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ON FUZZY PAIRWISE - T_0 AND FUZZY PAIRWISE - T_1 BITOPOLOGICAL SPACES

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Fuzzy pairwise-T_i (in short FP-T_i , i = 0, 1) bitopological spaces have been introduced earlier by Kandil and El-Shafee[3], Safiya *et al.*[10] and Kandil *et al.*[2]. Fuzzy pairwise-T_0 separation axiom has also been introduced by Choubey[1]. Here we add two more definitions of fuzzy pairwise-T_i separation axioms each for i = 0, 1. In all we have studied seven possible definitions of FP-T_0 and six possible definitions of FP-T_1 spaces. All these definitions satisfy 'good extension' property. Among these defining cpncepts, we choose those definitions of FP-T_i (i = 0, 1) spaces viz., FP-T_0 (i) and FP-T_1 (i) which seem to be most appropriate according to out point of view. On comparing these definitions with the remaining ones, it turns out that FP-T_0 (i) is the weakest and FP– T_1 (i) is the strongest among all. Finally, we prove that FP-T_0(i) and FP- T_1 (i) are hereditary, productive and projective properties.

Key Words : Fuzzy Topological Spaces; Fuzzy Bitopological Spaces; Fuzzy Pairwise-T_0 Bitopological Spaces; Fuzy Pairwise-T_1 Bitopological Spaces.

INTRODUCTION

FP-T_0 separation axiom has been introduced earlier by Choubey[1], Kandil and El-Shafee[3], Safiya *et al.*[10] (in two ways) and Kandil *et al.*[2]. We add two more definitions to this list. All these seven definitions are 'good extensions' of the corresponding concept P-T_0 in a bitopological space. Out of these definitions we have chose FP-T_0 (i) which is due to Choubey[1]. This seems to be most appropriate in view of the fact that FP-T_0 fuzzy bitopological spaces in this sense, are precisely the T_0-objects in the category of 'fuzzy bitopological spaces and fuzzy pairwise continuous maps'. On comparison, it turns out that FP-T_0 (i) is the weakest among all. We have proved that FP-T_0 (i) is a hereditary, productive and projective property.

Pairwise-T_1ness in a bitopological space has been defined in two ways in literature which we have mentioned here as P-T_1 (i) and P-T_1 (ii). Earlier, fuzzy pairwise-T_1 bitopological spaces have been introduced by Kandil and El-Shafee[3], Safiya *et al.*[10] (in two ways) and Kandil *et al.*[2]. They have introduced it as a generalization of P-T_1 (ii). Here we introduce two more definitions of FP-T_1ness which are generalizations of P-T_1(i). We have proved that FP-T_1 (i) and FP-T_1 (ii) are good extensions of P-T_1 (i) and it has already been shown in [3], [10] amd [2] respectively that

FP-T_1 (*iii*), ... FP-T_1 (*vi*) are 'good extensions' of P-T_1 (*ii*). Out of these six definitions, we choose FP-T_1 (*i*) as it is a natural generalization of definition 5.1 of [12] which seems to be the most appropriate definition of F-T_1ness in an fts for reasons given in [12].

Finally we have proved that FP-T_1 (i) is a hereditary, productive and projective property.

2. PRELIMINARIES

Here we shall follow Lowen's definition of a fuzzy topological space (in short, an fts)[5]. 'I' will denote the unit interval [0, 1] and I^X denotes the set of all fuzzy sets in X. A constant fuzzy set taking value $\alpha \in [0, 1]$ will be denoted by α and a fuzzy set A in $Y \subseteq X$ will be identified with that fuzzy set in X which takes value $A(y)$ for $y \in Y$ and zero elsewhere.

We recall from [11] : a '**fuzzy point**' 'x_r' in X is a fuzzy set in X, taking value $r \in (0, 1)$ at x and zero elsewhere. 'x' and 'r' are respectively called the '**support**' and '**value**' of the fuzzy point 'x_r'. A fuzzy point 'x_r' is said to belong to a fuzzy set A if $r < A(x)$. A fuzzy set A in a fuzzy topological space (X, τ) is fuzzy open iff $\forall$ fuzzy point $x_r \in A$, $\exists$ a basic fuzzy open set U such that $x_r \in U \subseteq A$.

A '**fuzzy singleton**' 'x_r' in X is a fuzzy set in X taking value $r \in (0, 1]$ at x and zero elsewhere. Two fuzzy points/fuzzy singletons are said to be '**distinct**' if their supports are distinct.

