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MIKUSIŃSKI TYPE PRODUCTS OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

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In this paper, the well-known equation: $x^{-1} \cdot x^{-1} - \pi^2 \delta(x) \cdot \delta(x) = x^{-2}, x \in \mathbb{R}$, is generalized for the distributions $x^{-p}, \delta^{(q-1)}(x)$, by arbitrary $p, q \in \mathbb{N}$, as they are embedded in Colombeau algebra of tempered generalized functions. Two more Colombeau products of this type concerning the distributions $x^{-p}, \delta^{(q-1)}(x)$, and $(x + i0)^{-p}$ are derived as well.

Key Words : Distribution; Product; Colombeau Algebra; Tempered Generalized Function

INTRODUCTION

Owing to the large employment of Schwartz distributions in the natural sciences and other mathematical fields, where products of distributions with coinciding singularities often appear, the problem of multiplication of distributions has been for a long time objective of many studies. Starting with the historically first construction of König[14] and the sequential approach developed by Mikusinski and co-authors[2], there have been numerous attempts to define products of distributions, or rather to enlarge the range of existing products (see [16] for complete review and bibliography).

Furthermore, several attempts have been made to include the distributions into algebras of generalized functions with differentiation always possible and subject to the Leibniz rule (differential algebras). According to the classical Schwartz counter-examples, however, in associative algebras of generalized functions, multiplication and differentiation cannot simultaneously extend the corresponding classical operations unrestrainedly. One, therefore, has to reduce the requirements on the multiplication. So far, most complete list of such properties possesses the differential algebra $\mathcal{G}$ of generalized functions of Colombeau[3]. The distribution space $\mathcal{D}'$ is $\mathbb{C}$-linearly embedded in it, and the multiplication is compatible with the differentiation and products with C^∞-differentiable functions. The notion of 'association' in Colombeau algebra, which is a faithful generalization of the equality in $\mathcal{D}'$, is particularly useful for evaluation of distribution products - as they are embedded in $\mathcal{G}$ - in terms of distributions again.

Recall now the following well-known result published by Mikusinski[15] in 1966 :

$$x^{-1} \cdot x^{-1} - \pi^2 \delta(x) \cdot \delta(x) = x^{-2}, x \in \mathbb{R}. \qquad \text{... (1)}$$

Though neither of the products on the left-hand side of (1) exists, their sum still has a correct meaning in the distribution space. Another formula of that type - in a nonstandard approach to distribution theory - was given by Raju[17] :

$$H \cdot \delta' (x) + \delta (x) \cdot \delta (x) \stackrel{*}{=} \frac{1}{2} \delta' (x), x \in \mathbb{R}. \qquad \dots (2)$$

Here H is the Heaviside function and '$\stackrel{*}{=}$' stands for equality up to an infinitesimal quantity.

Equations of this kind can be found in the mathematical and physical literature. We proposed the name 'products of Mikusinski type' for such equations in a previous work[6], where a generalization of (2) was obtained in the Colombeau algebra (see equation (7) below). In this paper, we generalize the basic Mikisinski formula (1) for the distributions x^{-p} and $\delta^{(q-1)}(x)\ (x \in \mathbb{R})$ for arbitrary $p, q \in \mathbb{N}$ (Theorem 4.2). This is done following the method of Mikusinski of applying a 'Fourier-product' formula, but in the setting of Colombeau algebra of tempered generalized functions. As intermediate results, we obtain the product in Colombeau algebra of the distribution $(x + i0)^{-p}\ (x \in \mathbb{R})$ for different $p \in \mathbb{N}$ (Theorem 3.3), and derive still another Mikusinski type product for the distributions x^{-p} and $\delta^{(q-1)}(x)$, by arbitrary $p, q \in \mathbb{N}$ (Theorem 4.1).

1. Definitions and Preliminary Results

We start by recalling the fundamentals of Colombeau theory, restricting ourselves to the algebra $\mathcal{G}_\tau$ of tempered generalized functions of one variable. It contains the space S' of tempered distributions on $\mathbb{R}$ and is appropriate for the considerations we envisage in this paper.

Notation — Let $\mathbb{N}$, $\mathbb{N}_0$ stand for the natural numbers, respectively, the nonnegative integers and $\delta_{mn} = \{1$ for $m = n$, $= 0$ otherwise; $m, n \in \mathbb{N}_0\}$. If q is $\in \mathbb{N}_0$, we put $A_q = \{\varphi(x) \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} x^n \varphi(x)\, dx = \delta_{0n}$ for each $n \in \mathbb{N}_0, \leq q\}$. We will denote $\varphi_\varepsilon = \varepsilon^{-1} \varphi(\varepsilon^{-1} x)$ for $\varphi \in A_0'$ and $\varepsilon > 0$.

Definition 1.1 — Let $\varepsilon[\mathbb{R}]$ be the set of functions $f(\varphi, x) : A_0 \times \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable with respect to x by a fixed 'parameter' φ; which is a $\mathbb{C}$-algebra with the pointwise function operations. Let its subalgebra $\varepsilon_{M,\tau}[\mathbb{R}]$ be the subset of $\varepsilon[\mathbb{R}]$ of 'moderate' functions $f(\varphi, x)$ in $\varepsilon[\mathbb{R}]$ such that for each $p \in \mathbb{N}_0$ there is a $q \in \mathbb{N}_0$ such that : for each $\varphi \in A_q$ there are $c > 0, \eta > 0$ satisfying $|\partial^p f(\varphi_\varepsilon, x)| \leq c\,(1 + |x|^q)\, \varepsilon^{-q}$ for all $x \in \mathbb{R}$ and $0 < \varepsilon < \eta$. The symbol τ stands for 'tempered'. The ideal $\mathcal{N}_\tau[\mathbb{R}]$ of $\varepsilon_{M,\tau}[\mathbb{R}]$ is the set functions $f(\varphi, x)$ such that for each $p \in \mathbb{N}_0$ there is $q \in \mathbb{N}$ such that : for every $r \geq q$ and each $\varphi \in A_r(\mathbb{R})$ there are $c > 0, \eta > 0$ satisfying $|\partial^p f(\varphi_\varepsilon, x)| \leq c(1 + |x|^r\, \varepsilon^{r-q}$, for all $x \in \mathbb{R}$ and $0 < \varepsilon < \eta$. Then the tempered generalized functions are defined as elements of the quotient algebra $\mathcal{G}_\tau = \varepsilon_{M,\tau}[\mathbb{R}]/\mathcal{N}_\tau[\mathbb{R}]$.

