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A FINITE ELEMENT METHOD FOR THE SIVASHINSKY EQUATION

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The Sivashinsky equation is a nonlinear evolutionary equation of fourth order in space. In this paper, we have analyzed a semidiscrete finite element method and completely discrete scheme based on the backward Euler method and Crank-Nicolson-Galerkin scheme. A linearized backward Euler method have been developed and error bounds are derived for an L^2 projection.

Key Words : Sivashinsky Equation; Finite Element; Semidiscrete; Commonly Discrete; Backward Euler Method; Cranck-Nicolson Scheme; Error Estimate

1. INTRODUCTION

We consider the Sivashinsky equation, (see [5], [1]); models a planar solid-liquid interface for a binary alloy. This situation enables one to derive an asymptotic nonlinear equation

$$u_t + D^4 u + D[(2-u)Du] + \alpha u = 0,$$

where α is constant > 0, D denotes $\frac{\partial}{\partial x}$ and $\frac{\partial u}{\partial t}$ is denoted by u_t.

We introduce the mathematical model of finite element approximation of the following initial-boundary value problem :

$$u_t + D^4 u + \alpha u = D^2(f(u)), x \in \Omega, t \in]0, T[\qquad \text{... (1.1)}$$

for $u(x, t)$, subject to the boundary conditions :

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0, t \in]0, T[\qquad \text{... (1.2a)}$$

and

$$\frac{\partial^3 u}{\partial x^3}(0, t) = \frac{\partial^3 u}{\partial x^3}(1, t) = 0, t \in]0, T[\qquad \text{... (1.2b)}$$

and the initial condition :

$$u(x, 0) = u_0(x), x \in \Omega, \qquad \dots (1.3)$$

where $f(u) = \frac{1}{2}u^2 - 2u, \Omega =]0, 1[$ and $T > 0$.

Let r and k integers with $r \geq 4$ and $1 \leq k \leq r-2$,

$0 = x_0 < x_1 < x_2 < \dots, x_N = 1$ a partition of $[0, 1]$ and $h = \max (x_j - x_{j-1})$.

Let S_h be the piecewise polynomial spline space :

$$S_h = \left\{ \chi \in C^k(\Omega), \chi_{/(x_{j-1}, x_j)} \in P_{r-1}, j = 1, \dots, N; D_\chi(0) = D_\chi(1) = 0 \right\},$$

where P_{r-1} denotes the set of polynomials on (x_{j-1}, x_j) of degree less or equal to $r - 1$.

In this paper, we shall denote the norms of $L^2(\Omega), L^\infty(\Omega)$ and $H^s(\Omega)$ by $\|\cdot\|, \|\cdot\|_\infty$ and $\|\cdot\|_s$. The semi-norm $\|D^s \upsilon\|$ is denoted by $|\upsilon|_s, (\upsilon, w)$ denoting the inner product $\int_\Omega \upsilon\, w\, dx$ in $L^2(\Omega)$.

We note $\tilde{H}^2(\Omega) = \left\{ u \in H^2(\Omega), \frac{\partial u}{\partial x} = 0, x \in \partial\Omega \right\}$ and $\tilde{S}_h = S_h \cap \{\chi, (\chi, 1) = 0\}$

we have in particular $S_h \subset \tilde{H}^2(\Omega)$.

For small h and $2 \leq s \leq r$, we have the following approximation : $\forall\, \upsilon \in H^s(\Omega) \cap \tilde{H}^2(\Omega)$

$$\text{Inf } \{\|\upsilon - \chi\| + h\|D(\upsilon - \chi)\| = h^2\|D^2(\upsilon - \chi)\|, \chi \in S_h\} \leq Ch^s\|\upsilon\|_s. \qquad \dots (2)$$

This problem is solved by a finite difference method in [6] and a wavelet Galerkin approximation of problem (1) is presented in [7]. The paper is organized as follows. In section 2, the classical semidiscrete algorithm is used for the system of equations (1), and an estimate is given for the approximation error. In section 3, we discretize the problem (1) by the backward Euler-Galerkin method, therefore we consider a linearized modification of the method and we prove the error bounds for an L^2 projection. Finally, we estimate the difference between the exact solution and the solution of the Cranck-Nicolson-Galerkin scheme.