Now we recall from [7]: a fuzzy singleton 'x_r' is said to be '**quasi-coincident**' with a fuzzy set A at x (notation: $x_r q A$) if $r + A(x) > 1$. If x_r is not quasi-coincident with A, we write $x_r \bar{q} A$. Two fuzzy sets A and B are said to be quasi-coincident with each other if $\exists\ x \in X$ such that $A(x) + B(x) > 1$. If A and B are not quasi coincident with each other, we write $A \bar{q} B$.

Definition 2.1 — A fuzzy topological space (X, τ) is called

(*i*) **F - T_0** if $\forall\ x, y \in X, x \neq y$, $\exists\ U \in \tau$ such that $U(x) \neq U(y)$[1]. .

(*ii*) **F - T_1** if $\forall\ x, y \in X, x \neq y$, $\exists\ U\perp$, 1 $V \in \tau$ such that $U(x) = 1$, $U(y) = 0$ and $V(y) = 1$, $V(x) = 0$[12].

Definition 2.2 — A biotopological space (X, T_1, T_2) is called :

(1) pairwise-T_0 (in short, ***P-T_0***) if $\forall\ x, y \in X, x \neq y$, $\exists\ U \in \tau_1 \cup \tau_2$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$.[8]

(2) pairwise-T_1 (*i*) (in short, ***P-T_1*** (*i*)) if $\forall\ x_1 y \in X, x \neq y$ $\exists\ U \in T_1$, $V \in T_2$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$.[9]

(3) pairwise T_1 (*ii*) (in short, ***P-T_1*** (*ii*)) if $\forall\ x, y \in X, x \neq y$, $\exists\ U\ T_1 \cup T_2$ such that $x \in U, y \notin U$.[8]

It can be checked easily that a bitopological space (X, T_1, T_2) is

(1) P - T_0 if (X, T_1) or (X, T_2) is T_0.

(2) P - T_1 (*i*) if (X, T_i) is T_1 for i = 1, 2 and

(3) P - T_1 (*ii*) if (X, T_i) is T_1 for i = 1 or 2.

Definition 2.3[10] — Let X be a set and τ_1, τ_2 be two fuzzy topologies on X.

Then (X, τ_1, τ_2) is called a **fuzzy bitopological space (in short, a fbts)**

Further, let $A \subseteq X$ and $\tau_{iA} = \{\lambda \mid A : \lambda \in \tau_i\}$ denote the subspace fuzzy topology on A induced by τ_i, $i = 1, 2$. Then $(A, \tau_{1A}, \tau_{2A})$ is called the **subspace** of (X, τ_1, τ_2) with the underlying set A.

A fuzzy bitopological property P is called **hereditary** if each subspace of a fbts with property P, also has property P.

Definition 2.4[14] — Let $\{(X_i, \tau_{1i}, \tau_{2i}) : i \in \mathcal{A}\}$ be a family of fbts'. Then the space $\left(\Pi X_i, \Pi \tau_{1i}, \Pi \tau_{2i}\right)$ is called the **product fbts** of the family of fbts' $\{(X_i, \tau_{1i}, \tau_{2i}) : i \in \mathcal{A}\}$ where $\Pi \tau_{1i}$ and $\Pi \tau_{2i}$ respectively denote the usual product fuzzy topologies of the families $\{\tau_{1i} : i \in \mathcal{A}\}$ and $\{\tau_{2i} : i \in \mathcal{A}\}$ of fuzzy topologies on X.

A fuzzy bitopological property P is called **productive** if the product fbts of a family of fbts', each having the property P, also has property P.

A property P in a fbts is called **projective** if for a family of fbts $\{(X_i, \tau_{1i}, \tau_{2i}) : i \in \mathcal{A}\}$, the product fbts $(\Pi X_i, \Pi \tau_{1i}, \Pi \tau_{2i})$ has property P implies that each coordinate space has property P.

Definition 2.5[2] — A family β of fuzzy sets in X, is called a **supra fuzzy topology** on X if it is closed under arbitrary union and contains X, ϕ. The space (X, β) is called a **supra fuzzy topological space**.

Theorem 2.5[2] — *Let (X, τ_1, τ_2) be a fbts. Then let us define a mapping $c_{12} : I^X \to I^X$ as follows :*

$$c_{12}(\mu) = \tau_1 - cl(\mu) \cap \tau_2 - cl(\mu)$$

Then the family $\tau_s = \{\mu \in I^X : c_{12}(\mu') = \mu'$ where $\mu' = 1 - \mu\}$, is a supra fuzzy topology on X associated with the fuzzy topologies τ_1 and τ_2.

Definition 2.6[10] — A property 'FP' in a fbts is said to be a **good extension** of the corresponding property 'P' in a bitopological space (X, T_1, T_2) if (X, T_1, T_2) possesses 'P' iff the fbts $(X, \omega(T_1), \omega(T_2))$ possesses 'FP' where $\omega(T_i)$ denotes the set of all l.s.c. maps from $(X, T_i) \to [0, 1], i = 0, 1$.

