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CHARACTERIZATIONS OF THE 0-DISTRIBUTIVE LATTICE

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The 0-distributive lattice is characterized zero using semiideals, ideals and filters.

Key Words : Characterization; 0-Distributive Lattice; Semiideal; Semilattice; Ideal; Filter

1. INTRODUCTION AND PRELIMINARIES

The 0-distributive lattice and semilattice have been studied by Varlet[6], Pawar and Thakare[3 & 4] and Jayaram[2]. In this paper, we obtain some characterizations of the 0-distributive lattice. For the lattice theoretic concepts which have now become commonplace the reader is referred to Szasz[5] and Gratzer[1].

We recall tnat a lattice is a partially ordered set in which any two-element subset has a greatest lower bound (*in finimum*) denoted by $\wedge$ (*meet*) and a least upper bound (*supremum*) denoted by $\vee$ (*join*). Let L be a lattice. A semiideal of L is a nonempty subset A of L such that $a \in A, b \leq a(b \in L) \Rightarrow b \in A$. An ideal of L is a semiideal A of L such that $a, b \in A \Rightarrow a \vee b \in A$. If $a \in L$, $\{x \in L \mid x \leq a\}$ is an ideal. It is called the principal ideal generated by a and is denoted by (a). A filter of L is a nonempty subset F of L such that (*i*) $a \in F, b \geq a\,(b \in L) \Rightarrow b \in F$ and (*ii*) $a, b \in F \Rightarrow a \wedge b \in F$. If $a \in L$, $x \in \mid x \geq a\}$ is a filter. It is called the principal filter generated by a and is denoted by $[a)$. A maximal ideal (filter) of L is a proper ideal (filter) which is not contained in any other proper ideal (filter). A prime semiideal (ideal) is a proper semiideal (ideal). A such that $a \wedge b \in A \Rightarrow a \in A$ or $b \in A$. A minimal prime semiideal (ideal) is a prime semiideal (ideal) which does not contain any other prime semiideal (ideal). A prime filter is a proper filter B such that $a \vee b \in B \Rightarrow a \in B$ or $b \in B$.

Let A be nonempty subset of a lattice L with 0, $A^* = \{x \in L \mid a \wedge x = 0$ for all $a \in A\}$ and $A^0 = \{x \in L \mid a \wedge x = 0$ for some $a \in A$. Then A^* is called the annihilator of A and A^0 is called the pseudoannihilator of A. If $a \in L$, we write $(a)^*$ for $\{a\}^*$ and $(a)^0$ for $\{a\}^0$. We say that a is dense if $(a)^* = \{0\}$. If sup $(a)^* \in (a)^*$, it is called the pseudocomplement of a and is denoted by a^*. A pseudocomplemented lattice is a lattice with 0 in which every element has a pseudocomplement. A normal element of a pseudocomplemented lattice is an element a such that $a = a^{**}$.

The following lemma is easily proved.

Lemma 1.1 — The set $I(L)$ of all ideals of a lattice L forms a lattice under set inclusion as partial ordering relation. For $A, B \in I(L), A \vee B = \{x \mid x \leq a \vee b$ for some $a \in A$ and $b \in B\}$ and $A \wedge B = A \cap B = \{a \wedge b \mid a \in A$ and $b \in B\}$. If $\cap A_i$ is nonempty for some subset $\{A_i\}$ of $I(L)$, $\cap A_i = \wedge A_i$.

Dualizing 1.1, we have :

Lemma 1.2 — The set $F(L)$ of all filters of a lattice L forms a lattice under set inclusion as partial ordering relation. For $A, B \in F(L), A \vee B = \{y \mid y \geq a \wedge b$ for some $a \in A$ and $b \in B\}$ and $A \wedge B = A \cap B = \{a \vee b \mid a \in A$ and $b \in B\}$. If $\cap A_i$ is nonempty for some subset $\{A_i\}$ of $F(L)$, $\cap A_i = \wedge A_i$.

The following two lemmas are contained in Venkatanarasimhan[7].

Lemma 1.3 — If $\{A_i \mid i \in I\}$ is a family of ideals of a lattice, then

$$\vee A_i = \{x \mid x \leq a_{i_1} \vee \ldots \vee a_{i_n}, a_{i_1} \in A_{i_1}, \ldots, a_{i_n} \in A_{i_n}; i_1, \ldots, i_n \in I\}.$$

A similar result holds for filters.