2. Semidiscrete Problem

We may then set the approximate problem of system (1) :

Find $u_h(t) \in S_h$ for $t \in [0, T]$, such that :

$$\begin{cases} \left(\frac{\partial u_h}{\partial t}, \chi\right) + \alpha(u_h, \chi) + (D^2 u_h, D^2 \chi) = (f(u_h), D^2 \chi), \forall \chi \in S_h \\ u_{(0)} = u_{0h}, \end{cases} \qquad \dots (3)$$

where $u_{0h} \in S_h$ is approximation to u_0.

We shall prove the following estimate for the error between the solutions of the semi-discrete and continuous problems. Then we introduce the so called elliptic projection $P_2 : \tilde{H}^2(\Omega) \to S_h$ defined by :

$$\begin{cases} (D^2(P_2 u - u), D^2 \chi) = 0, \ \forall \chi \in S_h, \\ (P_2 u - u, 1) = 0 \end{cases} \qquad \dots (4)$$

Lemma 1 — With P_2 defined by (4) we have :

$$| P_2 \upsilon - \upsilon |_2 \leq C h^{s-2} \| \upsilon \|_s, \ \forall \ \upsilon \in H^s(\Omega) \cap \tilde{H}^2(\Omega)$$

PROOF : We have :

$$| P_2 \upsilon - \upsilon |_2 \leq \text{Inf} \ \{ | \chi - v |_2, \chi \in S_h \text{ and } (\upsilon - \chi, 1) = 0 \}.$$

By the approximation property (2), we prove the lemma.

Lemma 2 — For $2 \leq s \leq r, 0 \leq q \leq r - 4$, then

$$\| P_2 \upsilon - \upsilon \|_{-q} \leq C h^{s+1} \| \upsilon \|_s, \ \forall \ \upsilon \in H^s(\Omega) \cap \tilde{H}^2(\Omega)$$

and with

$$\| w \|_{-q} = \sup \left\{ \frac{(x, \varphi)}{\| \varphi \|_q}, \varphi \in H^q(\Omega), q \geq 0 \right\}.$$

PROOF : Let φ arbitrary in $H^q(\Omega)$ and $(\varphi, 1) = 0$ then the problem :

$$\begin{cases} D^4 \psi = \varphi \ in \ \ \Omega, \quad (\psi, 1) = 0 \\ \dfrac{\partial \psi}{\partial x} = \dfrac{\partial^3 \psi}{\partial x^3} = 0 \ \ on \ \ \partial \Omega \end{cases}$$

has a unique solution $\psi \in H^{q+4}(\Omega)$, (see [2]). And we have the *a priori* inequality

$$\| \psi \|_{q=4} \leq C \| \varphi \|_q, \ \forall \ q \geq 0$$

By integration by parts we obtain in view of the boundary conditions,

$$(P_2 \upsilon - \upsilon, \varphi) = (P_2 \upsilon - \upsilon, D^4 \psi) \ = \ (D^2(P_2 \upsilon - \upsilon), D^2 \psi)$$

and hence using (4), (2) and lemma 1,

$$(P_2 \upsilon - \upsilon, \varphi) = (D^2(P_2 \upsilon - \upsilon), D^2(\psi - \chi))$$

$$\leq | P_2 \upsilon - \upsilon |_2 . \ \text{Inf} \ \{ | \psi - \chi |_2, \chi \in S_h, (\psi - \chi, 1) = 0 \}$$

$$\leq (C h^{s-2} \| \upsilon \|_s)(C h^{q+2} \| \psi \|_{q+4})$$

$$\leq Ch^{s-2} \| \upsilon \|_s \| \varphi \|_q$$

This yields that :

$$\| P_2 \upsilon - \upsilon \|_{-q} \leq Ch^{s+q} \| \upsilon \|_s, \ \forall \ q \geq 0$$

Lemma 3 — We have :

$$| P_2 \upsilon - \upsilon |_1 \leq Ch^{s-1} \| \upsilon \|_s, \ \forall \ \upsilon \in H^s(\Omega) \cap \tilde{H}^2(\Omega).$$

PROOF : From the equality :

$$\int_\Omega \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x} \right) dx = \int_\Omega \left(\frac{\partial u}{\partial x} \right)^2 dx + \int_\Omega u \frac{\partial^2 u}{\partial x^2} dx = 0, \ \forall \ u \in \tilde{H}^2(\Omega),$$

we have :

$$| u |_1^2 \leq \| u \| \, | u |_2, \ \forall \ u \in \tilde{H}^2(\Omega)$$

and hence, using lemma 2 and 1 respectively,

$$| P_2 \upsilon - \upsilon |_1^2 \leq \| P_2 \upsilon - \upsilon \| \, | P_2 \upsilon - \upsilon |_2$$

$$\leq Ch^s h^{s-2} \| \upsilon \|_s^2,$$

which completes the proof.

