Indian J. pure appl. Math., **32**(3) : 325-330, March 2001

BETWEEN SMPC-FUNCTIONS AND SUBMAXIMAL SPACES

R. A. MAHMOUD

Department of Mathematics, Faculty of Science, Menoufia University, Egypt

(*Received* 24 *December* 1998; *after revision* 15 *February* 2000; *accepted* 26 *June* 2000)

Submaximality is one of the important properties of spaces. SMPC is a function property on spaces. The prime aim in this paper is to establish some basic results on submaximal spaces. Also, new characterizations of the SMPC function depending on the concept of filters are obtained. Moreover, connections between SMPC and submaximal spaces are investigated.

Key Words and Phrases : Preopen Sets; Submaximal Spaces; Weakly-Precontinuous and SMPC Functions

1. INTRODUCTION

The original meaning of submaximal property on spaces was introduced by Bourbaki[3], in 1966. Since then, many topologists have been interested in this class of spaces and investigated several of its fundamental topological properties. Depending on the preopenness notion due to Mashhour *et al.*[11] in 1982, the concept of strongly M-precontinuous (SMPC) property of functions has been established by Abd El-Monsef *et al.*[1] and studied for a number of its properties. This paper is devoted to study both of submaximal spaces and SPMC functions. So, all useful definitions and preliminaries which will be used throughout the present work contained in the second section, nextly. Whenever, in the third one, we characterize SMPC using the filterbase concept. Furthermore, new properties and connections of this type of near continuous functions are presented. Finally, several characterizations and fundamental properties of submaximality have also been offered.

2. PRELIMINARIES

In this paper, given a topological space (X, τ) consisting of a non-empty set X with topology τ of subsets of X. All topological spaces here are without any separation properties, whenever such are needed it will be explicitly stated. For any subset A of X, its closure and interior with respect to τ will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Also, a space will always mean a topological space and "$\rightarrow$" for implies. $A \subseteq X$ is called preopen[11] if $A \subseteq \mathrm{Int}\ (\mathrm{Cl}(A))$. The collection of all preopen sets in (X, τ) will be denoted by $PO(X, \tau)$. For (X, τ), τ_p means the smallest topology on X containing $PO(X, \tau)$ due to Andrijevic[2]. Also, Reilly and Vamanamurthy[12] have proved that the class $\tau^{\alpha} = PO\ (X, \tau) \cap SO\ (X, \tau)$ is a topology on X, where $SO\ (X, \tau)$ is a collection of smei-open subsets of (X, τ), which consists of all sets $A \subseteq X$ satisfying $A \subseteq \mathrm{Cl}\ (\mathrm{Int}\ (A))$. Moreover, Jankovic[7] has proved that $PO(Y, \sigma^{\alpha}) = PO\ (Y, \sigma)$ for any space (Y, σ).

A space (X, τ) is said to be a submaximal space[3] if each of its dense subsets is open. But (X, τ) is resolvable[6] if there is a dense subset D of X for which its complement is also dense. A space which is not resolvable is called irresolvable. Any subset of X is resolvale (irresolvable) if it is resolvable (irresolvable) as a subspace. Also, (X, τ) is hereditarily irresolvable[6] if each of its non-empty subsets is irresolvable.

A function $f: (X, \tau) \to (Y, \sigma)$ is said to be preirresolute[12] (strongly M-precontinuous (SMPC)[1]) if the inverse image of each preopen in (Y, σ) is preopen (open) in (X, τ). Also f is precontinuous[11] if $f^{-1}(V) \in PO(X, \tau)$ for each $V \in \sigma$.

A property P which is preserved under semi-homemorphism (α-homemorphism) is called a semi-topological[4] (α-topological[8]) property, whenever semi and α-homemorphisms are defined likewise ordinary one except continuity and openness are replaced by semi and α-corresponding ones.

3. On SMPC Functions

The obvious implication : $\tau \subseteq t^{\alpha} \subseteq PO(X, \tau) \subseteq \tau_P$, is always satisfied, which is very useful throughout this paper.

