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THE BANACH ALGEBRAS OF FIXED POINTS AND BESOV NORM OF DERIVATIVE

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In this paper we give a constructive proof of existence of certain Banach algebras denominated of fixed points and Besov norm of derivative, in connection with certain Besov spaces. Also appear the denominated Banach spaces of outer points. The Banach algebras and spaces which arise in this paper are new and interesting by himself.

Key Words : Banach Algebras; Besov Spaces; Minimaxs; Polynomial Approximation

1. INTRODUCTION

Let K be a compact subset of $\mathbb{C}$. $C(K)$ denotes the set of all continuous functions in K. Π_n denotes the set of polynomials of degree n at most. For each $n = 0, 1, 2, \ldots$ the nth minimax of f is given by

$$E_n(f) = \inf_{p \in \Pi_n} \|f - p\|_\infty.$$

The minimax series of f is

$$S(f) = E_0(f) + E_1(f) + E_2(f) + \ldots$$

For $q > 0$ the q-minimax series of f is given by

$$S_q(f) = (E_0^q(f) + E_1^q(f) + E_2^q(f) + \ldots)^{1/q}.$$

We consider also the weighted $\gamma - q$ minimax series with the Besov's weights $(k+2)^{\gamma q - 1}$, namely

$$S_{\gamma, q}(f) = (2^{\gamma q - 1} E_0^q(f) + 3^{\gamma q - 1} E_0^q(f) + \ldots + (k+2)^{\gamma q - 1} (E_k^q(f) + \ldots)^{1/q}.$$

Such series appear directly in the norm of the Besov space $B_{\gamma, q}^{\infty}$ (see [1]),

$$\|f\|_{B^{\infty}_{\gamma,q}} = (\|f\|^q_\infty + S^q_{\gamma,q}(f))^{1/q}, f \in B^{\infty}_{\gamma,q} ,$$

and have been considered in a recent paper[5]. In such paper it is proved that

$$B^{\infty}_{\gamma,q}(K) = \left\{ f \in C(K) : S_{\gamma,q}(f) < \infty \right\}$$

are Banach algebras.

The key is an inequality for the product

$$S_{\gamma,q}(fg) \le \Omega\,[\|f\|_\infty\, S_{\gamma,q}(g) + \|g\|_\infty\, S_{\gamma,q}(f)], \qquad \text{... (1.1)}$$

where Ω is a constant depending of q and γ. As a consequence there exists a constant $K[\gamma, q]$ such that

$$S_{\gamma,q}(fg) \le K[\gamma,q]\, S_{\gamma,q}(f)\, S_{\gamma,q}(g) \qquad \text{... (1.2)}$$

for all f and $g \in B^{\infty}_{\gamma,q}$ with zeros in K. (See corollary 1 and Theorem 4 in [5]).

The set of functions for which $S^*(f) = \sum_{k=0}^{\infty} E^*_k(f) < \infty$, where $E^*_k(f)$ denotes the error of best approximation of $f \in C[0, 2\pi]$ with trigonometric polynomials of degree k at most was already studied by S. N. Bernstein. He proved that such functions are of class $C^1[0, 2\pi]$.

A. Pietsch describe certain analogies between spaces of a sequences, functions and operators. He give a method for to construct the so called approximation spaces where the Besov's weights are considered[6]. The results are applied to establish certain results about distributions of Fourier coefficients and eigenvalues. Such analogies was first time observed by Peetre. Similar concepts where already investigated by Butzer and Chererer, Bruding and Krugljak, as well as by Peetre and Sparr within the framework of their interpolation theory of Abelian groups, see the introduction in [6].

The main result of the paper is that if $x_0, x_1, \ldots, x_{n-1}$ are n fixed points of the interval $[a, b]$, the set

$$\left\{ f \in C^{(n)}([a,b]) : f(x_i) = 0, (0 \le i \le n-1) \text{ and } \|f^{(n)}\|_{B^{\infty}_{\gamma,q}} < \infty \right\}$$

is a Banach algebra under the norm

$$\|f\| = \|f^{(n)}\|_{B^{\infty}_{\gamma,q}} .$$

These algebras are denominated the Banach Algebras of fixed points and Besov norm of derivative.

