Indian J. pure appl. Math., 32(3) : 345-354, March 2001

SOLUTION OF THE RANDOM-PHASE EQUATIONS DESCRIBING UNSTABLE THREE WAVE COUPLING

I. R. DURRANI

Centre of Excellence in Solid State Physics, University of Punjab, Lahore, Pakistan
E-mail : id786@one.net.pk

AND

Z. R. BHATTI

Department of Mathematics, Govt. College of Science, Wahdat Road, Lahore, Pakistan
e-mail: zrbhatti@one.net.pk

(*Received* 29 *June* 1999; *after revision* 16 *June* 2000; *accepted* 26 *June* 2000)

The random-phase equations are studied in the comparatively general case with mutually different and time dependent dissipation and coupling strength. The time of explosion is calculated.

Key Words : Random-Phase Equations; Three Wave Coupling' Explosion Time; Nonlinearity in Plasma

1. INTRODUCTION

The phenomenon of explosive instability seems to be one of the most fascinating in the field of nonlinear instabilities. It occurs as a result of nonlinear wave coupling between waves having negative and positive wave energies and may lead to situations where all amplitudes of the interacting waves reach unlimited values in a finite time.

In [1], the possibility of having time-dependent coupling strength and dissipation in the non-linear coupled equations describing the evolution in time of unstable systems is discussed and a solution in the coherent-phase description is presented. We are here going to study the corresponding problem for the random phase equations. In this approach, the evolution of the individual phases of the waves is disregarded. This is achieved by averaging over the phases at every instant so that all effects due to the individual phases are cancelled. It is assumed that the time of coherence is much shorter than the characteristic time of interaction τ, between the waves. This is usually expressed by the condition $\tau \Delta \omega >> 1$, where $\Delta \omega$ is the uncertainty in frequency.

2. BASIC EQUATIONS

When collisional and resonant particle effects are neglected in the nonlinear terms, the set of random-phase equations can be written:

$$\frac{\partial N_0}{\partial t} + v_0(t) N_0 = c_0(t)(N_0 N_1 + N_0 N_2 + N_1 N_2), \quad \dots (1a)$$

$$\frac{\partial N_1}{\partial t} + v_1(t) N_1 = c_1(t)(N_0 N_1 + N_0 N_2 + N_1 N_2) \quad \dots (1b)$$

and

$$\frac{\partial N_2}{\partial t} + v_2(t) N_2 = c_2(t)(N_0 N_1 + N_0 N_2 + N_1 N_2). \quad \dots (1c)$$

Here N_f is the number density of the 'quanta' associated with mole "j", $v_j(t)$ is due to linear dissipation and $c_j(t)$ is the coupling strength. The signs of the terms in the right hand side have been chosen in such a way that nonlinear instability is possible.

This set of equations has been solved for an n-wave system with constant c_j and constant equal to y_j by [2] and has also been studied by [3].

3. The Structure of the Solution

We are now going to consider time-dependent $v_j(t)$ and $c_j(t)$ where the time dependence is slow, to be consistent with the method of approach. We can express N_j for one mode as :

$$N_j = \exp\left(-\int_0^t v_j(t')\,dt'\right)$$

$$\left[\int_0^t \exp\left(\int_0^{t'} v_j(t'')\,dt''\right) c_j(t')(N_0 N_1 + N_0 N_2 + N_1 N_2)\,dt' + N_j(0)\right]. \quad \dots (2)$$

This integral can be approximated for $t \approx t_\infty$ by substituting the asymptotic solution in the integrand[4]. The solution for all $v_j = \mu'$ independent of 't' with constant c_j can in the unstable case be written :

$$N_j = \exp(-\mu' t)\left[\frac{k_j}{1 - \exp\frac{6\gamma}{\mu'}[\exp(-\mu' t_\infty) - \exp(-\mu' t)]} + A_j\right]. \quad \dots (3)$$