Corollary 1 — With P_2 defined by (4) then, for all $\upsilon \in H^s(\Omega) \cap \tilde{H}^2(\Omega)$, $2 \leq s \leq r$ we have:

$$\| P_2 \upsilon - \upsilon \| + h \, | P_2 \upsilon - \upsilon |_1 + h^2 \, | P_2 \upsilon - \upsilon |_2 \leq Ch^s \| \upsilon \|_s.$$

We shall want to estimate the error in the semidicrete problem (see [3], [4] and [8]).

In this note, henceforth the solution of (1) and u_0 are sufficiently regular, we use the standard error decomposition, (see [8], [9] and [10]) :

$$u_h - u = (u_h + P_2 u) + (P_2 u - u) = \theta + \rho.$$

Theorem 1 — *Let u_h the solution of (3) and suppose that the solution u of (1) is sufficiently regular and $\| u_h(t) \|_\infty$ is bounded independently of h for $t \in [0, T]$. We have with $C = C(u, \alpha, T)$:*

$$\| u_h(t) - u(t) \|^2 \leq \| u_{0h} - u_0 \|^2 + Ch^{2r} \left(\| u_0 \|_r^2 + \int_0^T \| u_t \|_r^2 \, ds \right).$$

PROOF : We shall estimate θ, using (3) and (4) we have for $\chi \in S_h$

$$(\theta_t, \chi) + \alpha(\theta, \chi) + (D^2\theta, D^2\chi)$$

$$= (f(u_h) - f(u), D^2\chi) - (P_2 u_t - u_t, \chi) - \alpha(P_2 u - u, \chi).$$

We may choose $\chi = \theta$, then the above becomes

$$(\theta_t, \theta) + \alpha\|\theta\|^2 + \|D^2\theta\|^2$$

$$\leq \|f(u_h) - f(u)\|\,\|D^2\theta\| + \|P_2 u_t - u_t\|\,\|\theta\| + \alpha\|P_2 u - u\|\,\|\theta\|.$$

As $f(\cdot)$ is continuously differentiable, we have with $C = C(\|u_h\|_\infty, \|u\|_\infty)$

$$\|f(u_h) - f(u)\| \leq C\|u_h - u\|$$

$$\leq C(\|\theta\| + \|\rho\|).$$

Using the Schwarz inequality and for $\alpha > 0$, we obtain :

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \leq C(\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2).$$

Hence, using Gronwall's lemma,

$$\|\theta(t)\|^2 \leq \|\theta(0)\|^2 + C\int_0^t (\|\rho\|^2 + \|\rho_t\|^2)\,ds.$$

By Lemma 2 we have :

$$\|\rho(t)\| = \|u(t) - P_2 u(t)\|$$

$$\leq Ch^r\|u(t)\|_r = Ch^r\left\|u_0 + \int_0^t u_t\,ds\right\|_r$$

$$\leq Ch^r\left(\left\|u_0\right\|_r + \int_0^t \|u_t\|_r\,ds\right)$$

and $$\|\rho_t(t)\| = \|u_t - P_2 u_t\| \leq Ch^r\|u_t\|_r$$

and further

$$\|\theta(0)\| = \|u_{0h} - P_2 u_0\|$$

$$\leq \|u_{0h} - u_0\| + \|P_2 u_0 - u_0\|$$

$$\leq \|u_{0h} - u_0\| + Ch^r\|u_0\|_r.$$

Together these estimates show the theorem.

Remark : If the solution u of the problem (1) is sufficiently regular and the initial data satisfy $u_{0h} = P_2 u_0$ then the semidiscrete scheme is $0(h^r)$ accurate.

3. Completely Discrete Scheme

3.1. Backward Euler-Galerkin Method

We shall now study numerical approximation for (1), by completely discrete schemes with the backward Euler-Galerkin method.

Let k be the time step and U^n the approximation in S_h of $u(t)$ at $t = t_n = nk$, $n = 0, 1, \ldots, N$. We denote $\partial_t V^n$ by $\frac{1}{k}(V^n - V^{n-1})$.