Lemma 1.4 — Every proper filter of a lattice with 0 is contained in a maximal filter.

The following five lemmas are contained in Venkatanarasimhan[8].

Lemma 1.5 — Let A be a nonempty proper subset of a lattice L. Then A is a filter if and only if $L - A$ is a prime semiideal.

Lemma 1.6 — Let A be a nonempty subset of a lattice L. Then A is a maximal filter if and only if $L - A$ is a minimal prime semiideal.

Lemma 1.7 — Any prime semiideal of a lattice with 0 contains a minimal prime semiideal.

Lemma 1.8 — Let A be a nonempty subset of a lattice with 0. Then A^* is the intersection of all the minimal prime semiideals not containing A.

Lemma 1.9 — Any normal semiideal of a lattice with 0 is the intersection of all the minimal prime semiideals containing it.

The following lemma is contained in Pawar and Thakare[3].

Lemma 1.10 — Let A be a proper filter of a lattice L with 0. Then A is maximal if and only if for each x in $L - A$, there is some a in A such that $a \wedge x = 0$.

The following lemma is easily proved.

Lemma 1.11 — Let A be a nonempty subset of a lattice L with 0 and $x \in L$. Then A^* and A^o are semiideals of L and $[x]^* = [x)^o = (x)^o = (x)^*$.

Lemma 1.12 — Let A and B be filters of a lattice L with 0 such that A and B^o and disjoint from A.

PROOF : It is easily seen that $A \vee B$ is a proper filter of L. Hence by Lemma 1.4, $A \vee B \subseteq M$ for some maximal filter M. Now $B \subseteq M$ and consequently $M \cap B^o = \phi$. By Lemma 1.6, $L - M$ is a minimal prime semiideal. Clearly $B^o \subseteq L - M$ and $(L - M) \cap A = \phi$.

Lemma 1.13 — Let A be a filter of a lattice L with 0. Then A^0 is the intersection of all the minimal prime semiideals disjoint from A.

PROOF : Let N be any minimal prime semiideal disjoint from A. If $x \in A^0$, then $x \wedge a = 0$ for some $a \in A$ and so $x \in N$.

Let $y \in L - A^0$. Then $a \wedge y \neq 0$ for all $a \in A$. Hence, $A \vee [y) \neq L$. By Lemma 1.4, $A \vee [y) \subseteq M$ for some maximal filter M. By Lemma 1.6, $L - M$ is a minimal prime semiideal. Clearly $(L - M) \bigcap A = \phi$ and $y \notin L - M$.

Lemma 1.14[5] — Let $I(L)$ be the lattice of ideals of a lattice L. Then for any $A_1, \ldots, A_n \in I(L)$

$$A_1 \vee \ldots \vee A_n = \{x \in L \mid x \leq a_1 \vee \ldots \vee a_n, a_j \in A_j, j = 1, \ldots, n\}$$

and

$$A_1 \wedge \ldots \wedge A_n = A_1 \bigcap \cdots \bigcap A_n = \{a_1 \wedge \ldots \wedge a_n, a_j \in A_j, j = 1, .., n\}.$$

Dualizing 1.14 we have

Lemma 1.15 — Let $F(L)$ be the lattice of filters of a lattice L. Then for any $A_1, \ldots, A_n \in F(L)$

$$A_1 \vee \ldots \vee A_n = \{y \mid y \geq a_1 \wedge \ldots \wedge a_n, a_j \in A_j, j = 1, \ldots, n\}$$

and

$$A_1 \wedge \ldots \wedge A_n = A_1 \bigcap \cdots \bigcap A_n = \{a_1 \vee \ldots \vee a_n, a_j \in A_j, j = 1, \ldots, n\}.$$

Lemma 1.16 — Let A be a nonempty proper subset of a lattice L. Then A is a prime ideal if and only if $L - A$ is a prime filter.

PROOF : Suppose A is a prime ideal. By Lemma 1.5, $L - A$ is a filter. Let $x, y \in L - (L - A) = A$. Then $x \vee y \in A = L - (L - A)$. Thus $L - A$ is a prime filter. Suppose $L - A$ is a prime filter. By Lemma 1.5, $A = L - (L - A)$ is a prime semiideal. Let $x, y \in A = L - (L - A)$. Then $x \vee y \in L - (L - A) = A$ since $L - A$ is prime. Thus A is a prime ideal.

2. DEFINITION AND EXAMPLES

Definition 2.1 — A 0-distributive lattice is a lattice with 0 in which $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$.