Proposition 1 — (1) $f: (X, \tau) \to (Y, \sigma)$ is SMPC

$\to$ (2) $f: (X, \tau) \to (Y, \sigma^{\alpha})$ is continuous

$\to$ (3) $f: (X, \tau^{\alpha}) \to (Y, \sigma^{\alpha})$ is continuous.

Functions $f: (X, \tau) \to (Y, \sigma)$ satisfying (2) above are called strongly α-irresolute[9].

While functions satisfying (3) above are called α-irresolute functions[8] (or α-functions[13]).

The following examples show that in general the implications of the above proposition are irreversible.

Example 1 — If $\tau \neq \tau^{\alpha}$, the identity function $1_X: (X, \tau^{\alpha}) \to (X, \tau^{\alpha})$ is continuous whereas $1_X: (X, \tau) \to (X, \tau^{\alpha})$ is not continuous showing that strong α-irresoluteness is not implied by α-irresoluteness. Now let (R, σ) be the usual space of real numbers. Then (R, σ^{α}) is connected since for any space (X, τ), (X, τ) and (X, τ^{α}) share the same clopen (closed and open) subsets. Further, (R, σ) is resolvable so that no non-constant function from (R, σ^{α}) to (R, σ) can be SMPC. However, the identity $1_R: (R, \sigma^{\alpha}) \to (R, \sigma)$ is strongly α-irresolute and not SMPC.

Definition 1 — A filterbase F p-converges to x and we write $F \xrightarrow{p} x$ if for each preopen U with $x \in U$, there is an $F \in F$ with $F \subseteq U$.

Proposition 2 — If F is a filterbase in (X, τ), $F \xrightarrow{p} x$ iff $F \xrightarrow{\tau_p} x$. i.e. F p-converges to x iff F converges to x in (X, τ_p).

Proof : If $F \xrightarrow{p} x$ and $W \in \tau_p$ with $x \in W$, there exists $V = \bigcap_{k=1}^{n} A_k$ with each $A_k \in PO(X, \tau)$ and $x \in V \subseteq W$. There exists $F_k \in F$ with $F_k \subseteq A_k$ for each $k = 1, 2, ..., n$.

Since F is a filterbase, there is an $F \in F$ with $F \subseteq \bigcap_{k=1}^{n} F_k \subseteq V \subseteq W$. Thus, $F \xrightarrow{\tau_p} x$. The converse is clear since $PO(X, \tau) \subseteq \tau_p$.

Lemma 1[5] — Any $A \in PO(X, \tau)$ iff $A = U \cap D$ for some $U \in \tau$ and dense $D \subseteq X$.

Theorem 1 — *The following are equivalent :*

(i) $f : (X, \tau) \to (Y, \sigma)$ *is SMPC.*

(ii) $f : (X, \tau) \to (Y, \sigma_p)$ *is continuous.*

(iii) For each filterbase F on X, $f(F) \xrightarrow{P} f(x)$ *if* $F \to x$.

(iv) $f : (X, \tau) \to (Y, \sigma)$ *is continuous and* $f^{-1}(D) \in \tau$ *for each dense* $D \subseteq Y$.

(v) $f : (X, \tau) \to (Y, \sigma)$ *is continuous and* $f^{-1}(E)$ *is closed if* $IntE = \phi$.

PROOF : (i) $\Leftrightarrow$ (ii) Since a basic open set in σ_P has the form $V = \bigcap_{k=1}^{n} B_k$ where each $B_k \in PO(Y, \sigma)$. So if $f : (X, \tau) \to (Y, \sigma)$ is SMPC, and V is a basic open set in σ_p, $f^{-1}(V) = \bigcap_{k=1}^{n} f^{-1}(B_k) \in \tau$ so that $f : (X, \tau) \to (Y, \sigma_p)$ is continuous. The converse is clear. Also, (*ii*) and (*iii*) are equivalent in view of the previous proposition. Clearly (*iv*) and (*v*) are equivalent. Also (*i*) $\to$ (*iv*) since dense sets are preopen.