Then the approximate problem may be written as :

$$\begin{cases} \text{Find } U^n \in S_h, \quad n = 1, 2, \ldots, N \\ (\partial_t U^n, \chi + \alpha (U^n, \chi) + (D^2 U^n, CD^2 \chi) = (f(U^n), D^2 \chi), \chi \in S_h \\ U^0 = u_{0h}, \end{cases} \quad \ldots (5)$$

where u_{0h} is an approximation of u_0.

Theorem 2 — *Let U^n and u the solutions of (5) and (1), respectively, if we assume that the solution u is regular and $\| U^n \|_\infty$ is bounded for $n = 0, 1, \ldots, N$.*

We have for small k, with $C = C(u, \alpha, T)$:

$$\| U^n - U(t_n) \|^2 \le C \left[\| U^0 - u_0 \|^2 + k \sum_{j=0}^{n} \| \rho^j \|^2 + \int_0^T \| \rho_t (s) \|^2 ds + k^2 \int_0^T \| u_{tt} (s) \|^2 ds \right]$$

PROOF : We write :

$$U^n - u(t_n) = (U^n - P_2 u(t_n)) + (P_2 u(t_n) - u(t_n)) = \theta^n + \rho^n.$$

We shall estimate θ^n. We have for $\chi \in S_h$

$$(\partial_t \theta^n, \chi) + \alpha (\theta^n, \chi) + (D^2 \theta^n, D^2 \chi)$$

$$= (f(U^n) - f(u(t_n)), D^2 \chi) + (u_t (t_n) - \partial_t P_2 u(t_n), \chi) + \alpha (u(t_n) - P_2 u(t_n), \chi).$$

Taking $\chi = \theta^n$, this yields :

$$(\partial_t \theta^n, \theta^n) + \alpha \| \theta^n \|^2 + \| D^2 \theta^n \|^2$$

$$\leq \| f(U^n) - f(u(t_n)) \| \, \| D^2 \theta^n \| + \| u_t(t_n) - \partial_t P_2 u(t_n) \| \, \| \theta^n \|$$

$$+ \alpha \| u(t_n) - P_2 u(t_n) \| \, \| \theta^n \|$$

$$\leq C [\| f(U^n) - f(u(t_n)) \| + \| u_t(t_n) - \partial_t P_2 u(t_n) \| + \| u(t_n) - P_2 u(t_n) \|] \, \| D^2 \theta^n \|$$

$$= C (I_1 + I_2 + I_3) \| D^2 \theta^n \|.$$

In this case we have these estimates :

$$I_1 = \| f(U^n) - f(u\, t_n)) \|$$

$$\leq C \| U^n - u(t_n) \|$$

$$\leq C(\| \theta^n \| + \| \rho^n \|).$$

$$I_2 = \| u_t(t_n) - \partial_t P_2 u(t_n) \|$$

$$\leq \| u_t(t_n) - \partial_t u(t_n) \| + \| \partial_t u(t_n) - \partial_t P_2 u(t_n) \| = J_1 + J_2.$$

$$J_1 = \| u_t(t_n) - \partial_t u(t_n) \| = \left\| u_t(t_n) - \frac{1}{k} (u(t_n) - u(t_{n-1})) \right\|$$

$$= \left\| \frac{1}{k} \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{tt}(s) \, ds \right\|$$

$$\leq C k^{1/2} \left(\int_{t_{n-1}}^{t_n} \| u_{tt}(s) \|^2 ds \right)^{1/2}$$

$$J_2 = \| \partial_t u(t_n) - \partial_t P_2 u(t_n) \|$$

$$= \frac{1}{k} \| u(t_n) - P_2 u(t_n)) - (u t_{n-1}) - P_2 u(t_{n-1})) \|$$

$$= \frac{1}{k} \| \rho^n - \rho^{n-1} \|$$

$$\leq k^{-1/2} \left(\int_{t_{n-1}}^{t_n} \| \rho_t(s) \|^2 ds \right)^{1/2}$$

$$I_3 = \| u(t_3) - P_2 u(t_n) \| = \| \rho^n \|.$$

Applying the Cauchy-Schwartz inequality and the identity

$$(\partial_t \theta^n, \theta^n) = \frac{1}{2} \partial_t \| \theta^n \|^2 + \frac{k}{2} \| \partial_t \theta^n \|^2 \geq \frac{1}{2} \partial_t \| \theta^n \|^2.$$