Definition 2.2 — A 1-distributive lattice is a lattice with 1 in which $a \vee b = 1 = a \vee c$ implies $a \vee (b \wedge c) = 1$.

Definition 2.3 — A 0 & 1-distributive lattice is a lattice which is both 0-distributive and 1-distributive.

The following theorem is obvious.

Theorem 2.4 — *Any distributive lattice with 0(1) is 0-distributive (1-distributive). In particular every finite distributive lattice is 0 & 1-distributive.*

Theorem 2.5 — *Any pseudocomplemented lattice is 0-distributive.*

PROOF : Let L be a pseudocomplemented lattice and $a, b, c \in L$ such that $a \wedge b = 0 = a \wedge c$. Then $b \leq a^*$ and $c \leq a^*$. Hence, $b \vee c \leq a^*$. It follows that $a \wedge (b \vee c) = 0$. Thus L is 0-distributive.

Theorem 2.6 — *Let* $a, a_1, \ldots, a_n$ *be elements of a 0-distributive lattice L such that* $a \wedge a_1 = \ldots = a \wedge a_n = 0$. *Then* $a \wedge (a_1 \vee \ldots \vee a_n) = 0$.

PROOF : Since L is 0-distributive, $a \wedge (a_1 \vee a_2) = 0$. Assume $a \wedge (a_1 \vee \ldots \vee a_{k-1}) = 0$ for $2 < k \leq n$. Then $a \wedge (a_1 \vee \ldots \vee a_{k-1}) \vee a_k = a \wedge (b \vee a_k)$ where $b = a_1 \vee \ldots \vee a_{k-1}$. By our induction hypothesis $a \wedge b = 0$. Also $a \wedge a_k = 0$. Hence, $a \wedge (b \vee a_k) = 0$ as L is 0-distributive. That is $a \wedge (a_1 \vee \ldots \vee a_k) = 0$. Thus the theorem follows by induction.

Theorem 2.7 — *Any finite 0-distributive lattice is pseudocomplemented.*

PROOF : Let L be a finite 0-distributive lattice and $a \in L$. Since L is finite, $(a)^*$ is finite. Let $(a)^* = \{a_1, \ldots, a_n\}$ and $b = a_1 \vee \ldots \vee a_n$. By Theorem 2.6, $b \in (a)^*$. Hence, b is the pseudocomplement of a.

Remark 2.8 : Dualizing Theorems 2.5, 2.6 and 2.7 we get the corresponding theorems, for 1-distributive lattice.

We give below an example of a 0 & 1 - distributive lattice which is neither distributive nor pseudocomplemented.

Example 2.9 — Let $L = C_1 \cup C_2 \cup \{0, 1\}$ where C_1 and C_2 are two infinite chains without least and greatest elements. Define an ordering on L as follows. For all $c_1 \in C_1$ and $c_2 \in C_2$, $c_1 \parallel c_2$, $0 < c_1 < 1$ and $0 < c_2 < 1$. Clearly L is a 0 & 1 - distributive lattice. But L is neither distributive nor pseudocomplemented.

The following is an example of a bounded lattice which is neither 0-distributive nor 1-distributive.

Example 2.10 — Let $L = \{0, a, b, c, 1\}$. Define an ordering on L as follows. $0 \leq a, b, c \leq 1$, $a \parallel b$, $a \parallel c$ and $b \parallel c$. Clearly L is a lattice which is neither 0-distributive nor 1-distributive.

3. CHARACTERIZATIONS

Theorem 3.1 — *Let L be a lattice with 0. Then the following statements are equivalent :*

1. *L is 0-distributive.*
2. *If* $a, a_1, \ldots, a_n$ *are elements of L such that* $a \wedge a_1 = \ldots = a \wedge a_n = 0$, *then* $a \wedge (a_1 \vee \ldots \vee a_n) = 0$.
3. *If A is an ideal and* $\{A_i \mid i \in I\}$ *is a family of ideals of L such that* $A \cap A_i = (0]$ *for all i, then* $A \cap (\wedge_{i \in I} A_i) = (0]$.
4. *Every maximal filter of L is prime.*
5. *If M is a maximal filter of L,* $L - M$ *is a minimal prime ideal.*
6. *If M is a maximal filter of L,* $L - M$ *is an ideal.*
7. *Every minimal prime semiideal of L is a minimal prime ideal.*
8. *Every prime semiideal of L contains a prime ideal.*
9. *Every proper filter of L is disjoint from a minimal prime ideal.*
10. *Every proper filter of L is disjoint from a prime ideal.*
11. *Every proper filter of L is contained in a prime filter.*
12. *For each nonzero element a of L, there is a minimal prime ideal not containing a.*
13. *For each nonzero element a of L, there is a prime ideal not containing a.*
14. *Each nonzero element of L is contained in a prime filter.*

PROOF : (1) $\Rightarrow$ (2) — Follows by Theorem 2.6.