The algebra $\mathcal{G}_\tau$ contains the tempered distributions[4], canonically embedded by the map $i : S' \to \mathcal{G}_\tau : u \mapsto \{\tilde{u}(\varphi, x) = (u * \check{\varphi})(x)\}$, where $\check{\varphi}(x) = \varphi(-x)$ and φ is running the set A_0. Basic examples are the embeddings $\widetilde{x_+^p}$, $\widetilde{x^{-p}}$, and $\widetilde{\delta^{(p)}}(x)$ of the distributions $x_+^p = \{x^p$ for $x \geq 0, = 0$ for $x < 0\}$, $x^{-p} = (-1)^{p-1}/(p-1)!\, \partial^p(\ln |x|)$, and $\delta^{(p)}(x), p \in \mathbb{N}$. We note that similar, yet different schemes of 'new generalized functions' were introduced by Antonevich and Radyno[1] and by Egorov[7], and applied to various problems in Analysis.

We recall some properties of the parameter functions $\varphi \in A_q$, that will be in use later. From the equation for their Fourier transform

$$\hat{\varphi}_\varepsilon(x) = \int_{I\!R} e^{-ixy} \varphi_\varepsilon(y)\, dy = \varepsilon^{-1} \int_{I\!R} e^{-ixy} \varphi\left(\frac{y}{\varepsilon}\right) dy = \int_{I\!R} {}^{-ixt} \varphi(t)\, dt = \hat{\varphi}(\varepsilon x)$$

and the definition of A_q, it follows that, for any q in $I\!N$ and $\varphi \in A_q$,

$$\hat{\varphi}(0) = 1, \; \hat{\varphi}^{(j)}(0) = 0 \text{ for all } j \in I\!N_0, j < q. \qquad \text{... (3)}$$

Then, from the Taylor formula up to order $q + 1$, we easily obtain the estimation :

$$|\hat{\varphi}(\varepsilon x) - 1| \le \frac{c(\varphi)\, \varepsilon^{q+1}}{1 + |x|^r}, \text{ for each } r \in I\!N, x \in I\!R. \qquad \text{... (4)}$$

Further, for any $f \in \mathcal{G}_\tau$ with representative $f(\varphi_\varepsilon, x)$ the integral $\int_{I\!R} f(x)\, dx$ is defined as

$$\int_{I\!R} f(\varphi_\varepsilon, x)\, \hat{\varphi}_\varepsilon(x)\, dx = \int_{I\!R} f(\varphi_\varepsilon, x)\, \hat{\varphi}(\varepsilon x)\, dx; \qquad \text{... (5)}$$

which makes sense since $\hat{\varphi} \in S$. By this, $\hat{\varphi}(\varepsilon x)$ can be omitted in (5) whenever $f(\varphi_\varepsilon, x)$ is supported in a compact subset of $I\!R$, thus coming to the classical integral.

Since the equality in Colombeau algebra is very strict, the following weaker concepts for 'association' are introduced.

Definition 1.2 — Two generalized functions f, g of $\mathcal{G}_\tau$ are said to be strongly associated, written as $f \overset{s}{\approx} g$, if for any $\psi \in S$, denoting $d_\psi(\varphi_\varepsilon) = \int_{I\!R} [f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)]\, \psi(x)\, \hat{\varphi}(\varepsilon x)\, dx$, there exist $N \in I\!N_0$, such that for any $q \ge N$, $\varphi \in A_q$ there exist $c, \eta > 0$ such that $|d\,\psi(\varphi_\varepsilon)| \le c\, \varepsilon^{q-N}$ for all $\varepsilon \in (0, \eta)$.

Definition 1.3 — Two functions $f, g \in \mathcal{G}_\tau$ are said to be associated (in weak sense), denoted as $f \approx g$, if for some representatives of theirs $f(\varphi_\varepsilon, x), g(\varphi_\varepsilon, x)$ and for each $\psi(x) \in S$ there is $q \in I\!N_0$ such that, for all $\varphi(x) \in A_q$, $\lim_{\varepsilon \to 0_+} \int_{I\!R} [f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)]\, \psi(x)\, \hat{\varphi}(\varepsilon x)\, dx = 0$.

It is easily seen that strong association implies weak association in $\mathcal{G}_\tau$. Another version of the latter definition is that of associated distribution.

Definition 1.4 — A generalized function $f \in \mathcal{G}_\tau$ is said to admit some $u \in S'$ as an 'associated distribution', denoted by $f \approx u$, if for some representative $f(\varphi_\varepsilon, x) \in \varepsilon_{M,\tau}[I\!R]$ of the function f and for each $\psi(x) \in S$ there is a $q \in I\!N_0$ such that, for all $\varphi(x) \in A_q$,

$$\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x)\, \psi(x)\, \hat{\varphi}(\varepsilon x)\, dx = \langle u, \psi \rangle.$$

Remark : Definitions 1.2-4 are independent of the representative chosen. The distribution associated by Definition 1.4, if it exists, is unique. The embedding of every distribution in Colombeau algebra is associated with the latter[3], the association thus being a generalization of the equality of distributions in classical distribution theory. This fact also implies that Definitions 1.3 and 1.4 are equivalent on the embedding of distributions :

$$f \approx \tilde{u} \Leftrightarrow f \approx u \text{ for any } f \in \mathcal{G}_\tau,\, u \in \mathcal{S}'. \qquad \text{... (6)}$$

Now, by a distribution product in Colombeau algebra, sometimes called 'Colombeau product', is meant the product of some distributions as they are embedded in Colombeau algebra, whenver the result admits an associated distribution (see [12] for comparison with other distribution products, and [5] for particular results). The following Mikusinski type product in Colombeau algebra that generalizes the result in (2) for arbitrary $p \in \mathbb{N}_0$ can be now proved[6]:

$$\frac{(-1)^p}{p\,!}\, \widetilde{x_+^p} \cdot \widetilde{\delta^{(p+1)}}(x) + \tilde{\delta}(x) \cdot \tilde{\delta}(x) \approx \frac{(p+1)}{2}\, \delta'(x). \qquad \text{... (7)}$$

We recall also two results on Colombeau product that will be needed in the sequel. First, for any multiindex $p \in N^d$, it holds[6]:

$$\widetilde{x^{-p}} \cdot \widetilde{\delta^{(p-1)}}(x) \approx \frac{(-1)^p\,(p-1)\,!}{2^d\,(2p-1)\,!}\, \delta^{(2p-1)}(x),\, x \in \mathbb{R}^d. \qquad \text{... (8)}$$

Note that eq. (8) was derived in the one-dimensional case by Fisher[8] and Itano[11] as regularized model product[13, 16] under (different) additional requirements on the regularizing δ-nets.

The $\approx$-association is consistent with the linear operations in Colombeau algebra, but it holds only a 'weak' version of the formula for partial derivatives ∂_i (i = 1, ..., d) of the Colombeau product of distributions on $\mathbb{R}^d$. Namely, the second result we shall need is the following.