It remains only to show that (*iv*) implies (*i*). Let $B \in PO(Y, \sigma)$ by Lemma 1 $B = U \cap D$ for some $U \in \sigma$ and dense $D \subseteq Y$. Since $f^{-1}(D) \varepsilon \tau$, and f is continuous, $f^{-1}(B) = f^{-1}(U) \cap f^{-1}(D) \varepsilon \tau$ showing that f is SMPC if (*iv*) holds.

4. ON SUBMAXIMAL SPACES

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is weakly precontinuous if $f : (X, \tau_p) \to (Y, \sigma)$ is continuous.

Clearly because $PO(X, \tau) \subseteq \tau_p$, precontinuity implies weak precontinuity.

The next example shows that the converse does not hold.

Example 2 — Let (R, σ) be the usual space of real numbers and let $1_R : (R, \sigma) \to (R, \sigma_p)$ be the identity function. Because (R, σ) is resolvable, σ_p is the discrete topology. Also $PO(R, \sigma) \neq \sigma_p$ since nonempty nowhere dense sets exist in (R, σ) which are not preopen. Thus, $\frac{1}{R}$ is weakly precontinuous but not precontinuous.

Numerous equivalents of submaximality *via* SMPC and other topological concepts are established through the following fundamental theorem in this note.

Theorem 2 — *The following are equivalent :-*

(i) (Y, σ) *is submaximal.*

(ii) For every space (X, τ), *each continuous* $f : (X, \tau) \to (Y, \sigma)$ *is SMPC.*

(iii) The identity function $1_y : (y, \sigma) \to (Y, \sigma)$ *is SMPC.*

(iv) $\sigma = PO(Y, \sigma)$

(v) For all $B \subseteq Y$, $B\text{-}IntB \neq \phi \to B\text{-}IntC1B \neq \phi$.

(vi) For every space $(X, \tau), h : (X, \tau) \cong (Y, \sigma)$ *iff* $h : (X, \tau) \to (Y, \sigma)$ *and* $h^{-1} : (Y, \sigma) \to (X, \tau)$

are SMPC.

(vii) Every weakly precontinuous $g : (Y, \sigma) \to (Z, \rho)$ *is continuous.*

(viii) For every continuous $f : (X, \tau) \to (Y, \sigma), f^{-1}(E)$ *is closed whenever* $Int\ E = \phi$

(ix) $\sigma = \{U - E : U \in \sigma$ *and* $IntE = \phi\} = \{U \cap D : U \in \sigma$ *and* $ClD = Y\}$.

(x) There exists an open, dense, hereditarily irresolvable subspace $D \subseteq Y$ *and* $\sigma = \sigma^{\alpha}$.

(xi) $PO(Y, \sigma) \subseteq SO(Y, \sigma)$ *and* $\sigma = \sigma^{\alpha}$.

(xii) $A \subseteq Y$ *is nowhere dense iff* $IntA = \phi$, *and* $\sigma = \sigma^{\alpha}$.

(xiii) (Y, σ^{α}) *is submaximal and* $\sigma = \sigma^{\alpha}$.