3. Fuzzy Pairwise-T_0 Bitopological Spaces

We list below, seven possible defining concepts of fuzzy pairwise-T_0 (in short FP-T_0) bitopological spaces.

Definition 3.1 — A fbts (X, τ_1, τ_2) is called

(*a*) FP - T_0 (*i*) if $\forall\ x, y \in X, x \neq y$, $\exists\ U \in \tau_1 \cup \tau_2$ such that $U(x) \neq U(y)$.

(*b*) FP - T_0 (*ii*) if $\forall\ x, y \in X, x \neq y$, $\exists\ U \in \tau_1 \cup \tau_2$ such that $U(x) = 1$, $U(y) = 0$ or $U(x) = 0$, $U(y) = 1$.

(*c*) FP - T_0 (*iii*) if for any pair of distinct fuzzy points x_r, y_s in *X*, $\exists\ U \in \tau_1 \cup \tau_2$ such that $x_r \in U, y_s \notin U$ or $x_r \notin U, y_s \in U$.

(*d*) FP - T_0 (*iv*) if for all pairs of fuzzy singletons x_r, y_s in *X* such that $x_r \bar{q}\, y_s$, $\exists\ U \in \tau_1 \cup \tau_2$ such that $x_r \subseteq U, y_s \bar{q}\, U$ or $y_s \subseteq U, x_r \bar{q}\, U$.

(*e*) FP-T_0 (*v*) if for all pairs of distinct fuzzy points x_r, y_s in *X* $\exists$ a fuzzy set $U \in \tau_1 \cup \tau_2$ such that $x_r \in U, U \bar{q}\, y_s$ or $y_s \in U, U \bar{q}\, x_r$.

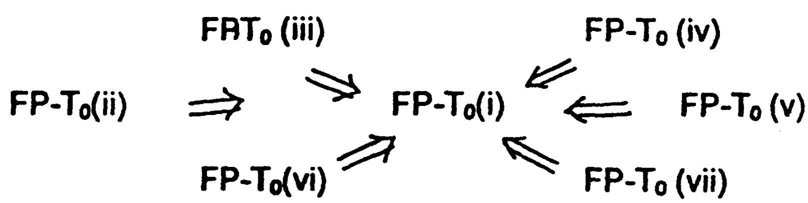
(*f*) FP-T_0 (*vi*) if for any pair of distinct fuzzy points x_r, y_s in *X*, $\exists\ U \in \tau_1 \cup \tau_2$ such that $x_r \in U, y_s, \cap U = \phi$ or $y_s \in U, x_r \cap U = \phi$.

(*g*) FP-T_0 (*vii*) if its associated supra fuzzy topological space (X, τ_s) is F-T_0 inthe sense of Kandil *et al.*[2].

We mention here that FP-T_0 (*i*) is from [1], FP-T_0 (*iv*) is due to Kandil and El-Shafee[3], FP-T_0 (*v*) and (*vi*) are due to Safiya *et al.*[10] and FP-T_0 (*vii*) is due to Kandil *et al.*[2] and FP-T_0 (*ii*) generalizes the concept of F-T_0 in a fts, given in [13].

A complete comparison of the definitions FP-T_0 (*ii*), FP-T_0 (*iii*), ... FP-T_0 (*vii*) with FP-T_0 (*i*) is establised in the following theorem :

Theorem 3.1 — *Let* (X, τ_1, τ_2) *be a fbts. Then we have the following implications* :



The reverse implications are not true in general.

PROOF : FP-T_0 (*ii*) $\Rightarrow$ FP-T_0 (*i*),

FP-T_0 (*iii*) $\Rightarrow$ FP-T_0 (*i*)

and FP-T_0 (*iv*) $\Rightarrow$ FP-T_0 (*i*) can be proved easily.

FP-T_0 (*v*) $\Rightarrow$ FP-T_0 (*i*) since FP-T_0 (*v*) $\Leftrightarrow$ FP-T_0 (*iii*),

FP-T_0 (*vi*) $\Rightarrow$ FP-T_0 (*i*) since FP-T_0 (*vi*) $\Rightarrow$ FP-T_0 (*v*)

and FP-T_0 (*vii*) $\Rightarrow$ FP-T_0 (*i*) since FP-T_0 (*vii*) $\Leftrightarrow$ FP-T_0 (*iv*) (see [2]).