Theorem 1.1[6] — *Let the embeddings of the distributions u, v and the distribution w satisfy* $\tilde{u} \cdot \tilde{v} \approx w$. *Then it holds*

$$\widetilde{\partial_i u} \cdot \tilde{v} + \tilde{u} \cdot \widetilde{\partial_i v} \approx \partial_i w,\; i = 1, 2, ..., d. \qquad \text{... (9)}$$

Remark : In general, only the sum on the left-hand side of (9) has an associated distribution, but not the individual summands in it. Clearly, this assertion extends to Mikusinski type products in Colombeau algebra as well.

Now, in compliance with definition (5) for integral in $\mathcal{G}_\tau$, the Fourier transform $\mathcal{F}f \equiv \hat{f}$, the inverse Fourier transform $\mathcal{F}^{-1} f \equiv \check{f}$, and the convolution of $f, g \in \mathcal{G}_\tau$ are introduced by the following equations for the representatives :

$$\hat{f}(\varphi_\varepsilon, x) = \int_{\mathbb{R}} e^{-ixy} f(\varphi_\varepsilon, y)\, \hat{\varphi}(\varepsilon y)\, dy,\; \check{f}(\varphi_\varepsilon, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixy}\, f(\varphi_\varepsilon, y)\, \hat{\varphi}(\varepsilon y)\, dy \qquad \text{... (10)}$$

$$(f * g)(\varphi_\varepsilon, x) = \int_{I\!R} f(\varphi_\varepsilon, x - y)\, g(\varphi_\varepsilon, y)\, \hat{\varphi}(\varepsilon y)\, dy. \qquad \dots (11)$$

It can be shown[18] that these definitions preserve the representative classes of generalized functions in $\mathcal{G}_\tau$ as well as the association relations. Moreover, the following basic properties were demonstrated by Colombeau[4] for any function f in $\mathcal{G}_\tau$:

$$\text{(a) } \mathcal{F}^{-1}\mathcal{F}f \overset{s}{\approx} f \text{ (b) } f * \tilde{\delta} \overset{s}{\approx} f. \qquad \dots (12)$$

Note that the corresponding strict equalities are not valid in the algebra $\mathcal{G}_\tau$.

2. Exchange Formulas in Colombeau Algebra $\mathcal{G}_\tau$

We shall need an 'exchange formula' between the operations of multiplication and convolution, via the Fourier transform. Such a formula was proved by Colombeau[4] in a strong-association sense when one of the multipliers is decreasing fast at infinity. Again, it is not valid as a strict equality in $\mathcal{G}_\tau$. We will prove an exchange formula in weak-association sense enlarging its validity for arbitrary generalized functions in $\mathcal{G}_\tau$. We formulate it for the inverse Fourier transform in order to get later a 'Fourier-product' equation appropriate for our purposes.

Theorem 2.1 — *For each two generalized functions* $f_{1,2}$ *in* $\mathcal{G}_\tau$ *it holds*

$$\bar{f}_1 \cdot \bar{f}_2 \approx \frac{1}{2\pi}\, \overline{f_1 * f_2}. \qquad \dots (13)$$

Proof : Taking into account Definition 1.4 and eqs. (10), (11), we have to evaluate the difference between the two sides of (13) for some representatives of $f_{1,2}$ and an arbitrary $\psi \in S$:

$$\Delta_\psi(\varphi_\varepsilon) := \int_{I\!R} \left[\frac{1}{2\pi}\overline{f_1 * f_2}(\varphi_\varepsilon, x) - \bar{f}_1(\varphi_\varepsilon, x)\bar{f}_2(\varphi_\varepsilon, x) \right] \psi(x)\, \hat{\varphi}(\varepsilon x)\, dx$$

$$= \frac{1}{4\pi^2} \int_{I\!R} \left[\int_{I\!R \times I\!R} e^{ixy} f_1(\varphi_\varepsilon, y - z) f_2(\varphi_\varepsilon, z)\, \hat{\varphi}(\varepsilon z)\, \hat{\varphi}(\varepsilon y)\, dz\, dy \right.$$

$$\left. - \int_{I\!R \times I\!R} e^{ix(y_1 + y_2)} f_1(\varphi_\varepsilon, y_1) f_2(\varphi_\varepsilon, y_2)\, \hat{\varphi}(\varepsilon y_1)\, \hat{\varphi}(\varepsilon y_2)\, dy_1\, dy_2 \right] \psi(x)\, \hat{\varphi}(\varepsilon x)\, dx$$

$$= \frac{1}{4\pi^2} \int_{(I\!R)^3} e^{ixy} f_1(\varphi_\varepsilon, y - z) f_2(\varphi_\varepsilon, z)\, \hat{\varphi}(\varepsilon z)$$

$$[\hat{\varphi}(\varepsilon y - \varepsilon z) - \hat{\varphi}(\varepsilon y)]\, \psi(x)\, \hat{\varphi}(\varepsilon x)\, dz\, dy\, dx, \qquad \dots (14)$$

where the change $y_1 + y_2 = y$, $y_2 = z$ is made.

Evaluate first the integral in z. Suppose that, in accordance with Definition 1.1, we have the estimate for the representatives

$$|f_1(\varphi_\varepsilon, z)| \le \varepsilon^{-l}(1+|z|^l), \; |f_2(\varphi_\varepsilon, z)| \le \varepsilon^{-m}(1+|z|^m), \qquad \ldots (15)$$

for some $l, m \in I\!N_0$ and all $z \in I\!R$. Supposing further that $\varphi \in A_q$, we fix some $p \in I\!N, p < q$. Replace then $\hat{\varphi}(\varepsilon y - \varepsilon z)$ with its Taylor expansion, first around the point εy and then - around 0, up to order p. Taking into account eq. (3), we get

$$\hat{\varphi}(\varepsilon y - \varepsilon z) - \hat{\varphi}(\varepsilon y) = (-\varepsilon)^p \sum_{j=1}^{p} \frac{(-1)^j}{j!\,(p-j)!} z^j y^{p-j} \hat{\varphi}_j^{(p)}(\varepsilon \theta_j y). \qquad \ldots (16)$$

Here each of the parameters θ_j ($j = 1, \ldots, p$) is a fixed number in the interval (0, 1). Further, the next estimate of any $\varphi \in A_0$ easily follows from (4) :

$$|\hat{\varphi}(\varepsilon z)| \le \frac{1+o(1)}{1+|z|^r}, \text{ as } \varepsilon \to 0_+, \text{ for all } r \in I\!N_0, z \in I\!R. \qquad \ldots (17)$$

Now, taking into account (15) and (17), we obtain the following estimate, for any $j = 1, \ldots, p$:

$$|I_j(y)| := \left| \int_{I\!R} f_1(\varphi_\varepsilon, y-z) f_2(\varphi_\varepsilon, z)\, \hat{\varphi}(\varepsilon z)\, z^j\, dz \right|$$

$$\le \varepsilon^{-l-m} \int_{I\!R} (1+|y-z|^l)(1+|z|^m)\, |z|^p (1+|z|^r)^{-1}\, dz$$

$$\le c \cdot \varepsilon^{-l-m-n}[1+o(1)](1+|y|^n), \text{ for some } c \in I\!R_+ \text{ and } n \in I\!N. \qquad \ldots (18)$$