(2) $\Rightarrow$ (3) — Suppose (2) holds. Let $A \in I(L)$ and $\{A_i, i \in I\} \subseteq I(L)$ such that $A \cap A_i = (0]$ for all i. Let $x \in A \cap \{\vee_{i \in I} A_i)\}$. Then $x \in A$ and $x \in \vee_{i \in I} A_i$. Hence, $x \leq a_{i1} \vee \ldots \vee a_{in}$ for some $a_{ij} \in A_{ij}$ and $x \wedge a_{ij} = 0$ for all a_{ij}. By (2) $x \wedge (a_{i1} \vee \ldots \vee a_{in}) = 0$. That is $x = 0$. Thus $A \cap (\vee^{i \in I} A_i) = (0]$.

(3) $\Rightarrow$ (2) — It suffices to note that $(a] \cap (a_i] = (a \wedge a_i]$ and $(a] \cap \{a_1] \vee \ldots \vee (a_n]\} = (a \wedge (a_1 \vee \ldots \vee a_n)]$.

(2) $\Rightarrow$ (4) — Suppose (2) holds. Let M be any maximal filter of L and $x, y \in L - M$. Then by Lemma 1.10, $a \wedge x = 0 = b \wedge y$ for some $a, b \in M$. Let $c = a \wedge b$. Clearly $c \wedge x = 0 = c \wedge y$ and $c \in M$. By (2) $c \wedge (x \vee y) = 0$. Hence $x \vee y \in L - M$. Thus M is prime.

(4) $\Rightarrow$ (5) — Suppose (4) holds and let M be any maximal filter of L. By Lemma 1.6, $L - M$ is a minimal prime semiideal. Let $x, y \in L - M$. By (4) $x \vee y \in L - M$. Hence, $L - M$ is an ideal. It follows that $L - M$ is a minimal prime ideal.

(5) $\Rightarrow$ (6) — Obvious.

(6) $\Rightarrow$ (7) — Let N be any animal prime semiideal of L. By Lemma 1.6, $L - N$ is a maximal filter. By (6), $N = L - (L - N)$ is an ideal. Thus N is a minimal prime ideal.

(7) $\Rightarrow$ (8) — Suppose (7) holds and let A be any prime semiideal of L. By Lemma 1.7, $A \supseteq N$ for some minimal prime semiideal N. By (7), N is a (minimal) prime ideal.

(8) $\Rightarrow$ (10) — Suppose (8) holds and let A be any proper filter of L. By Lemma 1.5, $L - A$ is a prime semiideal. By (8) $L - A \supseteq B$ for some prime ideal B. Clearly, $A \cap B = \phi$.

(10) $\Rightarrow$ (11) — Suppose (10) holds and let A be any proper filter of L. By (10), $A \cap B = \phi$ for some prime ideal B. By Lemma 1.16, $L - B$ is a prime filter and clearly $A \subseteq L - B$.

(11) $\Rightarrow$ (14) — Obvious.

(5) $\Rightarrow$ (9) — Suppose (5) holds and let A be any proper filter of L. By Lemma 1.4, $A \subseteq M$ for some maximal filter M. By (5), $L - M$ is a minimal prime ideal and clearly $A \cap (L - M) = \phi$.

(9) $\Rightarrow$ (12) — Suppose (9) holds and let a be any nonzero element of L. By (9), $[a)$ is disjoint from a minimal prime ideal N. Thus $a \notin N$.

(12) $\Rightarrow$ (13) — Obvious.

(13) $\Rightarrow$ (14) — Suppose (13) holds and let a be any nonzero element of L. By (13), there is a prime ideal A such that $a \notin A$. By Lemma 1.16, $L - A$ is a prime filter and clearly $a \in L - A$.