None of the reverse implications are true, it can be seen through the following counter example :

Counter Example 3.1 — Let $X = \{x, y\}$, τ_1 be the fuzzy topology on *X* generated by $\{\alpha : \alpha \in [0, 1]\} \cup \{U\}$ where $U(x) = \frac{1}{2}, U(y) = \frac{1}{3}$ and τ_2 be the fuzzy topology generated by $\{\alpha : \alpha \in [0, 1]\} \cup \{V\}$ where $V(x) = \frac{1}{4}$ and $V(y) = \frac{1}{5}$.

PROOF : FP-T_0 (*i*) $\not\Rightarrow$ FP-T_0 (*ii*)

Here the fbts (X, τ_1, τ_2) is clearly FP-T_0 (i) but it is not FP-T_0 (ii) since there is no non empty fuzzy set in $\tau_1 \cup \tau_2$ which takes zero value at x or y.

FPT_0 (i) $\not\Rightarrow$ FP-T_0 (iii)

For, if we take the distinct fuzzy points $x_{3/4}, y_{3/4}$ then $\nexists$ $U \in \tau_1 \cup \tau_2$ such that $x_{3/4} \in U, y_{3/4} \notin U$ or $y_{3/4} \in U, x_{3/4} \notin U$.

FP-T_0 (i) $\not\Rightarrow$ FP-T_0 (iv)

As, for the distinct fuzzy singletons $x_1, y_1 \nexists$ $U \in \tau_1 \cup \tau_2$ such that $x_1 \subseteq U, y_1 \bar{q} U$ or $y_1 \subseteq U, x_1 \bar{q} U$.

FP-T_0 (i) $\not\Rightarrow$ FP-T_0 (v)

This follows automatically from the fact that FP-T_0 (iii) $\Leftrightarrow$ FP-T_0 (v) and it has already been shown that FP-T_0 (i) $\not\Rightarrow$ FP-T_0 (iii)

FP-T_0 (i) $\not\Rightarrow$ FP-T_0 (vi)

Since for any two distinct fuzzy points x_r, y_s in X, $\nexists$ non-empty $U \in \tau_1 \cup \tau_2$ which is disjoint with x_r or y_s.

FP - T_0 (i) $\not\Rightarrow$FP-T_0 (vii)

This follows from the fact that FP-T_0 (iii) $\Leftrightarrow$ FP-T_0 (vii) and we have already seen that FP-$T_0(i) \not\Rightarrow$ FP-T_0 (iii).

All the definitions FP-T_0 (i), ... FP-T_0 (vii) are 'good extensions' of P-T_0, as seen below.

Theorem 3.2 — *Let (X, T_1, T_2) be a bitopological space. Then (X, T_1, T_2) is P-T_0 iff $(X, \omega(T_1), \omega(T_2))$ is FP-T_0 (i).*

PROOF : Let (X, T_1, T_2) be P-T_0. Choose $x, y \in X, x \neq y$. Then $\exists$ $U \in \tau_1 \cup \tau_2$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Now consider the characteristic function χ_U. Then χ_U is either in $\omega(T_1)$ or $\omega(T_2)$ and is such that $\chi_U(x) \neq \chi_U(y)$, showing that $(X, \omega(T_1), \omega(T_2))$ is FP-T_0 (i).

Conversely, let $(X, \omega(T_1), \omega(T_2))$ be FP-T_0 (i). To show that (X, T_1, T_2) is P-T_0, choose $x, y \in X, x \neq y$. Then $\exists$ $U \in \omega(T_1) \cup \omega(T_2)$ such that $U(x) \neq U(y)$. Let $U(x) < U(y)$. Choose r such that $U(x) < r < U(y)$ and consider $U^{-1}(r, 1]$. Then $U^{-1}(r, 1] \in \tau_1 \cup \tau_2$ and is such that $x \notin U^{-1}(r, 1]$ and $y \in U^{-1}(r, 1]$. Hence, (X, T_1, T_2) is $P - T_0$.

In a similar manner, it can be checked hat FP-T_0 (ii) and FP-T_0 (iii) are also 'good extensions'. Further, the 'good extension' property of FP-T_0 (iv), FP-T_0 (v) *and FP-T_0* (vi), *FP-T_0* (vii) *has been already shown in [3], [10] and [2] respectively.*

Finally, out of the seven possible defining concepts given here, we have selected FP-T_0 (i) keeping in view of proposition 2.4.3 of [1] where it has been proved that FP-T_0 bfts' in the sense of FP-T_0 (i) are precisely the T_0-objects in the category of 'fuzzy bitpological spaces and pairwise continuous maps'.

From now onwards we shall mean FP-T_0 in the sense of definition FP-$T_0(i)$.

The following proposition can be easily verified.

Proposition 3.1 — A fbts (X, τ_1, τ_2) is FP-T_0 if either the fts (X, τ_1) or (X, τ_2) is F-T_0.