Taking into account eqs. (16) and (18), we evaluate further

$$4\pi^2 |T(\varphi_\varepsilon)| := \left| \int_{I\!R \times I\!R} e^{ixy} f_1(\varphi_\varepsilon, y-z) f_2(\varphi_\varepsilon, z)\, \hat{\varphi}(\varepsilon z) - \hat{\pi}(\varepsilon y)]\, dz\, dy \right| \qquad \ldots (19)$$

$$= \left| (-\varepsilon)^p \int_{I\!R} e^{ixy} \sum_{j=1}^{p} \frac{(-1)^j y^{p-j} \hat{\varphi}^{(p)}(\varepsilon \theta_j y)}{j!\,(p-j)!} I_j(y)\, dy \right|$$

$$\le c \cdot \varepsilon^{p-l-m-n}[1+o(1)] \int_{I\!R} (1+|y|^n)\, |\hat{\varphi}_p(\varepsilon y)|\, dy, \text{ for some } n \in I\!N_0. \qquad \ldots (20)$$

Here we have put

$$\hat{\varphi}_p(\varepsilon y) := \sum_{j=1}^{p} \max_{0<\theta_j \le 1} \left| \frac{y^{p-j} \hat{\varphi}^{(p)}(\varepsilon \theta_j y)}{j!\,(p-j)!} \right|.$$

Now we can choose the parameter $q \in I\!N$ to be such that $q - l - m - n =: t > 0$. Then, $\hat{\varphi}_p(\varepsilon y)$ is a function in S such that $\hat{\varphi}_p^{(k)}(0) = 0$, for $k = 0, \ldots, q - p - 1$, and therefore $|\hat{\varphi}_p(\varepsilon y)| \leq c(\varphi_\varepsilon) \cdot \varepsilon^{q-p}(1 + |y|^s)^{-1}$, for any $s \in I\!N$. Taking then into account this latter estimate and equation (20), we obtain further

$$4\pi^2 |T(\varphi_\varepsilon)| \leq c \cdot \varepsilon^t [1 + o(1)] \int_{I\!R} (1 + |y|^n)(1 + |y|^s)^{-1} dy \leq c_1 \cdot \varepsilon^t [1 + o(1)]. \quad \ldots (21)$$

Applying successively equations (14), (19), (21) and (17), we finally get the estimate

$$|\Delta_\psi(\varphi_\varepsilon)| = \left| \int_{I\!R} T(\varphi_\varepsilon, x)\, \psi(x)\, \hat{\varphi}(\varepsilon x)\, dx \right|$$

$$\leq c_1 \cdot \varepsilon^t \int_{I\!R} |\psi(x) \cdot (1 + |x|^r)^{-1} [1 + o(1)]\, dx \leq c_2 \cdot \varepsilon^t [1 + o(1)].$$

Now, since $t > 0$, we obtain $\lim_{\varepsilon \to 0_+} \Delta_\psi(\varphi_\varepsilon) = 0$; which completes the proof of the theorem.

Clearly the above proof remains valid for the Fourier transform as well. Now, if we set $\bar{f}_i = g_i$, $i = 1, 2$, equations (12(*a*)) and (13) imply the Fourier-product formula given by this.

Corollary 2.1 — For each two generalized functions $g_{1,2}$ in $\mathcal{G}_\tau$, it holds

$$g_1 \cdot g_2 \approx \frac{1}{2\pi} \overline{\hat{g}_1 * \hat{g}_2}. \quad \ldots (22)$$

3. The Square of $(x + i0)^{-p}$ in Colombeau Algebra

We proceed further to particular tempered distributions embedded in Colombeau algebra $\mathcal{G}_\tau$. Before this, we will prove a basic property of their elements. It is known that the inverse Fourier transform of a distribution with support in $I\!R_+$ can be obtained as a weak limit of its (one-sided) Laplace transform, as the imaginary part of the variable tends to 0. The same holds for the elements of $\mathcal{G}_\tau$ having support in $I\!R_+$.

Lemma 3.1 — For each generalized function $f \in \mathcal{G}_\tau$ supported in $I\!R_+$, it holds

$$\lim_{x \to 0_+} \int_{I\!R_+} e^{izt} f(t)\, dt = 2\pi \bar{f}(x), \text{ where } z = x + i\chi \in \mathbb{C}. \quad \ldots (23)$$

Proof : According to definitions (5) and (10), we have to evaluate the difference between the two sides of (23) for some representative $f(\varphi_\varepsilon, t)$ of f :

$$\Delta(\varphi_\varepsilon) := \lim_{\chi \to 0_+} \int_{I\!R_+} e^{i(x + i\chi)t} f(\varphi_\varepsilon, t)\, \hat{\varphi}(\varepsilon t)\, dt - \int_{I\!R_+} e^{ixt} f(\varphi_\varepsilon, t)\, \hat{\varphi}(\varepsilon t)\, dt$$

$$= \lim_{\chi \to 0_+} \int_{I\!R_+} [e^{-\chi t} - 1]\, e^{ixt} f(\varphi_\varepsilon, t)\, \hat{\varphi}(\varepsilon t)\, dt.$$

Now, $e^{-\chi t} \to 1$, as $\chi \to 0_+$, uniformly regarding x. Since $|e^{-\chi t} - 1| \leq 0$ on the half-line, the last integral above is uniformly convergent at infinity, and we can therefore pass to the limit as $\chi \to 0_+$ under the integral sign. This gives $\Delta(\varphi_\varepsilon) = 0$, and the proof is complete.

Recall next the definition of the distribution

$$(x \pm i0)^{-p} : \psi \mapsto \lim_{\chi \to 0_+} \langle (x \pm i\chi)^{-p}, \psi \rangle, p \in I\!N, \qquad \text{... (24)}$$

for an arbitrary $\psi \in S(I\!R)$. Then we prove the following.