The paper is organized as follows, in section 2 is proved a fundamental lemma. In section 3 are constructed the so called Banach algebras of fixed points and Besov norm of derivative, in section 4 are considered the so called quasiproducts.

Finally in section 5 several examples are given.

The algebras and spaces of real functions in connection with Besov spaces which arise in this paper are interesting for the analysts and from the point of view of approximation theory.

2. Preliminary Result

Lemma 1 — Let $f \in C^{(1)}[a, b]$. Then

(*i*) If $\gamma q - 1 < 0$, then

$$S^q_{\gamma, q}(f) \le (b-a)^q \left[2^{\gamma q - 1} \|f'\|^q_\infty + S^q_{\gamma, q}(f') \right].$$

(*ii*) If $\gamma q - 1 \ge 0$, then

$$S^q_{\gamma, q}(f) \le 2^{\gamma q - 1}(b-a)^q \left[\|f'\|^q_\infty + S^q_{\gamma, q}(f') \right].$$

(*iii*) Moreover, if f' have a zero in $[a, b]$, then

$$S_{\gamma, q}(f) \le (b-a)\,(2^q + 1)^{1/q}\, S_{\gamma, q}(f')],\ (\gamma q - 1 < 0)$$

and

$$S_{\gamma, q}(f) \le (b-a)\,[2^{\gamma q - 1}(2^{1(1-\gamma)+1} + 1)]^{1/q}\, S_{\gamma, q}(f'),\ (\gamma q - 1 \ge 0).$$

Proof : Let p_k be the best approximation of f' in Π_k. Then

$$E_{k+1}(f) \le \left\| [f(a) + \int_a^x f'(t)\,dt - \int_a^x p_k(t)dt - f(a) \right\|_\infty$$

$$= \left\| \int_a^x (f'(t) - p_k(t))\,dt \right\|_\infty \le (b-a)\,\|f' - p_k\|_\infty = (b-a)\,E_k(f'),\ (\forall\ k \ge 0),$$

and

$$E_0(f) = E_0\left(f(a) + \int_a^x f'(t)dt \right) = E_0\left(\int_a^x f'(t)dt \right) \le \left\| \int_a^x f'(t)\,dt \right\|_\infty \le (b-a)\,\|f'\|_\infty$$

Then

$$S^q_{\gamma, q}(f) \le 2^{\gamma q - 1} E^q_0(f) + (b-a)^q \sum_{k=1}^{\infty} (k+2)^{\gamma q - 1} E^q_{k-1}(f').$$

Now we distinguish two cases

(*i*) $\gamma q - 1 < 0$. Then

$$S^q_{\gamma,q}(f) \le 2^{\gamma q-1} E^q_0(f) + (b-a)^q \sum_{k=1}^{\infty} (k+1)^{\gamma q-1} E^q_{k-1}(f')$$

$$\le 2^{\gamma q-1} E^q_0(f) + (b-a)^q S^q_{\gamma,q}(f')$$

$$\le (b-a)^q \left[2^{\gamma q-1} \|f'\|^q_\infty + S^q_{\gamma,q}(f') \right].$$

(*ii*) $\gamma q - 1 \ge 0$.

$$S^q_{\gamma,q}(f) \le 2^{\gamma q-1} E^q_0(f) + (b-a)^q \sum_{k=1}^{\infty} (k+2)^{\gamma q-1} E^q_{k-1}(f')$$

$$= 2^{\gamma q-1} E^q_0(f) + (b-a)^q \sum_{k=1}^{\infty} (2k+2)^{\gamma q-1} E^q_{k-1}(f')$$

$$= 2^{\gamma q-1} \left[E^q_0(f) + (b-a)^q \sum_{k=1}^{\infty} (k+1)^{\gamma q-1} E^q_{k-1}(f') \right]$$

$$\le 2^{\gamma q-1} (b-a)^q \left[\|f'\|^q_\infty + S^q_{\gamma,q}(f') \right].$$

(*iii*) Note that if f' have a zero in $[a, b]$, then

$$\|f'\|^q_\infty \le 2^q E^q_0(f') \le 2^{q(1-\gamma)+1} S^q_{\gamma,q}(f')$$

and the proof of (*iii*) follows from (*i*) and (*iii*).