The converse of this proposition is not true in general, as can be seen through the following counter example :

Counter Example 3.2 — Let $X = \{x, y, z\}$, τ_1 be the fuzzy topology on X generated by $\{\alpha : \alpha \in [0, 1]\} \cup \{\{x\}, \{y, z\}\}$ and τ_2 be generated by $\{\alpha : \alpha \in [0, 1]\} \cup \{\{z\}, \{x, y\}\}$. Here the fbts (X, τ_1, τ_2) is FP-T_0 but neither (X, τ_1) nor (X, τ_2) is F-T_0.

In the following lines, we prove that FP-T_0 is a hereditary, productive and projective property.

Theorem 3.3 — *Every subspace of a FP-T_0 fuzzy bitopological space is FP-T_0. The proof is easy, hence is omitted.*

Theorem 3.4 — Let $\{(X_i, \tau_{1i}, \tau_{2i}) : i \in \mathcal{A}) : i \in \mathcal{A}\}$ be a family of fbts'. Then the product fbts $(\Pi X_i, \Pi \tau_{1i}, \Pi \tau_{2i})$ is FP-T_0 iff each coordinate space is FP-T_0.

PROOF : Let (X, τ_1, τ_2) be FP-T_0 $\forall$ $i \in \mathcal{A}$. Then to show that the product fbts $(\Pi X_i, \Pi \tau_{1i}, \Pi \tau_{2i})$ is FP-T_0, take $x, y \in X, x \neq y$. Let $x = \Pi x_i, y = \Pi y_i$ then $x_j \neq y_j$ for some $j \in \mathcal{A}$. Now since $(X_j, \tau_{1j}, \tau_{2j})$ is FP-T_0, $\exists$ $U_j \in \tau_{1j} \cup \tau_{2j}$ such that $U_j(x_j) \neq U_j(y_j)$. Now take $U = \Pi U_i'$ where $U_j' = U_j$ and $U_i' = X_i$ for $i \neq j$. Then U is such that $U(x) \neq U(y)$. Hence, the product fbts is FP-T_0.

Conversely, let the product fbts be FP-T_0. Take any coordinate space (X_j, T_{ij}, τ_{2j}). Choose $x_j, y_j \in X, x_j \neq y_j$. Now construct $x, y \in X$ such that $x = \Pi x_i', y = \Pi y_i'$ where $x_i' = y_i'$ for $i \neq j$ and $x_j' = x_j, y_j' = y_j$. Then $x \neq y$ and hence $\exists$ $U \in \Pi \tau_{1i} \cup \Pi \tau_{2j}$ such that $U(x) \neq U(y)$. Now U must be a union of basic fuzzy open sets say $U = \bigcup_{k \in K} B_k$. Then $\bigcup B_k(x) \neq \bigcup B_k(y)$ which implies that $\exists$ at least one k such that $B_k(x) \neq B_k(y)$. Now let $B_k = \Pi V_i$ where $V_i = X_i$ except for finitely many i's. So $\Pi(V_i(x) \neq \Pi V_i(y)$, i.e., $\inf V_i(x_i') \neq \inf V_i(y_i')$ which implies that $V_j(x_j) \neq V_j(y_j)$ since $x_i' = y_i'$ for $i \neq j$. Thus $(X_j, \tau_{ij}, \tau_{2j})$ is FP-T_0.

4. FUZZY PAIRWISE - T_1 BITOPOLOGICAL SPACES

We mention here six possible definitions of fuzzy pairwise-T_1 (in short, FP-T_1) bitopological spaces.

Definition 4.1 — A fbts (X, τ_1, τ_2) is called :-

(a) FP-T_1 (i) if $\forall$ $x, y \in X, x \neq y, \exists$ $U \in \tau_1$ and $V \in \tau_2$ such that $U(x) = 1$, $U(y) = 0$ and $V(x) = 0$, $V(y) = 1$.

(*b*) FP-T_1 (*ii*) if $\forall$ pair of distinct fuzzy points x_r, y_s in X, $\exists\, U \in \tau_1, V \in \tau_2$ such that $x_r \in U, y_s \notin U$ and $y_s \in V, x_r \notin V$.

(*c*) FP-T_1 (*iii*) if $\exists\, U \in \tau_1 \cup \tau_2$ such that $x_r \subseteq U, y_s \bar{q}\, U$.

(*d*) FP-T_1 (*iv*) if $\forall$ pair of distinct fuzzy points x_r, y_s in X, $\exists\, U, V \in \tau_1 \cup \tau_2$ such that $x_r \in U, y_s \cap U = \phi$ and $y_s \in V, x_r \cap V = \phi$.