(14) $\Rightarrow$ (1) — Suppose (14) holds. Let $a, b, c \in L$ such that $a \wedge (b \vee c) \neq 0$. By (14), $a \wedge (b \vee c) \in B$ for some prime filter B. Clearly $a, b \vee c \in B$ and so $a, b \in B$ or $a, c \in B$. Hence, $a \wedge b \in B$ or $a \wedge c \in B$. Consequently $a \wedge b \neq 0$ or $a \wedge c \neq 0$. It follows that L is 0-distributive.

***Theorem* 3.2** — *Let L be a lattice with 0. Then the following statements are equivalent :-*

1. *L is 0-distributive.*

2. *If A is a nonempty subset of L and B is a proper filter intersecting A, there is a minimal prime ideal containing A^* and disjoint from B.*

3. *If A is a nonempty subset of L and B is a proper filter intersecting A, there is a prime filter containing B and disjoint from A^*.*

4. *If A is a nonempty subset of L and B is a prime semiideal not containing A, there is a minimal prime ideal containing A^* and contained in B.*

5. *If A is a nonempty subset of L and B is a prime semiideal not containing A, there is a prime filter containing $L - B$ and disjoint from A^*.*

6. *For each nonzero element a of L and each proper filter B containing a, there is a prime ideal containing $(a)^*$ and disjoint from B.*

7. *For each nonzero element a of L and each proper filter B containing a, there is a prime filter containing B and disjoint from $(a)^*$.*

8. *For each nonzero element a of L and each prime semiideal B not containing a, there is a prime ideal containing $(a)^*$ and contained in B.*

9. *For each nonzero element a of L and each prime semideal B not containing a, there is a prime filter containing $L - B$ and disjoint from $(a)^*$.*

PROOF : (1) $\Rightarrow$ (2) — Suppose (1) holds. Let A be a nonempty subset of L and B be any proper filter such that $B \cap A \neq \phi$. By Lemma 1.5, $L - B$ is a prime semiideal and by Lemma 1.7, $L - B \supseteq N$ for some minimal prime semiideal N. Clearly, $N \cap B = \phi$. Also $L - B \not\supseteq A$ and so $N \not\supseteq A$. By Lemma 1.8, $N \supseteq A^*$. Since L is 0-distributive N is a minimal prime ideal [See Theorem 3.1 (7)].

By Lemma 1.16, it follows that (2) $\Rightarrow$(3), (8) $\Rightarrow$ (9) and (6) $\Rightarrow$ (7).

By Lemma 1.5, it follows that (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) and (7) $\Rightarrow$ (9).

By Lemma 1.11, it follows that (5) $\Rightarrow$ (9), (4) $\Rightarrow$ (8) and (2) $\Rightarrow$ (6).

(9) $\Rightarrow$ (1) — Suppose (9) holds and let a be any nonzero element of L. By Lemma 1.5, $L - [a)$ is a prime semiideal and $a \notin L - [a)$. By (9) prime filter B containing $[a)$. Thus $a \in B$. It follows that L is 0-distributive [See Theorem 3.1 (14)].

***Theorem* 3.3** — *Let L be lattice with 0. Then the following statements are equivalent* —

1. *L is 0-distributive.*

2. *If A and B are filters of L such that A and B^o are disjoint, there is a minimal prime ideal containing B^o and disjoint from A.*

3. *If A and B are filters of L such that A and B^o are disjoint, there is a prime filter containing A and disjoint from B^o.*

4. *If A is a filter of L and B is a prime semiideal containing A^o, there is a minimal prime ideal containing A^o and contained in B.*

5. *If A is a filter of L and B is a prime semiideal containing A^o, there is a prime filter containing $L - B$ and disjoint from A^o.*

6. *For each nonzero element a in L and each filter A disjoint from $(a)^*$, there is a prime ideal containing $(a)^*$ and disjoint from A.*

7. *For each nonzero element a in L and each filter A disjoint from $(a)^*$, there is a prime filter containing A and disjoint from $(a)^*$.*

8. *For each nonzero element a in L and each prime semiideal B containing $(a)^*$, there is a prime ideal containing $(a)^*$ and contained in B.*

9. *For each nonzero element a in L and each prime semiideal B containing $(a)^*$, there is a prime filter containing $L - B$ and disjoint from $(a)^*$.*

PROOF : (1) $\Rightarrow$ (2) — Suppose (1) holds. Let A and B be filters of L such that $A \cap B^0 = \phi$. By Lemma 1.12, there is a minimal prime semiideal N such that $N \supseteq B^0$ and $N \cap A = \phi$. Since L is 0-distributive it follows that N is a minimal prime ideal [See Theorem 3.1 (7)].

