Indian J. pure appl. Math., **32**(3) : 355-360, March 2001

ITERATIVE TECHNIQUE FOR A INTEGRAL EQUATION IN BANACH SPACE AND APPLICATIONS

GUANGXING SONG

Department of Applied Mathematics, University of Petroleum, Dongying, Shandong, 257062, *P.R. China*

(*Received* 21 *March* 2000; *accepted* 26 *June* 2000)

In this paper, we use partial order theory to study nonlinear integral equations in general Banach spaces, existence and uniqueness of solution are obtained and error estimate of the iterative sequences of approximation solution is given. As an application, we utilize main results presented in this paper to study the existence and uniqueness problems of weakly Caratheodory's solution for a class of nonlinear differential equations. Our results generalize and improve the corresponding results given in some known literatures.

Key Words : Banach Space; Partial Ordering Theory; Integral Equation; Differential Equation; Iterative Technique

1. INTRODUCTION

The purpose of this paper is to study the following nonlinear integral equation

$$u(t) = h(t) + \int_0^t k(t, s)\, [f(s, u(s)) + Mu(s)]\, ds, \qquad \text{... (1)}$$

where $M > 0, h(t) \in C[I, E], k(t, s) \in C[D, R^+]$, $I = [0, 1]$, $D = \{(t, s) \in R^2 \mid 0 \le s \le t \le 1\}$, $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$, E is a real Banach space with norm $\|\cdot\|$. We assume that $f(t, u)$ is not continuous and only $f(t, u)$ satisfies the weak Caratheodory condition, we proved, however, that the solution of the integral equation (1) can be obtained by the iterative sequences and the error estimate of the iterative sequences of approximation solution is given. As applications, we can apply the main results obtained in this paper to study the initial value problem for nonlinear differential equation in Banach spaces

$$\begin{cases} u' = f(t, u) \\ u(0) = x_0 \end{cases}, \qquad \text{... (2)}$$

where $f(t, u)$ is not continuous.

2. PRELIMINARIES

Let E be a real Banach space and P a cone in E. The order $\le$ is introduced by cone P, i.e., $x, y \in E, x \le y$ if and only if $y - x \in P$. A cone P is said to be normal if there exists a constant

$N > 0$ such that $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; N is called the normal constant of P. Let $P_c = \{u \in C[I, E] \mid u(t) \geq \theta, \ \forall \ t \in I\}$, where $C[I, E]$ denotes the Banach space of all continuous mapping $u : I \to E$ with the norm $\|u\|_c = \max \{\|u(t)\| \ t \in I\}$. It is clear that P_c is a cone of the space $C[I, E]$ and it defines a partial ordering in $C[I, E]$. Obviously, the normality of P implies the normality of P_c and the normal constants of P_c and P are the same.

For any $u_0, v_0 \in C[I, E]$ such that $u_0 \leq v_0$, we define the ordered interval $[u_0, v_0] = \{u \in C[I, E] \mid u_0 \leq u \leq v_0\}$ and set

$$\Omega = \{x \in E \mid u_0(t) \leq x \leq v_0(t) \text{ for some } t \in I\}.$$

Let $f(t, x) : I \times \Omega \to E$. $f(t, x)$ is said to satisfy weak Caratheodory condition if

(*i*) $f(t, x)$ is strongly measurable on t for any $x \in \Omega$; and

(ii) for almost all $t \in I, f(t, x)$ demicontinuous on x, that is, there exist $I_0 \subset I$, mes $I_0 =$ mes I, such that when $\{x_n\}$ converges to x_0, $\{f(t, x_n\}$ converges weakly to $f(t, x_0)$ for any $t \in I_0$.

For any $1 \leq p < +\infty$, set

$$L_p[I, E] = \{x(t) : I \to E \mid x(t) \text{ is strongly measurable and } \int \|x(t)\|^p \, dt < +\infty\}.$$

Obviously, $L_p[I, E]$ is a Banach space with the norm $\|x\|_p = \left[\int \|x(t)\|^p \, dt \right]^{\frac{1}{p}}$.