Here

$$\gamma = c\sqrt{c^2 \overline{m}^2 - (m^2 - \sigma^2)}, \quad c^2 = \frac{1}{3}(c_0 c_1 + c_0 c_2 + c_1 c_2),$$

$$\overline{m} = \frac{1}{6c^2}\{N_0(0) + (c_1 + c_2) + N_1(0)(c_0 + c_2) + N_2(0)(c_0 + c_1)\},$$

$$m = \frac{1}{3}(N_0(0) + N_1(0) + N_2(0)),$$

$$\sigma = \left\{ \frac{1}{6} [N_0(0) - m)^2 + (N_1(0) - m)^2 + (N_2(0) - m)^2] \right\}^{1/2}$$

and $$k_j = 6\gamma \left(\frac{c_0}{c_0 c_1 + c_0 c_2 + c_1 c_2} \right)$$

and t_∞ is the time of explosion.

The constants A_j are determined by the initial values of N_j. If $c^2 \overline{m}^2 < (m^2 - \sigma^2)$, γ is imaginary.

In this case the solution should rather be expressed as

$$N_j = c_J \frac{\gamma}{c_2} \tan \left(\frac{3\gamma}{\mu'} (1 - \exp(-\mu')) + \text{arc tan} \frac{\overline{m} c^2}{\gamma} \right) - c_j \overline{m} + N_j(0), \quad \text{... (4)}$$

where γ^2 has been exchanged for $-\gamma^2$. However, for small γ which we will mainly consider in this expression the main structure of the solution should not differ very much from that of (3). For time-dependent $\mu'(t)$, $\exp(-\mu' t)$ should be exchanged for $\exp\left(-\int_0^t \mu'(t')\, dt' \right)$. The solution (3), with A_j neglected, can be used as an asymptotic solution for different $\gamma_j(t)$, if $\mu'(t)$ is made to represent the mean influence for all $\gamma_j(t)$. This solution is now put into (2). If {italic partial} integration are performed, and only the first term is kept we obtain :

$$N_j = \frac{k_j(t) \exp\left(-\int_0^t \mu'(t')\, dt' \right)}{1 - \exp\left(6\gamma \left(\int_0^t \exp\left(-\int_0^{t'} \mu'(t'')\, dt'' \right) dt' - \int_0^\infty \exp\left(-\int_0^{t'} \mu'(t'')\, dt'' \right) dt' \right) \right)} \quad \text{... (5)}$$

$$- \frac{k_j(0) \exp\left(-\int_0^t v_j(t')\, dt' \right)}{1 - \exp\left(-6\gamma \int_0^\infty \exp\left(-\int_0^{t'} \mu'(t'')\, dt'' \right) dt' \right) + N_j(0) \exp\left(-\int_0^t v_j(t')\, dt' \right)}$$

This solution is correct near $t = 0$ and $t = t_\infty$ and shows the main structure of the time evolution of N_j. The collective damping $\mu'(t)$ should here be chosen so as to have the average influence of $v_0(t)$, $v_1(t)$ and $v_2(t)$. The way of determining this will be discussed later. For small spread in $N_j(0)$ the denominator of (4) can be expanded for small γ. Since the main structure of the solution is not influenced by this simplification, it should be advisable to use this expansion irrespective of γ in most cases.

4. THE TIME OF EXPLOSION

In order to obtain an expression for the time of explosion we sum the equations ($1a - c$) obtaining:

$$\frac{\partial}{\partial t}(N_0 + N_1 + N_2) + v_0(t) N_0 + v_1(t) N_1 + v_2(t) N_2$$

$$= \left(\sum_j c_j(t) \right) (N_0 N_1 + N_0 N_2 + N_1 N_2).$$

Introducing the mean, M of N_0, N_1 and N_2 we have :

$$\frac{\partial M}{\partial t} + \left(v_0(t) \frac{N_0}{M} + v_1(t) \frac{N_1}{M} + v_2(t) \frac{N_2}{M} \right) \frac{1}{3} M$$

$$= \frac{1}{3} \sum_j c_j(t) \left(\frac{N_0 N_1}{M^2} + \frac{N_0 N_2}{M^2} + \frac{N_1 N_2}{M^2} \right) M^2$$