Corollary 1 — If $b - a$ is sufficiently small then

$$S_{\gamma,q}(f) < S_{\gamma,q}(f').$$

3. Banach Algebras of n Fixed Points and Besov Norm of Derivative

Let $x_0, x_1, \ldots, x_{n-1}$ be n fixed points of $[a, b]$. Let us consider the space

$$(B^\infty_{\gamma,q})^{(n)} = \left\{ f \in C^{(n)}[a, b] : f(x_0) = f(x_1) = f(x_{n-1}) = 0, \|f^{(n)}\|_{B^\infty_{\gamma,q}} < \infty \right\}.$$

Clearly $(\boldsymbol{B}^\infty_{\gamma,q})^{(n)}$ is a vector space. A norm in $(\boldsymbol{B}^\infty_{\gamma,q})^{(n)}$ is defined by :

$$\|f\|_{\boldsymbol{B}^\infty_{\gamma,q}} = \|f^{(n)}\|_{\boldsymbol{B}_{\gamma,q}}, f \in (\boldsymbol{B}^\infty_{\gamma,q})^{(n)}.$$

Note that $\|f^{(n)}\|_{\boldsymbol{B}^\infty_{\gamma,q}} = 0$ implies $f^{(n)} = 0$, thus f is a polynomial of degree $n - 1$ at most, and with n zeros result $f = 0$.

Proposition 1 : $(B^{\infty}_{\gamma, q})^{(n)}$ is a Banach space.

PROOF : Let $\left\{f_m\right\}_{m=0}^{\infty}$ be a Cauchy sequence in $(B^{\infty}_{\gamma, q})^{(n)}$ then $\left\{f_m^{(n)}\right\}_{m=0}^{\infty}$ is a Cauchy sequence in $B^{\infty}_{\gamma, q}$. Thus there exists a function $g \in B^{\infty}_{\gamma, q}$ such that $\| f_m^{(m)} - g \|_{B^{\infty}_{\gamma, q}}$ tends to zero as n tends to infinity.

Let us consider the function

$$f(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)\,!} g(t)\, dt + p(x),$$

where $p(x)$ is a polynomial of degree $n - 1$ at most such that

$$f(x_0) = f(x_1) = \ldots = f(x_{n-1}) = 0.$$

Then

$$f^{(n)} = g \text{ and } S_{\gamma, q}(f^{(n)}) = S_{\gamma, q}(g) < \infty.$$

Hence, $f \in (B^{\infty}_{\gamma, q})^{(n)}$ and $\| f_m - f \|_{(B^{\infty}_{\gamma, q})^{(n)}} = \| f_m^{(n)} - g \|_{B^{\infty}_{\gamma, q}}$ which tends to zero when n tends to infinity.

We will use the notation $S^n_{\gamma, q}(f)$ for $S_{\gamma, q}(f^{(n)})$.

Theorem* 1 — $(B^{\infty}_{\gamma, q})^{(n)}$ *is a Banach algebra with the punctual product by an scalar multiple.

PROOF : Note that if $f \in C^1[a, b]$ if f have a zero in $y_0 \in [a, b]$, then

$$\| f \|_{\infty} \le (b-a) \| f' \|_{\infty}.$$

Moreover $f^{(k)}$ and $g^{(n-k)}$ have zeros in $[a, b]$ for all f and $g \in (B^{\infty}_{\gamma, q})^{(n)}$, and k such that $1 \le k \le n-1$. Then

$$\| f \|_{\infty} \le (b-a)^n \| f^{(n)} \|_{\infty}, \ \| g \|_{\infty} \le (b-a)^n \| g^{(n)} \|_{\infty} \quad \ldots (3.1)$$

and

$$\| (fg)^{(n)} \|_g \le 2^n (b-a)^n \| f^{(n)} \|_{\infty} \| g^{(n)} \|_{\infty}$$

On the other hand from (1.2) and lemma 1, there exists a constant $M[\gamma, q]$ such that

$$S_{\gamma, q}(f^{(k)} g^{(n-k)}) \le K[\gamma, q] [S_{\gamma, q}(f^{(k)})] [S_{\gamma, q}(g^{(n-k)})]$$

$$\leq K[\gamma, q]\,(M[\gamma, q])^n\,[S^n_{\gamma, q}(f)]\,[S^n_{\gamma, q}(g)],\ (1 \leq k \leq n-1), \qquad \ldots (3.2)$$

for all f and $g \in (B^{\infty}_{\gamma, q})^{(n)}$.