Throughout this paper we always assume that E is a real Banach space, P is a normal cone in E and N is the normal constant of P.

3. Main Results

Theorem 1 — *Let $u_0, v_0 \in C[I, E]$ such that $u_0 < v_0$ and $f : I \times \Omega \to E$. Suppose that*

(i) $f(t, u) : I \times \Omega \to E$ satisfies weak Caratheodory conditions and suppose that $f(t, u_0(t)), f(t, v_0(t)) \in L_p[I, E]$;

(ii) there exist constants $L > 0$, $\beta \in [0, M]$ such that

$$-\beta(v - u) \leq f(t, v) - f(t, u) \leq L(v - u),$$

for $\quad u_0(t) \leq u \leq v \leq v_0(t), t \in I$; *and*

$$\text{(iii)} \quad u_0(t) \leq h(t) + \int_0^t k(t, s) [f(s, u_0(s)) + Mu_0(s)] \, ds,$$

$$v_0(t) \geq h(t) + \int_0^t k(t, s) [f(s, v_0(s)) + Mv_0(s)] \, ds,$$

then integral equation (1) has a unique solution $\bar{u}$ in $[u_0, v_0]$ and the iterative sequence for any $x_0(t) \in [u_0, v_0]$,

$$x_n(t) = h(t) + \int_0^t k(t, s)\,[f(s, x_{n-1}(s)) + Mx_{n-1}(s)]\,ds, \quad n = 1, 2, 3, \ldots \qquad \ldots (3)$$

converges uniformly to $\bar{u}$ *on* I *and have the following error estimate*

$$\| x_n - \bar{u} \|_c \leq \frac{2N\,[Nk_0\,(M + L)]^n}{n\,!} \cdot \| v_0 - u_0 \|_c, \quad n = 1, 2, 3, \ldots \qquad \ldots (4)$$

where $\qquad k_0 = max\ \{k(t, s) \mid (t, s) \in D\}$.

PROOF : For any $u(t) \in [u_0, v_0]$, we define the mapping A by

$$Au(t) = h(t) + \int_0^t k(t, s)\,[f(s, u(s)) + Mu(s)]\,ds.$$

It is easy to see by condition (i) that $A : [u_0, v_0] \rightarrow C[I, E]$ and $u(t)$ is a solution of integral equation (1) if and only if u is a fixed point of the A, i.e., $u = Au$.

Now, we prove that $A : [u_0, v_0] \rightarrow [u_0, v_0]$ is the increasing operator. In fact, suppose $u, v \in [u_0, v_0]$ such that $v \geq u$, then, by condition (ii), we have

$$Av(t) - Au(t) = \int_0^t k(t, s)\,[f(s, v(s)) + Mv(s) - f(s, u(s)) - Mu(s)]\,ds$$

$$\geq \int_0^t k(t, s)\,[-\beta\,(v(s) - u(s)) + M(v(s) - u(s))]\,ds$$

$$= (M - \beta) \int_0^t k(t, s)\,[v(s) - u(s)]\,ds \geq \theta,\ t \in I.$$

Therefore, the A is a increasing operator on $[u_0, v_0]$. By condition (*iii*), it is easy observe that $u_0 \leq Au_0, Av_0 \leq v_0$. Thus, $A : [u_0, v_0] \rightarrow [u_0, v_0]$ is the increasing operator.

Set $u_n = Au_{n-1}, v_n = Av_{n-1}, n = 1, 2, 3, \ldots$.

By induction and the monotonicity of A, it is easy to verify that

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0, \qquad \ldots (5)$$

by (5) and condition (ii), we have

$$\theta \leq v_n(t) - u_n(t) = Av_{n-1}(t) - Au_{n-1}(t)$$

$$= \int_0^t k(t, s)\,[f(s, v_{n-1}(s)) + Mv_{n-1}(s) - f(s, u_{n-1}(s)) - Mu_{n-1}(s)]\, ds$$

$$\leq (M + L)\, k_0 \int_0^t [v_{n-1}(s) - u_{n-1}(s)]\, ds, n = 1, 2, 3, \ldots . \quad \ldots (6)$$