By Lemma 1.5, it follows that (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) and (7) $\Rightarrow$ (9).

By Lemma 1.11, it follows that (5) $\Rightarrow$ (9), (4) $\Rightarrow$ (8) and (2) $\Rightarrow$(6).

By Lemma 1.16, it follows that (2) $\Rightarrow$ (3), (8) $\Rightarrow$ (9) and (6) $\Rightarrow$ (7).

(9) $\Rightarrow$ (1) — Suppose (9) holds and let a be any nonzero element of L. By Lemma 1.5, $L - [a)$ is prime semiideal not containing a. Since $(a] \cap (a)^* = (0] \subset L - [a)$ it follows that $L - [a)$ contains $(a)^*$. By (9), there is a prime filter B containing $[a) = L - (L - [a))$ and disjoint from $(a)^*$. Clearly $a \in B$. It follows that L is a 0-distributive [See Theorem 3.1 (14)].

Theorem 3.4 — *Let L be a lattice with 0. Then the following statements are equivalent :*

1. *L is 0-distributive.*

2. *For any nonempty subset A of L, A^* is the intersection of all the minimal prime ideals not containing A.*

3. *For any filter A of L, A^o is the intersection of all the minimal prime ideals disjoint from A.*

4. *For each a in L, $(a)^*$ is an ideal.*

5. *Every normal semiideal of L is an intersection of minimal prime ideals.*

6. *For any ideal A of L and any family of ideals $\{A_i \mid i \in I\}$ of L,*

$$\left(A \cap \left(\bigvee_{i \in I} A^i\right)\right)^* = \bigcap_{i \in I} (A \cap A_i)^*.$$

7. *For any three ideals A, B, C of L,*

$$(A \cap (B \vee C))^* = (A \cap B)^* \cap (A \cap C)^*.$$

8. *For any finite number of filters $A, A_1, \ldots, A_n$ of L,*

$$(A \vee (A_1 \cap \cdots \cap A_n))^o = (A \vee A_1)^o \cap \cdots \cap (A \vee A_n)^o.$$

9. *For any three filters A, B, C of L,*

$$(A \vee (B \cap C))^o = (A \vee B)^o \cap (A \vee C)^o.$$

10. *For all a, b, c in L, $(a \wedge (b \vee c))^* = (a \wedge b)^* \cap (a \wedge c)^*$.*

PROOF : (1) $\Rightarrow$ (2) — Follows by Lemma 1.8 and Theorem 3.1 (7).

(1) $\Rightarrow$ (3) — Follows by Lemma 1.13 and Theorem 3.7 (7).

(3) $\Rightarrow$ (4) — By Lemma 1.11, $(a)^* = [a)^o$. Hence result.

(4) $\Rightarrow$ (1) — Suppose (4) holds. Let $a, b, c \in L$ such that $a \wedge b = 0 = a \wedge c$. Then $b, c \in (a)^*$. By (4), $b \vee c \in (a)^*$. It follows that $a \wedge (b \vee c) = 0$. Thus L is 0-distributive.

(2) $\Rightarrow$ (5) — Since a semiideal A of L is normal if and only if $A = B^*$ for some semiideal B it follows that (2) $\Rightarrow$ (5).

(5) $\Rightarrow$ (4) — By Lemma 1.11, $(a)^* = (a]^*$ for all $a \in L$. Hence the result.

(2) $\Rightarrow$ (6) — Suppose (2) holds. Let $A \in I(L)$ and $\{A_i \mid i \in I\} \subseteq I(L)$. If Q is any minimal prime ideal of L such that $Q \not\supseteq A \cap \left(\bigvee_{i \in I} A_i\right)$, then $Q \not\supseteq A \cap A_j$ for some $j \in I$. By (2) it follows that $\left(A \cap \left(\bigvee_{i \in I} A_i\right)\right)^* \supseteq (A \cap A_i)^*$. The reverse inclusion is obvious.

(6) $\Rightarrow$ (7) — Obvious.

(7) $\Rightarrow$ (10) — Follows by Lemma 1.11.

(10) $\Rightarrow$ (1) — Suppose (10) holds. Let $a, b, c \in L$ such that $a \wedge b = 0 = a \wedge c$. Then $(a \wedge b)^* = L = (a \wedge c)^*$. By (10), $(a \wedge (b \vee c))^* = L$. It follows that $a \wedge (b \vee c) = 0$. Thus L is 0-distributive.