Here the quantities N_j/M can be expected to be slowly varying and we introduce the time-dependent parameters:

$$\mu(t) = \frac{1}{3} \left(v_0(t) \frac{N_0}{M} + v_1(t) \frac{N_1}{M} + v_2(t) \frac{N_2}{M} \right) \quad \dots (6)$$

and

$$C(t) = \frac{1}{3} \left(\sum_j c_j(t) \right) \left(\frac{N_0 N_1}{M^2} + \frac{N_0 N_2}{M^2} + \frac{N_1 N_2}{M^2} \right). \quad \dots (7)$$

We then obtain an equation for M :

$$\frac{\partial M}{\partial t} + \mu(t) M = C(t) M^2$$

with solution[9]

$$M(t) = \frac{M(0) \exp\left(-\int_0^{t'} \mu(t')\, dt' \right)}{1 - M(0) \int_0^t C(t') \exp\left(-\int_0^t \mu(t'')\, dt'' \right) dt'} \quad \dots (8)$$

and the time of explosion determined by

$$\int_0^\infty C(t') \exp\left(-\int_0^{t'} \mu(t'')\, dt'' \right) dt' = \frac{1}{M(0)} \quad \dots (9)$$

This expression is correct for arbitrary solution for N_j. However, for it to be useful we must be able to estimate the quantities N_j/M. This can be done with the formula (5) in cases where the time dependence of $v_j(t)$ and $c_j(t)$ is sufficiently slow, and when the quantities N_j/M varies slowly. From (5) we notice that for $t \rightarrow t_\infty$ we have

$$\frac{N_j}{M} = \frac{k_j(t_\infty)}{\frac{1}{3}(k_0(t_\infty) + k_1(t_\infty) + k_2(t_\infty))} = \frac{c_j(t_\infty)}{\frac{1}{3}(c_0(t_\infty) + c_1(t_\infty) + c_2(t_\infty))}$$

and for $t \rightarrow 0$ we have

$$\frac{N_j}{M} = \frac{N_j(0)}{\frac{1}{3}(N_0(0) + N_1(0) + N_2(0))}.$$

Here we must use some limiting value for large t in $c_j(t_\infty)$, since we do not know t_∞. For a first estimation of t_∞ we can use some weighted average of N_j/M for $t = 0$ and large t in (9). The value of t_∞ obtained is then used in (5) to obtain a better estimate of N_j/M in (9) and then the process is repeated. In (5) we have to calculate

$$\int_0^{t_\infty} \exp\left(-\int_0^{t'} \mu'(t'')\,dt''\right) dt'$$

in advance. We then must use some constant linear combination of v_0, v_1 and v_2 in μ', which must be estimated in advance. Then the same μ' must be used when calculating

$$\int^{t} \exp\left(-\int_0^{t'} \mu'(t'')\,dt''\right) dt'$$

in (5). It is convenient to use the same expression for μ' as for μ in the first calculation of t_∞ in (9).

From (9) we notice that the integrand is decreasing with t due to the exponential factor. Thus the value of N_j/M is weighted stronger for small t than for large t. However, for a first approximation we can use the mean of the quantities N_j/M for large and small t.

Since in the coherent-phase description very good estimates of t_∞ have been obtained for constant v_j, by considering just the mean of v_0, v_1 and v_2,[4-5] we can expect the estimates of t_∞ obtained by this method to converge rapidly.

If the values of the densities N_j are considerably different during the interaction, this method should give a better estimate of t_∞ than could be obtained by using

$$\mu(t) = \frac{1}{3}(v_0(t) + v_1(t) + v_2(t)),$$

which would correspond to the approach earlier used in the coherent-phase description. Of course this method also applies to the case of constant v_j and c_j and represents an improvement of earlier descriptions of the influence of mutually different v_j. In this case the expression (5) is considerably simplified and when the quantities N_j / M are considered as constants, the integrals can be evaluated analytically.