PROOF : If (Y, σ) is submaximal and $f : (X, \tau) \to (Y, \sigma)$ is continuous then f is SMPC by Theorem 3.1 in [1]. So (*i*) $\to$ (*ii*). Clearly (*ii*) $\to$ (*iii*) and (*iii*) $\to$ (*iv*) and since (*iv*) is equivalent to $PO(Y, \sigma) \subseteq \sigma$, (*iv*) $\Leftrightarrow$ (*v*). We now show that (*iv*) $\to$ (*i*). Suppose that $\sigma = PO(Y, \sigma)$ and let $D \subseteq Y$ be dense. Since every dense set is preopen, $D \in \sigma$ so that (Y, σ) is submaximal. Now (*i*) $\Leftrightarrow$ (*vi*) if $h : (x, \tau) \cong (Y, \sigma)$ then (X, τ) is submaximal and by (*ii*), h and h^{-1} are each SMPC. Also, if (Y, σ) is submaximal and h and h^{-1} are each SMPC, then clearly h is a homeomorphism so that (*i*) $\to$(*vi*). For the converse, (*vi*) $\to$ (*iii*) $\to$ (*i*). To see that (*i*) $\Leftrightarrow$ (*vii*) let (Y, σ) be submaximal and let $g : (Y, \sigma) \to (Z, \rho)$ be weakly precontinuous. Then $W \in \rho \to g^{-1}(W)\ \varepsilon\ \sigma_p = \sigma$ so that g is continuous. Conversely, if (*vii*) holds, than the identity $1_y : (Y, \sigma) \to (Y, \sigma_p)$ is weakly precontinuous and hence continuous so that $\sigma_p \subseteq \sigma$ and (Y, σ) is a preopen space and hence submaximal. By Theorem 1, previously (*viii*) $\Leftrightarrow$ (*ii*) and clearly (*ix*) $\Leftrightarrow$ (*iv*) by Proposition 1 of [5]. Since (*i*) $\to \sigma = \sigma^{\alpha}$ and by Proposition 1 of [10] (Y, σ^{α}) is submaximal and thus, $\sigma^{\alpha} = PO(Y, \sigma^{\alpha}) = PO(Y, \sigma) = \sigma$. Therefore, the equivalences of (*i*) with each of (*x*, (*xi*), (*xii*) and (*xiii*) follow from Theorems 2 and 4 of [5].

Immediately, every subspace of a hereditarily irresolvable space is hereditarily irresolvable and submaximal spaces are hereditarily irresolvable.

Now, which subspaces of submaximal spaces are submaximal?

It is known that open and hence preopen subsets of a submaximal space are submaximal and we show a bit more.

Theorem 3 — *If* (X, τ) *is submaximal and any* $A \subseteq X$, *then* $(A, \tau | A)$ *is submaximal.*

PROOF : Let B be a dense subset in $(A, \tau | A)$. To show that B is open in $(A, \tau | A)$ Let $W = B \cup (X - A)$ which is dense in X. But in fact, $Cl\ W = Cl\ (B \cup (X - A)) \supseteq Cl\,B \cup Cl\,(X - A) \supseteq A \cup (X - A) = X$. From the submaximality of (X, τ), it follows that W is open in X. But $B = W \cap A$, hence B is open in $(A, \tau/A)$. This means that $(A, \tau/A)$ is a submaximal space.

We conclude with an example showing that in general the intersection of two submaximal topologies is not submaximal.

Example 3 — Let $X = \{0, 0^*\} \cup \{1, 2, ...\}$ have topologies τ and τ^* defined by

$A \in \tau$ iff $0 \in A \to A = X$ and

$A \in \tau^*$ iff $0^* \in A \to A = X$.

Then (X, τ) and (X, τ^*) are submaximal. In particular, if D is τ-dense in X, either $D = \{0^*, 1, 2, ...\}$ or $D = X$ and in either case $D \in \tau$. Similarly, each τ^*-dense D is an element of τ^*.

Finally, as a subspace of $(X, \tau \cap \tau^*)$, $\{0, 0^*\}$ is indiscrete and hence resolvable so that $(X, \tau \cap \tau^*)$ is not even hereditarily irresolvable and hence not submaximal.

Obviously, each finer space of a submaximal space is also, more generalization of this fact established nextly.

Proposition 3 — For any space (X, τ), the space (X, τ_p) is submaximal.

PROOF : Let D be τ_p-dense. Then it is τ-dense and hence $D \in PO(X, \tau)$. Thus $D \in \tau_p$ and therefore (X, τ_p) is submaximal.

***Theorem* 4** — *Submaximality is preserved by open surjections.*

PROOF : If $f: (X, \tau) \to (Y, \sigma)$ is an open surjection and (X, τ) is submaximal and if $D \subseteq Y$ is dense, $f^{-1}(D)$ is dense and hence open in X so that $D = f(f^{-1}(D))$ is open.