We have for $\alpha > 0$

$$\partial_t \| \theta^n \|^2 \leq C \, [\| \theta^n \|^2 + \| \rho^n \|^2 + k \int_{t_{n-1}}^{t_n} \| u_{tt}(s) \|^2 \, ds + \frac{1}{k} \int_{t_{n-1}}^{t_n} \| \rho_t(s) \|^2 \, ds]$$

$$\leq C \, (\| \theta^n \|^2 + R_n).$$

Where the latter equality defines R_n, so that

$$\frac{1}{k} (\| \theta^n \|^2 - \| \theta^{n-1} \|^2 \|^2) \leq C \, (\| \theta^n \|^2 + R_n)$$

or $$(1 - Ck) \| \theta^n \|^2 \leq \| \theta^{n-1} \|^2 + CkR_n.$$

For small k

$$\| \theta^n \|^2 \leq (1 + Ck) \| \theta^{n-1} \|^2 + CkR_n$$

and, by repeated application

$$\| \theta^n \|^2 \leq (1 + Ck)^n \| \theta^0 \|^2 + Ck \sum_{j=1}^{n} (1 = Ck)^{n-j} R_j$$

$$\leq \| \theta^0 \| + Ck \sum_{j=1}^{n} R_j$$

here, as before,

$$\| \theta^0 \|^2 = \| U^0 - P_2 \, u(t_0) \|^2 \leq \| U^0 - u_0 \|^2 + \| \rho^0 \|^2$$

which completes the proof.

Corollary 2 — With U^n and u the solutions of (5) and (1), respectively and u is regular, we have for $n = 0, 1, \ldots, N$ and $C = C(u, \alpha T)$

$$\| U^n - u(t_n) \|^2 \leq C \left[\| U^0 - u_0 \|^2 + h^{2r} \left(\| u_0 \|_r^2 + \int_0^T \| u_t(s) \|_r^2 \, ds \right) + k^2 \int_0^T \| u_{tt}(s) \|^2 \, ds \right]$$

PROOF : By Lemma 2 we have :

$$\| \rho^n \| = \| P_2 u(t_n) - u(t_n) \| \le Ch^r \left(\| u_0 \|_r + \int_0^{t_n} \| u_t \|_r \, ds \right)$$

from which the results follows, in view of theorem 2.

3.2. Linearized Backward-Euler-Galerkin Method

We shall now present the linearized problem, in which the nonlinear term $f(U^n)$ in equation (5) is replaced by $f(U^{n-1})$

$$\begin{cases} \text{Find } U^n \in S_h, \quad n = 1, 2, \dots, N \\ (\partial_t u^n, \chi) + \alpha (U^n, \chi) + (D^2 U^n, D^2 \chi) = (f(U^{n-1}), D^2 \chi), \chi \in S_h \\ U^0 = u_{0h} \end{cases} \quad \dots (6)$$

Theorem 3 — *Let U^n and u the solutions of (6) and (1), respectively. We suppose that u is sufficiently regular and $\| U^n \|_\infty$ is bounded for $n = 0, 1, \dots, N$.*

We have for small k, with $C = C(u, \alpha, T)$

$$\| U^n - u(t_n) \|^2 \le C \left[\| U^0 - u_0 \|^2 + h^{2r} \left(\| u_0 \|_r^2 + \int_0^T \| u_t(s) \|_r^2 \, ds \right) \right.$$

$$\left. + k^2 \int_0^T (\| u_t(s) \|^2 + \| u_{tt}(s) \|^2 \, ds) \right].$$

PROOF : For $\chi \in S_h$ we have

$$(\partial_t \theta^n, \chi) + \alpha (\theta^n, \chi) + (D^2 \theta^n, D^2 \chi)$$

$$= (f(U^{n-1}) - f(u(t_n)), D^2 \chi) + (u_t(t_n) - \partial_t P_2 u(t_n), \chi) + \alpha (u(t_n) - P_2 u(t_n), \chi)$$

Taking $\chi = \theta^n$, we find :

$$(\partial_t \theta^n, \theta^n) + \alpha \| \theta^n \|^2 + \| D^2 \theta^n \|^2$$

$$\le C (\| f(U^{n-1}) - f(u(t_n)) \| + \| u_t(t_n) - \partial_t P_2 u(t_n) \| + \| u(t_n) - P_2 u(t_n) \|) \| D^2 \theta^n \|$$

$$= C (I_1 + I_2 + I_3) \| D^2 \theta^n \|.$$

We have the following estimates :

$$I_1 = \| f(U^{n-1}) - f(u(t_n)) \|$$

$$\leq C \| u(t_n) - U^{n-1} \|$$

$$\leq C (\| u(t_{n-1}) - U^{n-1} \| + \| u(t_n) - u(t_{n-1}) \|)$$

$$\leq C \left(\| \theta^{n-1} \| + \| \rho^{n-1} \| + k^{1/2} \left(\int_{t_{n-1}}^{t_n} \| u_t(s) \|^2 \, ds \right)^{1/2} \right).$$