Theorem 3.1 — *For each $p \in I\!N$, the embeddings in $\mathcal{G}_\tau$ of the distributions x_+^{p-1} and $(x + i0)^{-p}$ satisfy* :

$$\mathcal{F}^{1}\,(\widetilde{x_+^{p-1}}) \approx \frac{i^{(p\ 1)}}{2\pi}\, \widetilde{(x + i0) - p}. \qquad \text{... (25)}$$

PROOF : By their definition, the representatives of Colombeau generalized functions satisfy: $\partial_x f(\varphi_\varepsilon, x) = \partial_x [x \mapsto f(\varphi_\varepsilon, x)]$; which applied to the embeddings of the distributions gives $\partial_x \tilde{u}(\varphi_\varepsilon, x) = \widetilde{\partial_x u}(\varphi_\varepsilon, x)$ for each $u \in S'(I\!R)$. So, in particular, we have

$$\partial_x \widetilde{x_+^p}(\varphi_\varepsilon, x) = p\, \widetilde{x_+^{p-1}}(\varphi_\varepsilon, x), p \in I\!N;\ \partial_x \tilde{H}(\varphi_\varepsilon, x) = \tilde{\delta}(\varphi_\varepsilon, x). \qquad \text{... (26)}$$

Applying Lemma 3.1 and eq. (10), we now get for each $p \in I\!N$:

$$\mathcal{F}^{1}\,(\widetilde{x_+^{p-1}})(\varphi_\varepsilon, x) = \lim_{\chi \to 0_+} \frac{1}{2\pi} \int_{I\!R} e^{i(x + i\chi)y}\, \widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\, \hat{\varphi}(\varepsilon y)\, dy$$

$$= \lim_{\chi \to 0_+} \frac{i}{2\pi}(x + i\chi)^{-1} \left[\int_{I\!R} e^{ixy - \chi y}\, \widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\, \varepsilon \hat{\varphi}'(\varepsilon y)\, dy \right.$$

$$\left. + \int_{I\!R} e^{i(x + i\chi)y}\, \partial_y [\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)]\, \hat{\varphi}(\varepsilon y)\, dy \right]$$

$$=: \lim_{\chi \to 0_+} [I_{p-1}(\varphi_\varepsilon, x + i\chi) + J_{p-1}(\varphi_\varepsilon, x + i\chi)].$$

Here, in the integration by parts, we have used that $|t|^n \hat{\varphi}(t) \to 0$ as $|t| \to \infty$ for all $n \in I\!N$ and $t \in I\!R$, the integrated term thus being 0. For any $\psi \in S(I\!R)$ and $f \in \mathcal{G}_\tau$ with a representative $f(\varphi_\varepsilon, x)$ denote by $L_\psi(f) := \lim_{\varepsilon \to 0_+} \left\langle \lim_{\chi \to 0_+} \int_{I\!R} 1 \cdot e^{ixy}\, \widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\, \varepsilon \hat{\varphi}'(\varepsilon y)\, dy, \psi(x) \right\rangle$.

Then we have

$$L_{\psi}(I_{p-1}) = \frac{i}{2\pi}(x+i0)^{-1} \left\langle \lim_{\varepsilon \to 0_+} \int_{I\!R} 1 \cdot e^{ixy} \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y)\, \varepsilon \hat{\varphi}'(\varepsilon y)\, dy, \psi(x) \right\rangle .$$

We have passed to the limit as $\chi \to 0_+$ under the integral sign on the same argument as in the proof of Lemma 3.1. Now, it can be checked that (as required by Definition 1.1) the following estimate holds : $\left| \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y) \right| \leq c \cdot \varepsilon^{-p} (1 + |y|^p)$. Choosing some $q \in I\!N, q > p$, we have for any $\varphi \in A_q$, by the Taylor theorem and (3) : $\varepsilon \hat{\varphi}'(\varepsilon y) = \sum_{j=1}^{q-1} (j!)^{-1} \varepsilon^j y^{j-1} \hat{\varphi}^{(j)}(0) + \varepsilon^q \hat{\varphi}^{(q)}(\varepsilon \theta y)$, where $\theta \in (0, 1)$. Denote further $\hat{\varphi}_q(y) := \sup_{\varepsilon \in I\!R_+} \hat{\varphi}^{(q)}(\varepsilon \theta y)$, which is well defined since $\hat{\varphi} \in S(I\!R)$. Due to the estimate $\left| \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y)\, \varepsilon^q \hat{\varphi}^{(q)}(\varepsilon \theta y) \right| \leq \left| \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y)\, \varepsilon^q \hat{\varphi}_q(y) \right| \leq c \cdot \varepsilon^{q-p} (1 + |y|^p)$ $c_1 (1 + |y|^n)^{-1}$ for some $c \in I\!R_+$ and all $n \in I\!N, y \in I\!R$, and by the theorem for dominated convergence, we can pass to the limit as $\varepsilon \to 0_+$ under the integral sign. Since for any $p \in I\!N$ we can choose $q > p$, it holds:

$$L_{\psi}(I_{p-1}) = \frac{i}{2\pi}(x+i0)^{-1} \left\langle \lim_{\varepsilon \to 0_+} \int_{I\!R} e^{ixy}\, \widetilde{x_+^{p-1} (\varphi_\varepsilon, y)}\, \varepsilon^q \hat{\varphi}^{(q)}(\varepsilon \theta y)\, dy, \psi(x) \right\rangle = 0. \quad \ldots(27)$$

Consider next the term J_{p-1} above, by $p = 1$. Taking into relation (6), eqs. (26), and the fact that $\delta(\varphi_\varepsilon, y)$ is supported in a compact subset of $I\!R$, we get :

$$L_{\psi}(J_0) = \lim_{\varepsilon \to 0_+} \left\langle \lim_{\chi \to 0_+} \frac{i}{2\pi}(x+i\chi)^{-1} \int_{I\!R} e^{\chi y} e^{ixy} \tilde{\delta}(\varphi_\varepsilon, y)\, dy, \psi(x) \right\rangle$$

$$= \left\langle \frac{i}{2\pi}(x+i0)^{-1}, \psi(x) \right\rangle . \quad \ldots (28)$$

Employing further eq. (26) and integrating by parts, the integrated part being again 0, we obtain :

$$J_{p-1} = \frac{-i^2}{2\pi}(x = i\chi)^{-2} \int_{I\!R} \widetilde{x_+^{p-2}} (\varphi_\varepsilon, y)\, \hat{\varphi}(\varepsilon y)\, d\, [e^{i(x+\chi)y}]$$

$$= i(x+i\chi)^{-1} [I_{p-2} + J_{p-2}]. \quad \ldots (29)$$

By (27), which holds for each $p \in \mathbb{N}$, we get : $L_\psi(J_{p-1}) = L_\psi(i(x+i\chi)^{-1} J_{p-2})$.

Iterating the above manipulation $(p-2)$ times and taking into account (28), we obtain

$$\lim_{\varepsilon \to 0_+} \langle \mathcal{F}^1 (\widetilde{x_+^{p-1}}) (\varphi_\varepsilon, x), \psi \rangle$$

$$= \lim_{\varepsilon \to 0_+} \left\langle \lim_{\chi \to 0_+} i^{p-1} (x+i\chi)^{-p+1} J_0, \psi \right\rangle \left\langle \frac{i^p}{2\pi} (x+i0)^{-p}, \psi(x) \right\rangle.$$

Thus, with (6) taken into account, proves equation (25) for each $p \in \mathbb{N}$. The proof of the theorem is complete.