(*e*) FP-T_1 (*v*) if $\forall$ pair of distinct fuzzy points x_r, y_s in X $\exists\, U, V \in \tau_1 \cup \tau_2$ such that $x_r \in U, y_s \bar{q}\, U$ and $y_s \in V, x_r \bar{q}\, V$.

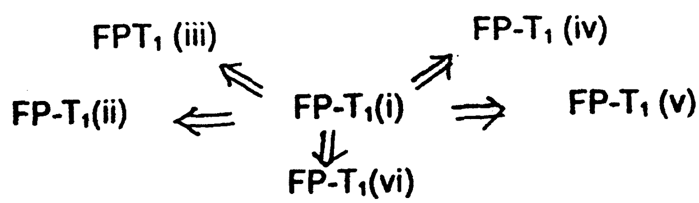
(*f*) FP-T_1 (*vi*) if the supra fuzzy topological space (X, τ_s) associated with the fbts (X, τ_1, τ_2) is F-T_1 in the sense of Kandil *et al.*[2].

We mention here that FP-T_1 (*iii*) is due to Kandil and El-Shafee[3], FP-T_1 (*iv*) FP-T_1 (*v*) are due to Safiya *et al.*[10] and FP-T_1 (*vi*) is due to Kandil *et al.* [2]

Remark 4.1 : In a fbts, the definitions FP-T_1 (i), FP-T_1 (*ii*) generalize the concept of P-T_1 (*i*) and FP-T_1 (*iii*), ... FP-T_1 (*vi*) generalize the concept of P-T_1 (*ii*) in a bitopological space.

A comparison of definitions FP-T_1 (*ii*), ... FP-T_1 (*vi*) with FP-T_1 (*i*) has been stated in the following theorem :

Theorem 4.1 — *Let (X, τ_1, τ_2) be a fbts. Then we have the following implication diagram*



The reverse implications are not true in general.

PROOF : FP-T_1 (*i*) $\Rightarrow$ FP-T_1 (*ii*) it can be easily verified.

FP-T_1 (*i*) $\Rightarrow$ FP-T_1 (*iii*)

Choose any two fuzzy singletons x_r, y_s such that $x_r \bar{q}\, y_s$. If $x = y$ then we can choose $r \in \tau_1$ and $s \in \tau_2$ satisfying the requirement that $x_r \subseteq r, y_s \subseteq s$ and $r \bar{q}\, s$.

Next let us choose two distinct fuzzy singletons x_r, y_s then $x \neq y$ and hence using FP-T_1 (*i*), $\exists\; U \in \tau_1, V \in \tau_2$ such that $U(x) = 1$, $U(y) = 0$ and $V(x) = 0$, $V(y) = 1$ which implies that $x_r \subseteq U, y_s \subseteq V, U \bar{q}\, y_s$ and $V \bar{q}\, x_r$. Hence, (X, τ_1, τ_2) is FP-T_1 (*iii*).

Further, one can easily verify that

FP-T_1 (*i*) $\Rightarrow$ FP-T_1 (*iv*)

and FP-T_1 (*i*) $\Rightarrow$ FP-T_1 (*v*)

Finally, FP-T_1 (*i*) $\Rightarrow$ FP-T_1 (*vi*)

follows from the fact that FP-T_1 (*vi*) $\Leftrightarrow$ FP-T_1 (*iii*) (see [2]).

Now we produce counter examples to show that none of the reverse implications are true in general.

Counter Example 4.1 — This counter example is based on the counter example given in [11] on page 445.

Let X be a non empty set. We define the mapping $\mu_{(x, N)} : X \to [0, 1]$ $\forall$ $x \in X, N \in (0, 1)$ as follows :

$$\mu_{(x, N)}(x) = N \text{ and } \mu_{(x, N)}(y) = 1 - N \text{ for } y \neq x,$$

Let τ_1 be the fuzzy topology on X generated by

$$\{\alpha : \alpha \in [0, 1]\} \cup \{\mu_{(x, N)} : x \in X, N \in (0, 1)\}$$

and τ_2 be the discrete fuzzy topology on X. Then (X, τ_1, τ_2) is FP-T_1 (ii) but not FP-T_1 (i) since $\nexists$ any non empty fuzzy open set in τ_1 which takes value zero at any $x \in X$.

Further, the following counter example shows that none of FP-T_1 (*iv*), FP-T_1 (*v*) or FP-T_1 (*vi*) imply FP-T_1 (*i*).

Counter Example 4.2 — Let X be any infinite set and τ_1 be the fuzzy topology on X generated by $\{\alpha : \alpha \in [0, 1]\} \cup \{U \subseteq X : X - U \text{ is finite}\}$ and τ_2 be the indiscrete fuzzy topology on X, i.e., $\{\alpha : \alpha \in [0, 1]\}$.