(3) $\Rightarrow$ (8) — Suppose (3) holds and let $A, A_1, \ldots, A_n$ be filters of L. If Q is any minimal prime ideal of L such that $Q \cap (A \vee (A_1 \cap \cdots \cap A_n)) = \phi$, then $Q \cap A = \phi = Q \cap (A_1 \cap \cdots \cap A_n)$ and so $Q \cap (A \vee A_j) = \phi$ for some $j \in \{1, \ldots, n\}$. By (3) it follows that $(A \vee (A_1 \cap \cdots \cap A_{in}))^0 \supseteq (A \vee A_1)^0 \cap \cdots \cap (A \vee A_n)^0$. The reverse inclusion is obvious.

(8) $\Rightarrow$ (9) — Obvious.

(9) $\Rightarrow$ (10) — Suppose (9) holds and let $a, b, c \in L$. Then $([a) \vee ([b) \cap [c)))^0 = ([a) \vee [b))^0 \cap ([a) \vee [c))^0$. Now $([a) \vee ([b) \cap [c)))^0 = ([a) \vee [b \vee c))^0 = [a \wedge (b \vee c))^0 = (a \wedge (b \vee c))^*$ and $([a) \vee [b))^0 \cap ([a) \vee [x))^0 = [a \wedge b)^0 \cap [a \wedge c)^0 = (a \wedge b)^* \cap (a \wedge c)^*$ by Lemma 1.11.

Thus $(a \wedge (b \vee c))^* = (a \wedge b)^* \cap (a \wedge c)^*$.

Theorem 3.5 — *Let L be a lattice with 0. Then the following statements are equivalent :*

1. *L is 0-distributive.*
2. *For any finite number of ideals $A, A_1, \ldots, A_n$ of L,*

$$((A \vee A_1) \cap \cdots \cap (A \vee A_n))^* = A^* \cap (A_1 \cap \cdots \cap A_n)^*.$$

3. *For any three ideals A, B, C of L,*

$$((A \vee B) \cap (A \vee C))^* = A^* \cap (B \cap C)^*.$$

4. *For any filter A and any family of filters $\{A_i \mid i \in I\}$ of L,*

$$\left(\bigvee_{i \in I} (A \cap A_i)\right)^0 = A^0 \cap \left(\bigvee_{i \in I} A_i\right)^0.$$

5. *For any three filters A, B, C of L,*

$$((A \cap B) \vee (A \cap C))^0 = A^0 \cap (B \vee C)^0.$$

6. *For all a, b, c in L,*

$$((a \vee b) \wedge (a \vee c))^* = (a)^* \cap (b \wedge c)^*.$$

7. *For any family of ideals* $\{A_i, i \in I\}$ *of* L,

$$\left(\bigvee_{i \in I} A_i\right)^* = \bigcap_{i \in I} A_i^*.$$

8. *For any two ideals* A, B *of* L, $(A \vee B)^* = A^* \cap B^*$.

9. *For any finite number of filters* $A_1, \ldots, A_n$ *of* L, $(A_1 \cap \ldots \cap A_n)^0 = A_1^0 \cap \ldots \cap A_n^0$.

10. *For any two filters* A, B *of* L, $(A \cap B)^0 = A^0 \cap B^0$.

11. *For all* a, b *in* L, $(a \vee b)^* = (a)^* \cap (b)^*$.

12. $I(L)$ *is pseudocomplemented.*

13. $I(L)$ *is 0-distributive.*

PROOF : (1) $\Rightarrow$ (2) — Suppose (1) holds and let $A, A_1, \ldots, A_n$ be ideals of L. If Q is any minimal prime ideal of L such that $Q \not\supseteq A_1 \cap \ldots \cap A_n$. It follows that $((A \vee A_1) \cap \ldots \cap (A \vee A_n))^* \supseteq A^* \cap (A_1 \cap \ldots \cap A_n)^*$. The reverse inclusion is obvious.

(2) $\Rightarrow$ (3) — Obvious.

(3) $\Rightarrow$ (6) — Suppose (3) holds and let $a, b, s \in L$. Then $(((a] \vee (b]) \cap ((a] \vee (c]))^* = (a]^* \cap ((b] \cap (c])^*$. Now $(((a] \vee (b]) \cap ((a] \cap ((a] \vee (c])\,)^* = ((a \vee b] \cap (a \vee c])^* = (((a \vee b) \wedge (a \vee c)])^* = ((a \vee b) \wedge (a \vee c)\,)^*$ and $(a]^* \cap ((b] \cap (c])^* = (a)^* \cap (b \wedge c]^* = (a)^* \cap (b \wedge c)^*$ by Lemma 1.11. Thus $((a \vee b) \wedge (a \vee c))^* = (a)^* \cap (b \wedge c)^*$.

