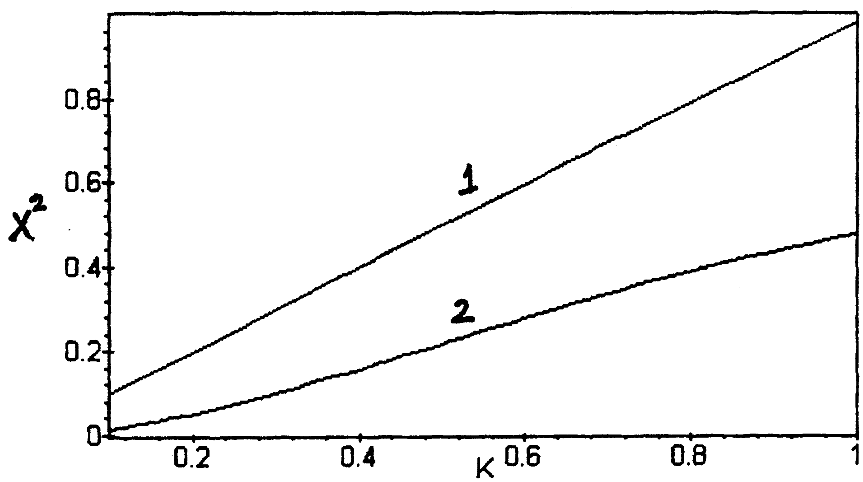

FIG. 5. The dependence of X^2 (the square of the dimensionless growth rate) on K (dimensionless wave number) when τ = 0.2, α = 0.2 and 0.1 $\leq K \leq$ 1. (The curves labelled 1, 2 are for the absence and for the presence of two rigid planes respectively).

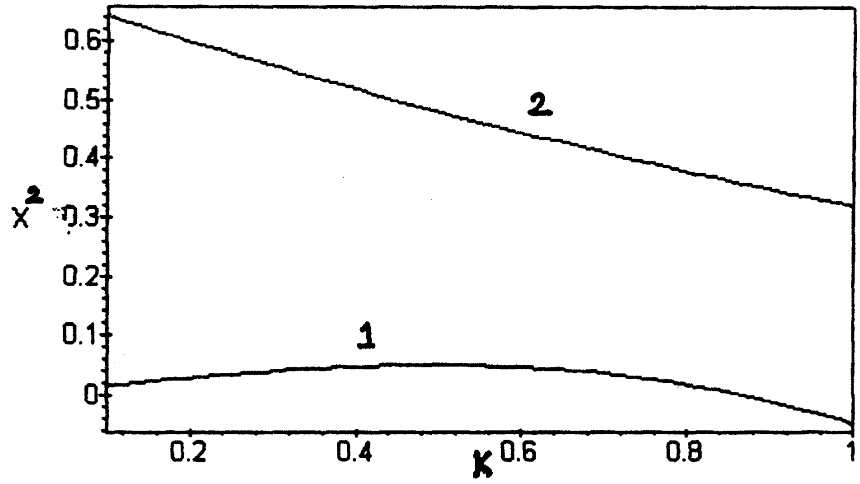

FIG. 6. The dependence of X^2 (the square of the dimensionless growth rate) on K when τ = 0.2, α = 0.15 and 0.1 $\leq K \leq$ 1. (The curves labelled 1, 2 are for the absence and for the presence of stratifications respectively).

$$X_1 = \sigma^2/g\beta, K_1 = k/\beta$$

and $$\lambda = \rho_1/\rho_2 \text{ and } \tau_2 = \beta^2 T/g\,\rho_2. \quad \text{... (67)}$$

SUMMARY AND CONCLUSIONS

We have presented analytical and numerical results for the problem of Rayleigh Taylor instability in the presence of surface tension, stratifications and two rigid planes. This problem is of special interest in oil industry[10] and also in Aluminium reduction cells[11]. It is found from figures 2 that the surface tension have a stabilizing effect. Also it is clear from Figs. 3 and 6 that the stratifications has a destabilizing effect. Finally, curves in Figs. 4 and 5 show that the presence of one rigid plane or two rigid planes has a stabilizing effect. All curves were plotted by using Maple *V*.

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