It is easy to see that this fbts is FP-T_1 (*iii*), FP-T_1 (*iv*), FP-T_1 (*v*) and also FP-T_1 (*vi*) (since FP-T_1 (*vi*) $\Leftrightarrow$ FP-T_1 (*iii*), see [2]) but it is not FP-T_1 (*i*) since $\nexists$ any non empty fuzzy open set U in τ_2 which takes zero value at any $x \in X$.

Now we discuss about the 'good extension' property of the definitions of FP-T_1ness given here.

FP-T_1 (iii), FP-T_1 (*iv*) and FP-T_1 (*v*) are 'good extensions' of P-T_1 (*ii*), as shown in [3] and [10] respectively. FP-T_1 (*vi*) is also a 'good extension' of P-T_1 (*ii*) since it is equivalent to FP-T_1 (iii) see [2]). In the following proposition we prove that FP-T_1 (*i*) and FP-T_1 (*ii*) are 'good extensions' of P-T_1 (*i*).

Proposition 4.1 — Let (X, T_1, T_2) be a bitopological space. Then (X, T_1, T_2) is P-T_1 (i) iff $(X, \omega(T_1), \omega(T_{2)})$ is FP-T_1 (i).

PROOF : Let (X, T_1, T_2) be P-T_1 (*i*). Choose $x, y \in X, x \neq y$. Then $\exists$ $U \in T_1$ and $V \in T_2$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Now consider the characteristic functions χ_U, χ_V. Then $\chi_U \in \omega(T_1), \chi_v \in \omega(T_2)$ and are such that $\chi_U(x) = 1, \chi_U(y) = 0$ and $\chi_V(y) = 1, \chi_V(x) = 0$. Hence $(X, \omega(T_1), \omega(T_2))$ is FP-T_1 (*i*).

Conversely, let $(X, \omega(T_1), \omega(T_2))$ be FP-T_1 (*i*). To show that (X, T_1, T_2) is P-T_1 (*i*), choose $x, y \in X, x \neq y$. Then $\exists$ $U \in \omega(T_1)$ and $V \in \omega(T_2)$ such that $U(x) = 1, U(y) = 0$ and $V(x) = 0$, $V(y) = 1$. Consider now $U^{-1}(0, 1] \in T_1$ and $V^{-1}(0, 1] \in T_2$. Then $x \in U^{-1}(0, 1], y \notin U^{-1}(0, 1]$ and $y \in V^{-1}(0, 1], x \notin V^{-1}(0, 1]$ showing that (X, T_1, T_2) is P-T_1 (*i*).

Proposition 4.2 — A bitopological space (X, T_1, T_2) is P-T_1 (i) iff the fbts $(X, \omega(T_1), \omega(T_2))$ is FP-T_1 (ii).

PROOF : Let (X, T_1, T_2) be P-T_1 (*i*). Then to show that $(X, \omega(T_1), \omega(T_2))$ is FP-T_1 (*ii*), choose any pair of distinct fuzzy points x_r, y_s in X. Then $x \neq y$ and hence since (X, T_1, T_2) is P-T_1 (*i*), $\exists$ $U \in T_1, V \in T_2$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Now the characteristic functions χ_U and χ_V are such that $\chi_U \in \omega(T_1), \chi_V \in \omega(T_2)$ and $X_r \in \chi_U, y_s \notin \chi_U, y_s \in \chi_v$ and $x_R \notin \chi_V$.

Conversely, let $(X, \omega(T_1), \omega(T_2))$ be FP-T_1 (*ii*). To show that (X, T_1, T_2) is P-T_1 (*i*), choose $x, y \in X, x \neq y$. Consider now the distinct fuzzy points x_r and y_r. Since $(X, \omega(T_1), \omega(T_2))$ is FP-T_1 (*ii*), $\exists$ $U \in \omega(T_1), V \in \omega(T_2)$ such that $x_r \in U, y_r \notin U$ and $y_r \in V$ and $x_r \notin V$. Now consider $U^{-1}(r, 1] \in T_1$ and $V^{-1}(r, 1] \in T_2$. Then $x \in U^{-1}(r, 1], y \notin U^{-1}(r, 1]$ and $y \in V^{-1}(r, 1]$ $x \notin V^{-1}(r, 1]$, showing that (X, T_1, T_2) is P-T_1 (*i*).

Out of the six definitions of FP-T_1 mentioned here, we choose FP-T_1 (*i*) as it is a natural generalization of the F-T_1 concept in a fts in the sense of definition 5.1 given in [12] which seems to be the most suitable definition of fuzzy T_1ness in a fts for reasons given in [12].

From now onwards we shall mean FP-T_1 in the sense of FP-T_1 (*i*). One can easily verify the following.