From the (6) and the normality of cone P, it is easy to prove by induction that

$$\| v_n(t) - u_n(t) \| \leq \frac{[Nk_0 (M + L)]^n}{n!} \cdot \| v_0 - u_0 \|_c \cdot t^n, t \in I, n = 1, 2, 3, \ldots . \quad \ldots (7)$$

Hence, by (7) we get

$$\| v_n - u_n \|_c \leq \frac{[Nk_0 (M + L)]^n}{n!} \cdot \| v_0 - u_0 \|_c, n = 1, 2, 3, \ldots . \quad \ldots (8)$$

It follows from (5) that for any positive integer m

$$\theta \leq u_{n+m} - u_n \leq v_n - u_n, \ \theta \leq v_n - v_{n+m} \leq v_n - u_n, \quad \ldots (9)$$

hence by (8), (9) and the normality of cone P, we get

$$\max \left\{ \| u_{n+m} - u_n \|_c, \| v_n - v_{n+m} \|_c \right\} \leq N \| v_n - u_n \|_c$$

$$\leq \frac{N[Nk_0 (M + L)]^n}{n!} \cdot \| v_0 - u_0 \|_c, \ n = 1, 2, 3, \ldots . \quad \ldots (10)$$

The results imply that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $C[I, E]$, hence there exist $u^*, v^* \in C[I, E]$ such that $u_n(t) \to u^*(t), v_n(t) \to v^*(t)$ as $n \to \infty$ uniformly on $t \in I$.

Obviously, $u_n \leq u^* \leq v^* \leq v_n$ $(n = 1, 2, 3, \ldots)$, therefore,

$$\theta \leq v^* - u^* \leq v_n - u_n, \ n = 1, 2, 3, \ldots,$$

hence by the normality of cone P and from (8), we get

$$u^* = v^* \equiv \bar{u} \in [u_0, v_0] \text{ and } u_{n-1} \leq \bar{u} \leq v_{n-1} \ (n = 1, 2, 3, \ldots). \quad \ldots (11)$$

By the monotonicity of A and (11), we have

$$u_n = Au_{n-1} \leq A\bar{u} \leq Av_{n-1} = v_n, \ n = 1, 2, 3, \ldots . \quad \ldots (12)$$

Thus, letting $n \to \infty$ in (12) and by the normality of cone P, we know that

$$\bar{u} \leq A\,\bar{u} \leq \bar{u},$$

i.e., $\bar{u} = A\,\bar{u}$. This implies $\bar{u}$ is a solution of Eq. (1) in $[u_0, v_0]$.

Next we prove that $\bar{u}$ is the unique solution of eq. (1) in $[u_0, v_0]$. In fact, suppose $\bar{v} \in [u_0, v_0]$ is also a solution of eq. (1), then from $u_0 \leq \bar{v} = A\bar{v} \leq v_0$ we have that $u_1 \leq \bar{v} \leq v_1$. By induction, it is easy to prove that

$$u_n \leq \bar{v} \leq v_n \ (n = 1, 2, 3, \ldots). \qquad \ldots (13)$$

Letting $n \to \infty$ in (13) and by the normality of cone P, we have

$$\bar{u} \leq \bar{v} \leq \bar{u},$$

i.e., $\bar{v} = \bar{u}$. Therefore, $\bar{u}(t)$ is the unique solution of eq. (1) in $[u_0, v_0]$.

Finally, it is easy to prove by (3) and induction that

$$u_n \leq x_n \leq v_n,\ n = 1, 2, 3, \ldots, \qquad \ldots (14)$$

for any $x_0 \in [u_0, v_0]$. Hence by (10), (14), we have

$$\| x_n - \bar{u} \|_c \leq \| x_n - u_n \|_c + \| u_n - \bar{u} \|_c \leq 2N \| v_n - u_n \|_c$$

$$= \frac{2N\,(Nk_0\,(M+L))^n}{n\,!} \cdot \| v_0 - u_0 \|_c,\ n = 1, 2, 3, \ldots,$$

i.e., error estimate (4) holds. Thus the proof is complete.