(6) $\Rightarrow$ (11) — Follows by taking $c = b$ in (6).

(11) $\Rightarrow$ (1) — Suppose (11) holds. Let $a, b, c \in L$ such that $a \wedge b = 0 = a \wedge c$. Then $a \in (b)^* \cap (c)^* = (b \vee c)^*$ by (11). It follows that $a \wedge (b \vee c) = 0$.

(1) $\Rightarrow$ (4) — Suppose (1) holds. Then for any filter A of L, A^0 is the intersection of all the minimal prime ideals disjoint from A [See Theorem 3.4 (3)]. Let $A \in F(L)$ and $\{A_i \mid i \in I\} \subseteq F(L)$. If Q is any minimal prime ideal of L such that $Q \cap \left(\bigvee_{i \in I} (A \cap A_i\right) = \phi$, then $Q \cap A = \phi$ or $Q \cap \left(\bigvee_{i \in I} A_i\right) = \phi$. It follows that $\left(\bigvee_{i \in I} (A \cap A_i)\right)^0 = A^0 \cap \left(\bigvee_{i \in I} A_i\right)^0$. The reverse inclusion is obvious.

(4) $\Rightarrow$ (5) — Obvious.

(5) $\Rightarrow$ (6) — Similar to (3) $\Rightarrow$ (6) with obvious modifications.

(1) $\Rightarrow$ (7) — Suppose (1) holds. Then for any ideal A and any family of ideals $\{A_i \mid i \in I\}$ of L, $\left(A \cap \left(\bigvee_{i \in I} A_i\right)\right)^* = \bigcap_{i \in I} (A \cap A_i))^*$ [See Theorem 3.4(6)]. Taking $A = \bigvee_{i \in I} A_i$, we get (7).

(7) $\Rightarrow$ (8) — Obvious.

(8) $\Rightarrow$ (11) — Suppose (8) holds and let $a, b \in L$. Then $((a] \vee (b])^* = (a]^* \cap (b]^*$. Now $((a] \vee (b])^* = (a \vee b]^* = (a \vee b)^*$ and $(a]^* \cap (b]^* = (a)^* \cap (b)^*$ by Lemma 1.11. Thus $(a \vee b)^* = (a)^* \cap (b)^*$.

(1) $\Rightarrow$ (9) — Suppose (1) holds. Then for any finite number of filter $A, A_1, \ldots, A_n$ of L, $(A \vee (A_1 \cap \cdots \cap A_n))^0 = (A \vee A_1)^0 \cap \cdots \cap (A \vee A_n)^0$ [See Theorem 3.4(8)]. Taking $A = A_1 \cap \cdots \cap A_n$, we get (9).

(9) $\Rightarrow$ (10) — Obvious.

(10) $\Rightarrow$ (11) — Suppose (10) holds and let $a, b \in L$. Then $([a) \cap [b))^0 = [a)^0 \cap [b)^0$. Now $([a) \cap [b))^0 = [a \vee b)^0 = (a \vee b)^*$ and $[a)^0 \cap [b)^0 = (a)^* \cap (b)^*$ by Lemma 1.11. Thus $(a \vee b)^* = (a)^* \cap (b)^*$.

(1) $\Rightarrow$ (12) — Suppose (1) holds. Let $A \in I(L)$. Then A^* is an ideal [See Theorem 3.4 (2)]. If $B \in I(L)$ such that $A \cap B = (0]$ and $x \in B$, then $a \wedge x = 0$ for all $a \in A$ and so $x \in A^*$. Thus $B \subseteq A^*$. It follows that A^* is the pseudocomplement of A.

(12) $\Rightarrow$ (13) — Follows by Theorem 2.5.

(13) $\Rightarrow$ (1) — Suppose (13) holds. Let $a, b, c \in L$ such that $a \wedge b = 0 = a \wedge c$. Then $(a] \cap (b] = (0] = (a] \cap (c]$ and hence by (13), $(a] \cap \{(b] \vee (c]\} = (0]$. That is $(a \wedge (b \vee c)] = (0]$. Consequently $a \wedge (b \vee c) = 0$.

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