Combining now eqs. (12*a*) (in weak-association sense) and (25), we get this.

Corollary 3.1 — For each $p \in \mathbb{N}$, it holds

$$\widetilde{\mathcal{F}(x+i0)^{-p}} \approx 2\pi i^{-p} \widetilde{x_+^{p-1}}. \qquad \text{... (30)}$$

We also need the result of the following.

Theorem 3.2 — *For each $p \in \mathbb{N}$, the embedding in $\mathcal{G}_\tau$ of the distribution x_+^{p-1} satisfies* :

$$\widetilde{x_+^{p-1}} * \widetilde{x_+^{p-1}} \approx \widetilde{x_+^{2p-1}}. \qquad \text{... (31)}$$

PROOF : Applying the defining eq. (11), we obtain for each $p \in \mathbb{N}$:

$$C_{p-1}(x) := ((\widetilde{x_+^{p-1}} * \widetilde{x_+^{p-1}}) (\varphi_\varepsilon, x) = \int_{\mathbb{R}} \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y) \widetilde{x_+^{p-1}} (\varphi_\varepsilon, x-y) \hat{\varphi}(\varepsilon y)\, dy$$

$$= \int_{\mathbb{R}} \widetilde{x_+^{p}} (\varphi_\varepsilon, x-y) \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y)\, \varepsilon \hat{\varphi}'(\varepsilon y)\, dy + \int_{\mathbb{R}} \widetilde{x_+^{p}} (\varphi_\varepsilon, x-y) \widetilde{x_+^{p-2}} (\varphi_\varepsilon, y) \hat{\varphi}(\varepsilon y)\, dy$$

$$=: I_p + J_{p,\, p-2}. \qquad \text{... (32)}$$

In the integration by parts, we have again used that $|t|^n \hat{\varphi}(t) \to 0$ as $|t| \to \infty$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, the integrated term therefore being 0. For an arbitrary $\psi \in S(\mathbb{R})$, we next write

$$\lim_{\varepsilon \to 0_+} \langle I_p, \psi \rangle = \left\langle \lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} \widetilde{x_+^{p}} (\varphi_\varepsilon, x-y) \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y)\, \varepsilon \hat{\varphi}'(\varepsilon y)\, dy, \psi(x) \right\rangle$$

The following estimate holds: $\left| \widetilde{x_+^{p}} (\varphi_\varepsilon, x-y) \widetilde{x_+^{p-1}} (\varphi_\varepsilon, y) \right| \le c \cdot \varepsilon^{-2p+1} \; (1 + |x-y|^p)(1 + |y|^{p-1})$.

Choosing some $q \in \mathbb{N}$, $q > 2p-1$, we have for any $\varphi \in A_q$, by the Taylor theorem and (3): $\varepsilon \hat{\varphi}'(\varepsilon y) = \varepsilon^q \hat{\varphi}^{(q)}(\varepsilon \theta y)$, $\theta \in (0, 1)$. Denote $\hat{\varphi}_q(y) := \sup_{\varepsilon \in \mathbb{R}_+} \hat{\varphi}^{(q)}(\varepsilon \theta y)$. Now, on the same argument as in the proof of (25), we get the estimate :

$$\left| \widetilde{x_+^p}(\varphi_\varepsilon, x-y)\, \widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\, \varepsilon^q\, \hat{\varphi}^{(q)}(\varepsilon\,\theta y) \right|$$

$$\leq c \cdot \varepsilon^{q-2p+1}(1+|x-y|^p(1+|y|^{p-1})(1+|y|^n)^{-1},$$

for some $c \in I\!R_+$ and all $n \in I\!N, x, y \in I\!R$. Since $\psi(x) \in S$, by the theorem for dominated convergence, we can pass to the limit as $\varepsilon \to 0_+$ under the integral signs above. For we can choose $q > 2p - 1$, it therefore holds for each $p \in I\!N$:

$$\lim_{\varepsilon \to 0_+} \langle I_p, \psi \rangle = 0. \qquad \text{... (33)}$$

Employing further eq. (26) and integrating by parts - the integrated part being again 0- we obtain:

$$J_{p,p-2} = - \int_{I\!R} \widetilde{x_+^{p-2}}(\varphi_\varepsilon, y)\, \hat{\varphi}(\varepsilon y)\, d\left[\widetilde{x_+^{p+1}}(\varphi_\varepsilon, x-y)\right] = I_{p+1} = J_{p+1,p-3}.$$

Thus, by (33), which holds for each $p \in I\!N$, we get $\lim_{\varepsilon \to 0_+} \langle J_{p,p-2}, \psi \rangle = \lim_{\varepsilon \to 0_+} \langle J_{p+1,p-3}, \psi \rangle$. Iterating the above manipulation $(p - 1)$ times, taking into account successively eqs. (32), (33), (26), and (12*b*) we eventually get

$$\lim_{\varepsilon \to 0_+} \langle C_{p-1}(x), \psi(x) \rangle = \lim_{\varepsilon \to 0_+} \langle J_{2p-1,-1}, \psi(x) \rangle$$

$$= \lim_{\varepsilon \to 0_+} \left\langle \int_{I\!R} \widetilde{x_+^{2p-1}}(\varphi_\varepsilon, x-y)\, \tilde{\delta}(\varphi_\varepsilon, y)\, \hat{\varphi}(\varepsilon y)\, dy, \psi(x) \right\rangle$$

$$= \lim_{\varepsilon \to 0_+} \left\langle (\widetilde{x_+^{2p-1}} * \tilde{\delta})(\varphi_\varepsilon, x), \psi(x) \right\rangle = \left\langle \widetilde{x_+^{2p-1}}, \psi(x) \right\rangle.$$

This proves eq. (31) for each $p \in I\!N$ and the theorem.

We are now in position to apply the Fourier-product formula to the embedding of the distribution $(x + i0)^{-p}$.

Theorem 3.3 — *For arbitrary* $p, q \in I\!N$, *the embedding in* $\mathcal{G}_\tau$ *of the distribution* $(x+i0)^{-p}$ *satisfies*

$$\widetilde{(x+i0)^{-p}} \cdot \widetilde{(x+i0)^{-q}} \approx \widetilde{(x+i0)^{-p-q}}. \qquad \text{... (34)}$$

PROOF : Applying consecutively the results given by eqs. (22), (30), (31), (25), we have :

$$\widetilde{(x+i0)^{-p}} \cdot \widetilde{(x+i0)^{-q}} \approx \frac{1}{2\pi} \mathcal{F}^{-1}\left[(\mathcal{F}\widetilde{(x+i0)^{-p}}) * (\mathcal{F}\widetilde{(x+i0)^{-q}})\right]$$

$$\approx \frac{1}{2\pi}\frac{4\pi^2}{i^{p+q}}\,\mathcal{F}^{-1}\left(\widetilde{x_+^{p-1}} * \widetilde{x_+^{q-1}}\right) \approx \frac{2\pi}{i^{p+q}}\,\mathcal{F}^{-1}\left(\widetilde{x_+^{p+q-1}}\right) \approx \widetilde{(x+i0)^{-p-q}}.$$

The theorem is proved.