Remark 1 : It should be pointed out that we do not use any conditions of compactness and continuity.

4. Application to Nonlinear Differential Equations

In this section, we shall use the results in section 3 to study existence and uniqueness problem of weak caratheodory solutions for the initial value problem (2). For this purpose, we first introduce the following notion:

An abstract function $x(t)$ is called a weak Caratheodory solution of eq. (2), if $f : I \times E \to E$ satisfies weak Caratheodory conditions and $x(t)$ satisfies Eq. (2).

Theorem 2 — *Let $u_0, v_0 \in C[I, E]$ such that $u_0 < v_0$, suppose that the hypothesis (i) and (ii) of Theorem 1 are satisfied, and suppose further that*

$$\text{(H)}\ \ u_0'(t) \leq f(t, u_0(t)),\ u_0(0) \leq x_0;\ \ v_0'(t) \geq f(t, v_0(t)),\ v_0(0) \geq x_0.$$

Then initial value problem (2) has a unique weak Caratheodory solution w^* in $[u_0, v_0]$ and the iterative sequence for any $w_0(t) \in [u_0, v_0]$

$$w_n(t) = e^{-Mt}\left[x_0 + \int_0^t (f(s, w_{n-1}(s)) + Mw_{n-1}(s))\, e^{Ms}\, ds \right],\ n = 1, 2, 3, \ldots \ \ldots (15)$$

converges uniformly to w^* on I and have the following error estimate

$$\| w_n - w^* \|_c \leq \frac{2N\,[N(M+L)]^n}{n\,!} \cdot \| v_0 - u_0 \|_c,\ n = 1, 2, 3, \ldots . \qquad \ldots (16)$$

PROOF : Obviously, the initial value problem (2) is equivalent to the following integral equation

$$u(t) = x_0\, e^{-Mt} + \int_0^t e^{-M(t-s)}\, [f(s, u(s)) + Mu(s)]\, ds. \qquad \ldots (17)$$

Set $h(t) = x_0\, e^{-Mt}$, $k(t, s) = e^{-M(t-s)}$, $(t, s) \in D$. Evidently, $h(t) \in C[I, E]$, $k(t, s) \in C[D, R^+]$ and $k_0 = \max\ \{k(t, s) \mid (t, s) \in D\} \leq 1$.

It is easy to see by condition (H) that

$$[u_0\,(t)\, e^{Mt}]' \leq e^{Mt}\, [f(t, u_0\,(t)) + Mu_0\,(t)],\ u_0\,(0) \leq x_0;$$

and

$$[v_0\,(t)\, e^{Mt}]' \geq e^{Mt}\, [f(t, v_0\,(t)) + Mv_0\,(t)],\ v_0\,(0) \geq x_0.$$

Thus, we have

$$u_0\,(t) \leq x_0\, e^{-Mt} + \int_0^t e^{-M(t-s)}\, [f(s, u_0(s)) + Mu_0\,(s)]\, ds$$

and

$$v_0\,(t) \leq x_0\, e^{-Mt} + \int_0^t e^{-M(t-s)}\, [f(s, v_0(s)) + Mv_0\,(s)]\, ds.$$

By above discussion it is easy know that conditions (i)-(iii) in Theorem 1 are satisfied for integral equation (17). The conclusion of Theorem 2 follows from Theorem 1.

This completes the proof of the Theorem 2.

Remark 2 : Obviously, Theorem 2 is a popularization and improvement of Theorem 1 in [3].

Remark 3 : In order to investigate differential and integral equations in Banach spaces, the compactness type conditions and continuity type conditions are used widely. But, the results of this paper, do not use any condition of the aspects.

REFERENCES

1. Adrian Constantin, *J. math. Anal. Appl.*, **205** (1997), 280-83.
2. Liu Lishan, *Indian J. pure appl. Math.*, **27** (1996), 959-72.
3. Song Fumin, *Acta Mathematica Sinica*, **41** (1998), 1265-72 (*in Chinese*).
4. N. Dunford and J. Schwartz, *Linear Operator (I)*, New York, 1958.
5. D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston and New York, 1988.
6. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.