Corollary 1 — If $\prod_{\alpha \in V} X_\alpha$ is submaximal then each X_α is submaximal.

The converse of the result in Corollary 1 need not hold in general as the next example shows.

Example 4 — Let $X = \{0\} \cup \left\{ \frac{1}{n} : n = 1, 2, \dots \right\}$ have the usual real subspace topology. Then the only proper dense subset of X is $D = \left\{ \frac{1}{n} : n = 1, 2, \dots \right\}$ which is open so that X is submaximal. But the product space X^2 is not submaximal. For $\{(0, 0)\} \cup (D \times D)$ is dense and hence preopen in X^2 but not open.

Lemma 2[5] — (X, τ^α) is submaximal iff (X, τ) contains an open, dense and hereditarily irresolvable subspace.

Submaximality of $(X, \overline{\tau})$ satisfies in its finer space (X, τ^α) as a corollary of Proposition 1[6]. While its converse does not hold as the following example illustrates.

Example 5 — The non-submaximal space (X^2, τ) of Example 4 had an open, dense, discrete and hence hereditarily irresolvable subspace so that (X^2, τ^α) is submaximal. So in general, (Y, σ^α) submaximal does not imply that (Y, σ) is submaximal.

The following example shows that even though (X, τ^α) is submaximal, (X, τ) may fail to be hereditarily irresolvable.

Example 6 — Let $X = \{0\} \cup \left\{ \frac{1}{n} : n = 1, 2, \dots \right\}$ have topology τ. Let each $A \subseteq \left\{ \frac{1}{n} : n = 2, 3, \dots \right\}$ be open and if U is open and $0 \in U$ and $\left\{ \frac{1}{n} : n = 2, \dots \right\} - U$ is finite. Clearly, $\left\{ \frac{1}{n} : n = 2, \dots \right\}$ is an open, dense, hereditarily irresolvable subspace so that (X, τ^α) is submaximal whereas the subspace $\{0, 1\}$ is indiscrete and hence resolvable.

Theorem 5 — *Submaximality is a topological property which is not semitopological.*

PROOF : It has been shown recently in [14] that the semitopological properties are precisely the α-topological properties. Since (Y, σ^{α}) submaximal while (Y, σ) is not submaximal for some space (Y, σ), submaximality is not an α-topological property.

REFERENCES

1. M. E. Abd El-Monsef, R. A. Mahmoud and A. A. Nasef, *Qatar Univ. Sci Bull.* **9** (1989), 17-25.
2. D. Andrijevic, On the topology generated by preopen sets, (*To appear in Math. Vesnik.*)
3. N. Bourbaki, *General Topology*, Addison-Wesley, Mass., 1966.
4. S. G. Crossley and S. K. Hildebrand, *Fund. Math.* **74** (1972), 233-54.
5. M. Ganster, *Kyung. Math. J.* **27** (2) (1987), 135-43.
6. E. Hewitt, *Duke Math. J.*, **10** (1943), 309-33.
7. D. S. Jankovic, *Acta Math. Hung.* **46** (1-2) (1985). 83-92.
8. S. N. Maheshwari and S. S. Thakur, *Tamkang J. Math.*, **11** (1980), 209-14.
9. R. A. Mahmoud, M. E. Abd El-Monsef and A. A. Nasef, *Kyung Math. J.* **36** (1) (1996), 143-50.
10. R. A. Mahmoud and D. A. Rose, *Tamkang J. Math.*, **24** (1993), 333-39.
11. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *Proc. Math. phys. Soc. Egypt* **53** (1982), 47-53.
12. I. L. Reilly and M. K. Vamanamurthy, *Acta Math. Hungar*, **45** (1985), 27-32.
13. D. A. Rose, *Int. J. Math. math. Sci.* **11** (1988), 713-20.
14. D. A. Rose and R. R. Hamlett, *Math. Chonicle* **20** (1991), 149-56.