We use the proof of Theorem 2, we have

$$I_2 = \| u_t(t_n) - \partial_t P_2 u(t_n) \| \leq k^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} \| u_{tt}(s) \|^2 \, ds \right)^{\frac{1}{2}} + k^{\frac{-1}{2}} \left(\int_{t_{n-1}}^{t_n} \| \rho_t(s) \|^2 \, ds \right)^{\frac{1}{2}}$$

and $$I_3 = \| u(t_n) - P_2 u(t_n) \| = \| \rho^n \|.$$

By estimates of I_1, I_2 and I_3, we have for $\alpha > 0$

$$\partial_t \| \theta^n \|^2 \leq C [\| \theta^{n-1} \|^2 = \| \rho^{n-1} \|^2 \| \rho^n \|^2$$

$$+ k \left(\int_{t_{n-1}}^{t_n} \| u_t(s) \|^2 \, ds + \int_{t_{n-1}}^{t_n} \| u_{tt}(s) \|^2 \, ds \right) + \frac{1}{k} \int_{t_{n-1}}^{t_n} \| \rho_t(s) \|^2 \, ds \Bigg].$$

This gives,

$$\frac{1}{k} (\| \theta^n \|^2 - \| \theta^{n-1} \|^2) \leq C (\| \theta^{n-1} \|^2 + R_n).$$

Where the latter equality defines R_n, and hence, for small k

$$\| \theta^n \|^2 \leq (1 + Ck) \| \theta^{n-1} \|^2 + R_n)$$

and by repeated application, we have

$$\| \theta^n \|^2 \leq C \| \theta^0 \|^2 + Ck \sum_{j=1}^{n} R_j$$

by Lemma 2, we prove the theorem.

3.3 Crank-Nicholson Method

The Crank-Nicholson scheme in which the semidiscrete equation is discretized in a symmetric fashion around the point $t_{n-\frac{1}{2}} = \left(n - \frac{1}{2}\right)k$ to produce a second-order precision in time but the backward Euler method is only first order accurate in time.

Noting $\overline{V}^n = \frac{1}{2}(V^n + V^{n-1})$. Then we define the approximate problem :

$$\begin{cases} \text{Find } U^n \in S_h, n = 1, 2, \dots, N \\ (\partial_t U^n, \chi) + \alpha(\overline{U}^n, \chi) + (D^2 \overline{U}^n, D^2 \chi) = (f(\overline{U}^n), D^2 \chi), \chi \in S_h \\ U^0 = u_{0h}. \end{cases} \quad \dots (7)$$

***Theorem* 4** — *Let U^n and u the solutions of (7) and (1), respectively. We suppose that the solution u is sufficiently regular and $\| U^n \|_\infty$ is bounded.*

We have for $n = 0, 1, \dots, N$ and $C = C(u, \alpha, T)$

$$\| U^n - u(t_n) \|^2 \le C \left[\| U^0 u_0 \|^2 + h^{2r} \left(\| u_0 \|_r^2 + \int_0^T \| u_t(s) \|_r^2 \, ds \right) \right.$$

$$\left. + k^4 \left(\int_0^T (\| u_{ttt}(s) \|^2 + \| u_{tt}(s) \|_2^2 \right) ds \right].$$

PROOF : We shall estimate θ^n, we have for $\chi \in S_h$

$$(\partial_t \theta^n, \chi) + \alpha(\overline{\theta}^n, \chi) + (D^2 \overline{\theta}^n, D^2 \chi)$$

$$= \left(f\left(\overline{U}^n) - f\left(u\left(t_{n-\frac{1}{2}} \right) \right) \right), D^2 \chi \right) - \alpha \left(\overline{P}_2 u(t_n) - r\left(t_{n-\frac{1}{2}} \right), \chi \right)$$

$$- \left(\partial_t P_2 u(t_n) - u_t\left(t_{n-\frac{1}{2}} \right), \chi \right) - \left(D^2 \left(\frac{1}{2} u(t_n) + \frac{1}{2} u(t_{n-1}) - u\left(t_{n-\frac{1}{2}} \right) \right), D^2 \chi \right)$$