Remark : In view of relation (6), eq. (34) gives also the Colombeau product of the distribution $(x+i0)^{-p}$ for different $p \in \mathbb{N}$. We note that Fisher and Kilicman[9] have obtained eq. (34) as a neutrix product - a setting that allows one to neglect terms diverging with a polynomial growth. Though, working in the framework of Colombeau theory of generalized functions and numbers, one can correctly deal with such terms, yet eq. (34) and its applications are obtained by using only classical numbers and 'converging procedures'.

4. Generalization of the Basic Mikusinski Function

In order to generalize (1), we need to derive a particular product of Mikusinski type. It will be convenient to denote the distribution $x^{-1} = \partial_x (\ln |x|), x \in \mathbb{R}$ by

$$\rho(x) := x^{-1} \text{ and } \rho^{(p)}(x) := \partial^p (x^{-1}) = (-1)^p p!\, x^{-p-1}. \qquad \text{... (35)}$$

With this notation, it now holds the following.

Theorem 4.1 — *For any $p, q \in \mathbb{N}_0$, the embeddings in $\mathcal{G}_\tau$ of the distributions $\rho^{(p)}(x)$ and $\delta^{(q)}(x), x \in \mathbb{R}$, satisfy*

$$\widetilde{\rho^{(p)}}(x) \cdot \widetilde{\delta^{(q)}}(x) + \widetilde{\rho^{(q)}}(x) \cdot \widetilde{\delta^{(p)}}(x) \approx c_{p,q}\, \delta^{(p+q+1)}(x),$$

where

$$c_{p,q} = \frac{-p!\,q!}{(p+q+1)!}. \qquad \text{... (36)}$$

Proof : For any $p \in \mathbb{N}_0$, with the above notations employed, eq. (8) reads

$$\widetilde{\rho^{(p)}} \cdot \widetilde{\delta^{(p)}} \approx \frac{-(p!)^2}{2(2p+1)!}\, \delta^{(2p+1)} = \frac{1}{2} c_{p,p}\, \delta^{(2p+1)}. \qquad \text{... (37)}$$

Since

$$\frac{1}{2} c_{p,p} = \frac{-(p!)^2}{2(2p+1)!} = \frac{-(p!)^2 (2p+2)}{2(2p+1)!\,(2p+2)} = \frac{-p!\,(p+1)!}{(2p+2)!} = c_{p,p+1},$$

applying Theorem 1.1 for the derivative of the product (37), we get

$$\widetilde{\rho^{(p)}} \cdot \widetilde{\delta^{(p+1)}} + \widetilde{\rho^{(p+1)}} \cdot \widetilde{\delta^{(p)}} \approx c_{p,p+1}\, \delta^{(2p+2)}. \qquad \text{... (38)}$$

Differentiate further (38) according to (9):

$$\widetilde{\rho^{(p)}} \cdot \widetilde{\delta^{(p+2)}} + \widetilde{\rho^{(p+2)}} \cdot \widetilde{\delta^{(p)}} + 2\,\widetilde{\rho^{(p+1)}} \cdot \widetilde{\delta^{(p+1)}} \approx c_{p,p+1}\, \delta^{(2p+3)}.$$

Taking now into account eq. (37) with p replaced by $p + 1$ and that

$$c_{p,p+1} - c_{p+1,p+1} = \frac{-p\,!\,(p+1)\,!}{(2p+2)\,!}\left(1 - \frac{p+1}{2p+3}\right) = c_{p,p+2},$$

we obtain

$$\widetilde{\rho^{(p)}} \cdot \widetilde{\delta^{(p+2)}} + \widetilde{\rho^{(p+2)}} \cdot \widetilde{\delta^{(p)}} \approx c_{p,p+2}\, \delta^{(2p+3)}, \qquad \text{... (39)}$$

for any $p \in \mathbb{N}_0$.

Having checked eq. (36) for the pairs (p, p), $(p, p + 1)$ and $(p, p + 2)$, we now suppose that the following equations hold for any $p \in \mathbb{N}_0$ and some $n \in \mathbb{N}_0, n \geq 5$:

$$\widetilde{\rho^{(p)}} \cdot \widetilde{\delta^{(p+n-2)}} + \widetilde{\rho^{(p+n-2)}} \cdot \widetilde{\delta^{(p)}} \approx c_{p,p+n-2}\, \delta^{(2p+n-1)}, \qquad \text{... (40)}$$

and

$$\widetilde{\rho^{(p)}} \cdot \widetilde{\delta^{(p+n-1)}} + \widetilde{\rho^{(p+n-1)}} \cdot \widetilde{\delta^{(p)}} \approx c_{p,p+n-1}\, \delta^{(2p+n)}. \qquad \text{... (41)}$$

We shall prove eq. (36) for the case $(p, p + n)$. Applying Theorem 1.1 to the product (41), we get

$$\widetilde{\rho^{(p+1)}} \cdot \widetilde{\delta^{(p+n-1)}} + \widetilde{\rho^{(p+n-1)}} \cdot \widetilde{\delta^{(p+1)}} + \widetilde{\rho^{(p)}} \cdot \widetilde{\delta^{(p+n)}} + \widetilde{\rho^{(p+n)}} \cdot \widetilde{\delta^{(p)}}$$

$$\approx c_{p,p+n-1}\, \delta^{(2p+n+1)}$$

Taking then into account (40) with p replaced by $p + 1$, we obtain

$$\widetilde{\rho^{(p+1)}} \cdot \widetilde{\delta^{(p+n)}} + \widetilde{\rho^{(p+n)}} \cdot \widetilde{\delta^{(p)}} \approx (c_{p,p+n-1} - (c_{p+1,p+n-1})\, \delta^{(2p+n+1)} \qquad \text{... (42)}$$

$$= \frac{-p\,!\,(p+n-1)\,!}{2p+n)\,!}\left(1 - \frac{p+1}{2p+n+1}\right)\widetilde{\delta^{(2p+n+1)}} = c_{p,p+n}\, \widetilde{\delta^{(2p+n+1)}}$$

Now, since (42) holds by induction for any pair $(p, p+n), p, n \in \mathbb{N}_0$, it therefore holds for any $(p, q) \in (\mathbb{N}_0)^2, p < q$. Due to the symmetry of eq. (42) in the indexes $p, q\ (c_{p,q} - c_{q,p})$, it also holds for any (p, q), p, q, and the proof is complete.