5. Solution for Individual Modes

The expression (2) can be rewritten as:

$$N_j(t) = \exp\left(-\int_0^t v_j(t')\,dt' \right)$$

$$\left[\int_0^{t'} \exp\left(\int_0^{t} v_j(t'')\,dr'' \right) c_j(t') \times M^2 \left(\frac{N_0 N_1}{M^2} + \frac{N_0 N_2}{M^2} + \frac{N_1 N_2}{M^2} \right) dt' + N_j(0) \right]$$

Substituting the solution (8) into this expression we obtain :

$$N_j(t) = \exp\left(-\int_0^t v_j(t')\,dt' \right)$$

$$\times \left[\int_0^t \frac{M(0)^2\, c_j(t') \exp\left(\int_0^{t'} (v_j(t'') - 2\mu(t''))\,dt'' \right) \propto (t')\,dt'}{\left(1 - M(0) \int_0^{t'} C(t'') \exp\left(-\int_0^{t''} \mu(t')\,dt' \right) dt'' \right)^2} + N_j(0) \right] \quad \ldots (10)$$

where $$\alpha(t) = \frac{N_0 N_1}{M^2} + \frac{N_0 N_2}{M^2} + \frac{N_1 N_2}{M^2}$$

is expected to be slowly varying and can for a first approximation be considered as a constant. When the time of explosion has been calculated in the way described above, we can use (5) to calculate the quantities N_j/M in $\mu(t)$, $c(t)$ and $\alpha(t)$.

It is possible to perform partial integrations in (10). The expansion thus obtained can be written:

$$N_j(t) = \frac{M(0) \frac{c_j(t)}{C(t)} \alpha(t) \exp\left(-\int_0^t \mu(t')\,dt' \right)}{1 - M(0) \int_0^t C(t') \exp\left(-\int_0^t \mu(t'')\,dt'' \right) dt'}$$

$$+\frac{1}{C(t)}\left[\frac{c_j(t)}{C(t)}\alpha(t)(v_j(t)-\mu(t)+\frac{c_j(t)}{C(t)}\frac{d\,\alpha(t)}{dt}+\alpha(t)\frac{d}{dt}\left(\frac{c_j(t)}{C(t)}\right)\right]$$

$$\times\ \ln\left[1-M(0)\int_0^t C(t')\exp\left(-\int_0^{t'}\mu(t''\,dt''\right)dt'\right]+\ldots \qquad \ldots (11)$$

where only the first two terms have been included. We note that the second term contains the small derivatives of $\alpha(t)$ and $c_j(t)/C(t)$ and the difference between $v_j(t)$ and $\mu(t)$ which will vanish if $v_0(t)=v_1(t)=v_2(t)$. If we perform further partial integrations we will obtain higher derivatives of $\alpha(t)$ and $c_j(t)/C(t)$ which we may drop. We also will obtain differences between higher powers of $v_j(t)$ and $\mu(t)$ which we in general may not drop.

We also note that when all $v_j(t)$ and $N_j(0)$ are equal, the first term is the exact solution and no approximations in the calculation of $\mu(t)$, $C(t)$ and $\alpha(t)$ are needed.

6. Calculation of the Time of Explosion

The time of explosion has been calculated in the way described above and the results have been compared with the explosion time obtained from numerical integration of (1). In Table I, the parameters v_j and c_j are constants.

TABLE I : Time of explosion

$N_0(0)$	$N_1(0)$	$N_2(0)$	v_0	v_1	v_2	c_0	c_1	c_2	Time of explosion Num. Int.	Calculated
2.7	1.0	0.7	0.3	0.3	1.1	0.1	0.3	0.1	2.5	2.5
2.7	1.0	0.7	0.4	0.4	1.3	0.1	0.3	0.1	3.3	3.3
3.0	1.5	0.7	0.4	0.4	1.3	0.1	0.3	0.1	2.2	2.2
3.0	1.0	1.0	0.2	0.2	0.6	0.1	0.3	0.1	1.7	1.7
3.0	1.0	0.7	0.5	0.5	1.5	0.1	0.3	0.1	4.3	4.6
3.0	1.0	1.0	0.6	0.6	1.8	0.1	0.3	0.1	∞	∞