Choosing this time $\chi = \overline{\theta}^n$, we find,

$$(\partial_t \theta^n, \overline{\theta}^n) + \alpha \| \overline{\theta}^n \|^2 + \| D^2 \overline{\theta}^n \|^2$$

$$\le C \left[\left\| f(\overline{U}^n) - f\left(u\left(t_{n-\frac{1}{2}} \right) \right) \right\| + \left\| \overline{P}_2 u(t_n) - u\left(t_{n-\frac{1}{2}} \right) \right\| \right.$$

$$+ \left\| \partial_t P_2 u(t_n) - u_t\left(t_{n-\frac{1}{2}} \right) \right\|$$

$$+ \left\| \frac{1}{2} u(t_n) + \frac{1}{2} u(t_{n-1}) - u\left(t_{n-\frac{1}{2}}\right) \right\|_2 \Bigg] \| \overline{\theta}^n \|_2$$

$$= C\,(I_1 + I_2 + I_3 + I_4)\, \| \overline{\theta}^n \|_2.$$

We have the following estimates :

$$I_2 = \left\| \overline{P}_2\, u(t_n) - u\left(t_{n-\frac{1}{2}}\right) \right\|$$

$$\leq \frac{1}{2} (\| \rho^n \| + \| \rho^{n-1} \|) + \left\| \frac{1}{2} u(t_n) + \frac{1}{2} u(t_{n-1}) - u\left(t_{n-\frac{1}{2}}\right) \right\|$$

$$I_1 = \left\| f(\overline{U}^n) - f\left(u\left(t_{n-\frac{1}{2}}\right)\right) \right\|$$

$$\leq C \left\| f(\overline{U}^n) - f\left(u\left(t_{n-\frac{1}{2}}\right)\right) \right\|$$

$$= C \left\| (\overline{U}^n - \overline{P}_2\, u(t_n)) + \left(\overline{P}_2\, u(t_n) - u\left(t_{n-\frac{1}{2}}\right)\right) \right\|$$

$$\leq C\,(\| \overline{\theta}^n \| + I_2)$$

$$I_3 = \left\| \partial_t P_2\, u(t_n) - u_t\left(t_{n-\frac{1}{2}}\right) \right\|$$

$$\leq \left\| \partial_t P_2\, u(t_n) - \partial_t u(t_n) \| + \| \partial_t u(t_n) - u_t\left(t_{n-\frac{1}{2}}\right) \right\|$$

$$= J_1 + J_2$$

$$J_1 = \| \partial_t P_2 u(t_n) - \partial_t u\,(t_n) \|$$

$$= \frac{1}{k} \| (P_2\, u(t_n) - u(t_n)) - (P_2\, u(t_{n-1}) - u(t_{n-1})) \|$$

$$= \frac{1}{k} \| \rho^n - \rho^{n-1} \|$$

$$\leq k^{-\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} \| \rho_t(s) \|^2\, ds \right)^{\frac{1}{2}}.$$

$$J_2 = \left\| \partial_t u(t_n) - u_t\left(t_{n-\frac{1}{2}}\right) \right\|$$

$$= \left\| \frac{1}{k} (u(t_n) - u(t_{n-1})) - u_t\left(t_{n-\frac{1}{2}}\right) \right\|$$

$$= \frac{1}{2k} \left\| \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (s - t_{n-1})^2 u_{ttt}(s)\, ds = \int_{t_{n-\frac{1}{2}}}^{t_n} (s - t_n)^2 u_{ttt}(s)\, ds \right\|$$

$$\leq C k^{\frac{3}{2}} \left(\int_{t_{n-1}}^{t_n} \| u_{ttt}(s) \|^2\, ds \right)^{\frac{1}{2}}.$$

$$I_4 = \left\| \left(\frac{1}{2} u(t_n) + \frac{1}{2} u(t_{n-1}) \right) - u\left(t_{n-\frac{1}{2}} \right) \right\|_2$$

$$= \frac{1}{2} \left\| \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (s - t_{n-1})\, u_{tt}(s)\, ds - \int_{t_{n-\frac{1}{2}}}^{t_n} (s - t_n)\, u_{tt}(s)\, ds \right\|_2$$

$$\leq C k^{\frac{3}{2}} \left(\int_{t_{n-1}}^{t_n} \| u_{tt}(s) \|_2^2\, ds \right)^{\frac{1}{2}}.$$