We return back to the notation x^{-p} instead of $\rho^{(p)}(x)$ in equation (36), with p replaced by $p - 1$ and multiplied by the factor $(-1)^{p+q}/(p-1)\,!\,(q-1)\,!$ Then the result of Theorem 4.1 reads

$$\frac{(-1)^{q-1}}{(q-1)\,!}\, \widetilde{x^{-p}} \cdot \widetilde{\delta^{(q-1)}}(x) + \frac{(-1)^{p-1}}{(p-1)\,!}\, \widetilde{x^{-q}} \cdot \widetilde{\delta^{(p-1)}}(x)$$

$$\approx \frac{(-1)^{p+q-1}}{(p+q-1)\,!}\, \delta^{(p+q-1)}(x), p, q \in \mathbb{N}\,. \qquad \text{... (43)}$$

We are now ready for the final step towards the extension of the basic Mikusinski equation.

***Theorem* 4.2** — *For arbitrary $p, q \in \mathbb{N}$, the embeddings in $\mathcal{G}_\tau$ of the distributions x^{-p} and $\delta^{(q-1)}(x)$ satisfy* :

$$\widetilde{x^{-p}} \cdot \widetilde{x^{-q}} - \pi^2 \frac{(-1)^{p+q}}{(p-1)!\,(q-1)!}\, \widetilde{\delta^{(p-1)}}(x) \cdot \widetilde{\delta^{(q-1)}}(x) \approx x^{-p-q}. \qquad \text{... (44)}$$

PROOF : (*i*) We first proof the following formula that translates exactly from distribution theory[10] into Colombeau algebra.

$$\widetilde{(x+i0)^{-p}} = \widetilde{x^{-p}} - i\pi \frac{(-1)^{p-1}}{(p-1)!}\, \widetilde{\delta^{(p-1)}}(x), x \in \mathbb{R}. \qquad \text{... (45)}$$

By definition, $x^{-p} = (-1)^{p-1}/(p-1)!\, d^p/dx^p \ln x), p \in \mathbb{N}$. Thus, for $x \in \mathbb{R}$, we have the representation

$$\widetilde{x^{-p}}(\varphi_\varepsilon, x) = \frac{(-1)^{2p-1}}{(p-1)!\,\varepsilon^{p+1}} \int_{-\varepsilon l+x}^{\varepsilon l+x} \ln|y|\, \varphi^{(p)}\left(\frac{y-x}{\varepsilon}\right) dy. \qquad \text{... (46)}$$

Here, it is taken into account that, if supp $\varphi(x) \subseteq [-l, l]$ for some $l \in \mathbb{R}$, then suppl. $\varphi((y-x)/\varepsilon) \subseteq [-\varepsilon l + x, \varepsilon l + x]$. Also,

$$\widetilde{\delta^{(p-1)}}(\varphi_\varepsilon, x) = (-1)^{p-1}\varepsilon^{-p} \langle \delta_y, \varphi^{(p-1)}((y-x)/\varepsilon) \rangle$$

$$= (-1)^{p-1}\varepsilon^{-p}\varphi^{(p-1)}(-x/\varepsilon). \qquad \text{... (47)}$$

Now, replacing $(y-x)/\varepsilon = v$, and taking into account (46) and (47), we have :

$$\widetilde{(x+i0)^{-p}}(\varphi_\varepsilon, x) = \lim_{\chi \to 0_+} \frac{(-1)^{2p-1}}{(p-1)!\,\varepsilon^{p+1}} \int_{-\varepsilon l+x}^{\varepsilon l+x} \ln(y+i\chi)\, \varphi^{(p)}\left(\frac{y-x}{\varepsilon}\right) dy$$

$$= \lim_{\chi \to 0_+} \frac{(-1)^{2p-1}}{(p-1)!\,\varepsilon^{p}} \int_{-l}^{l} (\ln|\upsilon + \varepsilon x + i\chi| + i\, arg\,(\upsilon + \varepsilon x + i\chi))\, \varphi^{(p)}(\upsilon)\, d\upsilon$$

$$= \frac{(-1)^{2p-1}}{(p-1)!\,\varepsilon^{p}} \int_{-l}^{l} (\ln|\upsilon + \varepsilon x| + i\pi[1 - H(\upsilon + \varepsilon x)])\, \varphi^{(p)}(\upsilon)\, d\upsilon$$

$$= \frac{(-1)^{2p-1}}{(p-1)!\,\varepsilon^{p}} \left(\int_{-l}^{l} \ln|\upsilon + \varepsilon x|\, \varphi^{(p)}(\upsilon)\, d\upsilon - i\pi \int_{-x/\varepsilon}^{l} \varphi^{(p)}(\upsilon)\, d\upsilon \right)$$

$$= \widetilde{x^{-p}}(\varphi_\varepsilon, x) = i\pi \frac{(-1)^{p-1}}{(p-1)!}\, \widetilde{\delta^{(p-1)}}(\varphi_\varepsilon, x).$$

This proves eq. (45) for a fixed parameter function φ. When φ is running the set A_q, we get the representative class of the embeddings in $\mathcal{G}_\tau$ of the distributions on the left-hand side, respectively, right-hand side of (45). This establishes a one-to-one correspondence between these classes, which amounts to an equality in $\mathcal{G}_\tau$ of the corresponding generalized functions.

(ii) For arbitrary $p, q \in I\!N$, consider now the difference of the two sides of (34), taking into account relation (6) :

$$\widetilde{(x+i0)^{-p}} \cdot \widetilde{(x+i0)^{-q}} - \widetilde{(x+i0)^{-p-q}} \approx 0.$$

Applying eq. (45) and taking into account (43), we have for the latter equation

$$\widetilde{x^{-p}} \cdot \widetilde{x^{-q}} - \pi^2 \frac{(-1)^{p+q-2}}{(p-1)!\,(q-1)!} \widetilde{\delta^{(p-1)}} \cdot \widetilde{\delta^{(q-1)}}$$

$$- i\pi \left[\frac{(-1)^{q-1}}{(q-1)!} \widetilde{x^{-p}} \cdot \widetilde{\delta^{(q-1)}} + \frac{(-1)^{p-1}}{(p-1)!} \widetilde{x^{-q}} \cdot \widetilde{\delta^{(p-1)}} \right]$$

$$- \widetilde{x^{-p-q}} + i\pi \frac{(-1)^{p+q-1}}{(p+q-1)!} \widetilde{\delta^{(p+q-1)}} = \widetilde{x^{-p}} \cdot \widetilde{x^{-q}} - \pi^2$$

$$\frac{(-1)^{p+q}}{(p-1)!\,(q-1)!} \widetilde{\delta^{(p-1)}} \cdot \widetilde{\delta^{(q-1)}} - \widetilde{x^{-p-q}} \approx 0.$$

This gives equation (44) for arbitrary $p, q \in I\!N$, and the theorem is proved.

Remark — As is specific for the Mikusinski products, the individual summands in (44) do not admit associated distribution, but their sum considered as a single entity is associated with the distribution x^{-p-q}.

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