TABLE II : Time of explosion

$v_0(0)$	$v_1(0)$	$v_2(0)$	k_{v0}	k_{v1}	k_{v2}	k_{c0}	k_{c1}	k_{c2}	Time of explosion Num. Int.	Calculated
0.3	0.3	1.1	–0.2	–0.2	–0.16	–0.05	–0.2	–0.05	2.7	2.7
0.3	0.3	1.1	0.4	0.5	0.6	–0.06	–0.2	–0.06	3.2	3.2
0.3	0.3	1.1	0.4	0.5	0.6	–0.08	–0.2	–0.08	3.3	3.2
0.4	0.4	1.3	0.3	0.3	0.5	–0.08	–0.2	–0.08	4.7	5.0

When using (5) to calculate N_j/M, the denominator has been expanded for small γ and only the first two terms have been kept.

In Table II, the following parameters are common :$N_0(0)=2.7$, $N_1(0)=1$, $N_2(0)=0.7$, $c_0(0)=0.1$, $c_1(0)=0.3$ and $c_2(0)=0.1$.

The time dependencies of the parameters $v_j(t)$ and $c_j(t)$ are expressed by :

$$v_j(t) = \begin{cases} v_j(0), t \le t_{0j} \\ v_j(0) + k_{vj}(t - t_{0j}) \exp(-[(t - t_{1j})/(t_{1j} - t_{0j})]^2), t \ge t_{0j} \end{cases}$$

and

$$c_j(t) = \begin{cases} c_j(0), t \le t_{c0j} \\ c_j(0) + k_{cj}(t - t_{c0j}) \exp(-[(t - t_{c1j})/(t_{c1j} - t_{c0j})]^2), t \ge t_{c0j} \end{cases},$$

where the values $t_{00} = 1.0$, $t_{10} = 1.5$, $t_{01} = 1.0$, $t_{11} = 1.5$, $t_{02} = 1.0$, $t_{12} = 1.5$, $t_{c00} = 1.5$, $t_{c10} = 2.0$, $t_{c0} = 1.5$, $t_{c11} = 2.0$, $t_{c02} = 1.5$ and $t_{c12} = 2.0$ have been used.

In Table III, the parameters $N_j(0)$ are changed to $N_0(0) = 3$, $N_1 = (0) = 1.5$ and $N_2(0) = 0.7$. All other parameters are the same as in Table II.

From Tables I and II, we note that when the dissipation approaches the value necessary to eliminate the instability, approximating the influence of individual v_j becomes doubtful. This was to be expected since here a small change in μ can cause an infinite change in t_∞. In Table I, the prameters in the first three cases are equal to the constant parts of the parameters in Tables II and III. We can thus study the influence of the time variation in $v_j(t)$ and $c_j(t)$ on t_∞. The step used in the numerical calculations has been 0.05. The number of iterations performed in calculating (9) has been four. However, no significant change in t_∞ has occurred after the third iteration. In the first iteration the constant value of N_j/M the value of t_∞ from the first iteration has not differed more than 10% from the final value. However, no change in the final value of t_∞ is obtained by using the weights 1/2 and 1/2 in the first iteration. The values of t_∞ obtained with constant N_j/M and weights 4/5 and 1/5 are so good that one iteration should often be enough. In the case of constant v_j and c_j, (9) can be evaluated analytically which constant N_j/M. The result is:

$$t_\infty = -\frac{1}{\mu} \ln\left(1 - \frac{\mu}{M(0)C}\right)$$

TABLE III : Time of explosion

$v_0(0)$	$v_1(0)$	$v_2(0)$	k_{v0}	k_{v1}	k_{v2}	kc_0	k_{c1}	k_{c2}	Time of explosion Num. Int.	Time of explosion Calculated
0.3	0.3	1.1	0.6	0.6	1.0	–0.06	–0.2	–0.06	2.1	2.0
0.4	0.4	1.3	0.4	0.6	0.9	–0.05	–0.2	–0.05	2.8	2.7
0.4	0.4	1.3	0.4	0.5	0.6	–0.08	–0.2	–0.08	2.8	2.7