Proposition 4.3 — A fbts (X, τ_1, τ_2) is FP-T_1 iff the fts (X, τ_1) and (X, τ_2) are F-T_1.

Now we discuss about the hereditary, productive and projective properties of FP-T_1ness.

The following proposition can be easily verified.

Proposition 4.4 — Every subspace of a FP-T_1 fbts is FP-T_1.

***Theorem* 4.2** — *Let $\{X_i, \tau_{1i}, \tau_{2i}) ; i \in \mathcal{A}\}$ be a family of fbts'. Then the product fbts $(\Pi X_i, \Pi \tau_{1i}, \Pi \tau_{2i})$ is FP-T_1 iff each coordinate space $(X_i, \tau_{1i}, \tau_{2i})$ is FP-T_1.*

PROOF : Let each coordinate space $(X_i, \tau_{1i}, \tau_{2i})$, $i \in \mathcal{A}$ be FP-T_1. Then to show that the product space is FP-T_1, let $x, y \in X, x \neq y$. Let $x = \Pi x_i, y = \Pi y_i$ then $x_j \neq y_j$ for some $j \in \mathcal{A}$. Now consider x_j, y_j in X_j. Since $(X_j, \tau_{ij}, \tau_{2j})$ is FP-T_1, $\exists$ $U_j \in \tau_{ij}$ and $V_j \in \tau_{2j}$ such that $U_j(x_j) = 1, U_j(y_j) = 0$ and $V_j(y_j) = 1, V_j(x_j) = 0$, Now consider $U = \Pi U_i'$ and $V = \Pi V_i'$ where $U_i' = V_i' = X$ for $i \neq j$ and $U_j' = U_j, V_j' = V_j$. Then $U \in \Pi \tau_{1i}, V \in \Pi \tau_{2i}$ and are such that $U(x) = 1$, $U(y) = 0$, $V(y) = 1$, $V(x) = 0$, showing that the product fbts is FP-T_1.

Conversely, let $(\Pi X_i, \Pi \tau_{1i}, \Pi \tau_{2i})$ be FP-T_1. Consider any coordinate space say $(X_i, \tau_{1i}, \tau_{2i})$. Choose $x_i, y_i \in X_i, x_i \neq y_i$. Construct x, y in X such that $x = \Pi x_j', y = \Pi y_j'$ where $x_j' = y_j'$ for $j \neq i$ and $x_i' = x_i, y_i' = y_i$. Then $x \neq y$. Now using that the product space is FP-T_1, $\exists$ $U \in \Pi \tau_{1i}$ and $V \in \Pi \tau_{2i}$ such that $U(x) = 1$, $U(y) = 0$, $V(y) = 1$, $V(x) = 0$. Now choose any fuzzy point x_r in U. Then $\exists$ a basic fuzzy open set say $\Pi U_j^r \in \Pi \tau_{1i}$ such that $x_r \in \Pi U_j^r \subseteq U$ which implies that $r < \Pi U_j^r(x)$ or that $r < \inf_j U_j^r(x_j')$ and hence

$$r < U_j^r(x_j') \ \forall_j \in \mathcal{A} \quad \ldots (i)$$

and $$U(y) = 0 \Rightarrow \Pi U_j = 0. \quad \ldots (ii)$$

Similarly, corresponding to a fuzzy point y_s in V, $\exists$ a basic fuzzy open set $\Pi V_j^s \in \Pi \tau_{2j}$ such that $y_s \in \Pi V_j^s \subseteq V$ which implies that

$$s < V_j^s(y_j) \ \forall_j \in \mathcal{A} \quad \ldots (iii)$$

and $$\Pi V_j^{\ s}(y) = 0. \quad \ldots (iv)$$

Further, $\Pi U_j^r(y) = 0 \ \Rightarrow \ U_i^r(y_i) = 0$ since for $j \neq i, x_j' = y_j'$ and hence from (*i*), $U_j^r(y_j') = U_j(x_j') > r$. Similarly $\Pi V_j^s(x) = 0 \Rightarrow V_i^s(x_i) = 0$ using (*iii*).

Thus we have

$$U_i^r(x_i) > r, U_i^r(y_i) = 0$$

and $$V_i^s(y_i) > s, V_i^s(x_i) = 0.$$

Now consider $\sup_r U_i^r = U_i \in \tau_{1i}$ and $\sup_s V_i^s = V_i \in \tau_{2i}$ then $U_i(x_i) = 1$,

$$U_i(y_i) = 0, V_i(y_i) = 1, V_i(x_i) = 0 \text{ showing that } (X_i, \tau_{1i}, \tau_{2i}) \text{ is FP-}T_1.$$

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