Using the identity

$$(\partial_t \theta^n, \bar{\theta}^n) = \frac{1}{2k} (\| \theta^n \|^2 - \| \theta^{n-1} \|^2) = \frac{1}{2} \partial_t \| \theta^n \|^2$$

as $\alpha > 0$ and the estimates of I_1, I_2, I_3 and I_4, we obtain :

$$\partial_t \| \theta^n \|^2 \leq C \left[\| \bar{\theta}^n \|^2 + \| \rho^n \|^2 + \| \rho^{n-1} \|^2 + \frac{1}{k} \int_{t_{n-1}}^{t_n} \| \rho_t(s) \|^2\, ds \right.$$

$$\left. + k^3 \int_{t_{n-1}}^{t_n} \| u_{ttt}(s) \|^2\, ds = k^3 \int_{t_{n-1}}^{t_n} \| u_{tt}(s) \|_2^2\, ds \right]$$

and hence

$$\frac{1}{2k} (\| \theta^n \|^2 - \| \theta^{n-1} \|^2) \leq C (\| \bar{\theta}^n \|^2 + R_n),$$

where the latter equality defines R_n, we have :

$$(1 - Ck) \| \theta^n \| \leq (1 + Ck) \| \theta^{n-1} \|^2 + CkR_n.$$

And for small k,

$$\| \theta^n \|^2 \leq \left(\frac{1 + Ck)}{1 - Ck} \right) \| \theta^{n-q} \|^2 + CkR_n$$

after repeated application this yields

$$\| \theta^n \|^2 \leq \left(\frac{1 + Ck}{1 - Ck} \right)^n \| \theta^0 \|^2 + Ck \sum_{j=1}^{n} \left(\frac{1 + Ck}{1 - Ck} \right)^{n-j} R_j$$

$$\leq C \| \theta^0 \|^2 + Ck \sum_{j=1}^{n} R_j$$

We obtain :

$$\| \theta^n \|^2 \leq C \left[\| U^0 - u_0 \|^2 + k \sum_{j=1}^{n} \| \rho^j \|^2 + \int_0^T \| \rho_t (s) \|^2 ds \right.$$
$$\left. + \, k^4 \left(\int_0^T (\| u_{ttt} (s) \|^2 + \| u_{tt} (s) \|_2^2 ds \right) \right]$$

Noting that :

$$\| \theta^0 \|^2 \leq \| \rho^0 \|^2 + \| U^0 - u_0 \|^2$$

and by Lemma 2, we have

$$\| \rho^n \| \leq Ch^r \left(\| u_0 \|_r + \int_0^{t_n} \| u_t(s) \|_r ds \right)$$

which completes the proof.

CONCLUSIONS

In this paper, we use a finite element approximation of the Sivashinsky equation, if the initial data satisfy $u_{0h} = P_2 u_0$, we prove an optimal-order error bound in L^2 for the semidiscretisation method, and we analyse the convergence of the fully discrete scheme. The backward Euler method has the disadvantage that the implementation of problem (5) requires that the solution of a nonlinear system equations has to be solved at each time step. Then we have a linearized problem (6) of system (5) in which this difficulty is avoided by replacing U^n by U^{n-1} in the nonlinear term. For the purpose of obtaining higher accuracy in time we study the Cranck-Nicolson-Galerkin scheme.

REFERENCES

1. V. G. Gertsberg and G. I. Sivashinsky, *Prog. theor. Phys.* **66** (1981) 1219.

2. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Advanced Publications program, Pitman, Boston, (1985).
3. H. S. price and R. S. Varga, *Numerical Solution of Field Problems in Continuum Physics* AMS, Providence, R.I., 74-94, 1970.
4. H. H. Rachford, Jr., *SIAM J. numer. Anal.* **10** (1973) 1010-26.
5. G. I. Sivashinsky, *Physica* **8D** (1983) 243-48, North-Holland Publishing Company.
6. K. Omrani, A finite difference scheme for the Sivashinsky equation, (*to appear*).
7. K. Omrani, Approximation des equations de Sivashinsky et application des ondelettes; Thèse de Doctorat, *Facult. Sci. Tunis (to appear).*
8. V. Thomée, In: Matematical Aspects of Finite Element Methods in Partial Differnetial Equasion (*Ed*: C. de Boor) Academic Press, p 55-84, 1974.
10. V. Thomée and L. B. Wahlbin, *J. numer. Ana.* **12** (1975) 378-89.