where

$$M(0) = \frac{1}{3} \sum_j N_j(0),$$

$$\mu = \frac{1}{3}\left(\frac{N_0}{M} v_0 + \frac{N_1}{M} v_1 + \frac{N_2}{M} v_2\right),$$

$$C = \left(\frac{N_0 N_1}{M^2} + \frac{N_0 N_2}{M^2} + \frac{N_1 N_2}{M^2} \right) \frac{1}{3} \sum_j c_j$$

and

$$\frac{N_j}{M} = \frac{4}{5} \frac{N_j(0)}{M(0)} + \frac{1}{5} \frac{N_j(t_\infty)}{M(t_\infty)} = \frac{4}{5} \frac{N_j(0)}{M(0)} + \frac{c_j}{\frac{5}{3}(c_0 + c_1 + c_2)}.$$

Thus in this case the result corresponding to the first iteration where N_j/M is considered constant can be calculated analytically, and the result so obtained is likely to be a good estimated of t_∞.

7. Concluding Remarks

In the paper by Wilhelmsson[1] the time-dependent coupling factors have been transformed to equivalent time-dependent $v_j(t)$.

In the random-phase description it is not possible to do so, without obtaining new time-dependent factors in the right hand sides of the equations (1). However, the random-phase description is more convenient for creating averages in the way performed in this paper. For the calculation of an individual mode N_j, the solution (10) will be the most suitable if we do not have $v_j(t) \approx \mu(t)\, j = 0, 1, 2,$ or if we are interested in the solution far from the time of explosion. Otherwise we only need a few terms in the expansion (11).

For calculation of explosion time and individual modes in the range of parameter values where the problem becomes badly conditioned, the estimation of the quantities N_j/M must be improved. This can be done by using more terms in the expansion (5) and by considering the dependence of γ.

The authors in an earlier paper[10] based on coherent phase description for three interacting waves an analytical and computer investigation presented which considers the combined effect of dissipation and nonlinear frequency shifts. The computer results in [10] verified the analytic predictions. The asymptotic values for the saturated wave amplitudes reached, when time approaches infinity were determined by the amount of dissipation, and solitons formed turn out to be oscillatory structures. Further the problem of the integration of a three-wave set and five-wave set of equation was/is settled in [11] and [12].

Following [10 & 11] in the present paper, we have found the solution of the random phase equations describing unstable three-wave coupling and calculated the time of explosion which we find is in favourable comparison to the one obtained through numerical calculations. The random phase description as performed here is more convenient for deriving averages of the parameters introduced for the calculations relating to individual modes are for more tortuous.

References

1. H. Wilhelmsson, *Phys. Rev.* **A6** (1972) 1973.
2. L. Stenflo and H. Wilhelmsson, *Phys. Lett.* **29A** (1969) 217.
3. B. Coppi, M. N. Rosenbluth and R. N. Sudan, *Ann. Phys. (N.Y.)* **55** (1969) 207.
4. J. Weiland, *J. Plasma Phys.* **7** (1972) 375.
5. H.Wilhelmsson, *Physica Scripta (Sweden)* **2** (1970) 113.

6. H. Wilhelmsson, L. Stenflo and F. J. Engelmann, *J. math. Phys.* **11** (1970) 1738.
7. R. C. Davidson, *Methods in Nonlinear Plasma Theory* (*Ed*: H.S.W. Massey and K. A. Brueckner). *Pure. appl. Phys.*, **37**, Academic Press, New York and London, 1972.
8. V. N. Tsytovich, *Nonlinear effects in Plasma* (*Ed*: S. M. Hamberger). Plenum Press, New York, 1970.
9. H. Wilhelmsson, *Physica Scripta (Sweden)* **5** (1972) 116.
10. Z. R. Bhatti, I. R. Durrani and S. Asghar, *Sci. Int.* **11** (2) (1999) 187-94.
11. Z. R. Bhatti, I. R. Durrani and M. Sarif, *Bull. pure appl. Sci.*, **18**(2) D (2000).
12. I. R. Durrani and Z. R. Bhatti, *Bull. pure appl. Sci.*, **19**(1) E (2000).