Then from (1.1) (1.2) and (3.2)

$$S_{\gamma, q}((fg)^{(n)}) = S_{\gamma, q}\left(\sum_{k=0}^{n}\binom{n}{k} f^{(k)}\, g^{(n-k)}\right)$$

$$\leq \sum_{k=1}^{n-1}\binom{n}{k} S_{\gamma, q}(f^{(k)}\, g^{(n-k)}) + S_{\gamma, q}(f\, g^{(n)}) + S_{\gamma, q}(f^{(n)}\, g)$$

$$\leq K[\gamma, q]\,(M[\gamma, q])^n\,(2^n - 2]\,[S^n_{\gamma, q}(f)]\,[S^n_{\gamma, q}(g)] +$$

$$+\ \Omega\left[\,\|f\|_\infty\, S^n_{\gamma, q}(g) + \|g^{(n)}\|_\infty\, S_{\gamma, q}(f) + \|f^{(n)}\|_\infty\, S_{\gamma, q}(g) + \|g\|_\infty\, S^n_{\gamma, q}(f)\,\right].$$

Moreover taking into account that if f have a zero in $[a, b]$ then

$$\|f\|^q_\infty \leq 2^q\, E^q_0(f) \leq 2^{q(1-\gamma)+1}\, S^q_{\gamma, q}(f),$$

and using Lemma 1, there exists a constant $L[\gamma, q]$ such that

$$\|(f\,g)^{(n)}\|^q_\infty = \left(\sum_{k=0}^{n}\binom{n}{k}\|f^{(k)}\|_\infty\,\|g^{(n-k)}\|_\infty\right)^q \leq L[\gamma, q]\,[S^n_{\gamma, q}(g)]^q$$

Then

$$\left(\|fg\|_{(B^{\infty}_{\gamma, q})^{(n)}}\right)^q = 2^{\gamma q - 1}\,\|(f\,g)^{(n)}\|^q_\infty + S^q_{\gamma, q}((fg)^{(n)}))$$

$$\leq 2^{\gamma q - 1}\, L[\gamma, q]\,[S^n_{\gamma, q}(f)]^q\,[S^n_{\gamma, q}(g)]^q$$

$$+ \Bigg(K[\gamma, q]\,(M[\gamma, q])^n\,(2^n - 2)\,[S^n_{\gamma, q}(f)]\,[S^n_{\gamma, q}(g)]$$

$$+\ \Omega\left[\,\|f\|_\infty\, S^n_{\gamma, q}(g) + \|g^{(n)}\|_\infty\, S_{\gamma, q}(f) + \|f^{(n)}\|_\infty\, S_{\gamma, q}(g) + \|g\|_\infty\, S^n_{\gamma, q}(f)\,\right]\Bigg)^q$$

Finally note that from Lemma 1 there exists a constant $J[\gamma, q]$ such that

$$S_{\gamma, q}(f) \leq J[\gamma, q]\, S^n_{\gamma, q}(f) \text{ for all } f \in (B^{\infty}_{\gamma, q})^{(n)},$$

which with (3.1) imply

$$\left(\|fg\|_{(B^{\infty}_{\gamma, q})^{(n)}}\right)^q \leq 2^{\gamma q - 1}\, L[\gamma, q]\,[S^n_{\gamma, q}(f)]^q\,[S^n_{\gamma, q}(g)]^q$$

$$+\ 2^{2(q-1)}\,((K[\gamma,q]\,(M[\gamma,q])^n\,(2^n-2)\,[S^n_{\gamma,q}(f)]\,[S^n_{\gamma,q}(g)])^q$$

$$+\ (\Omega\,[(b-a)^n\,\|f^{(n)}\|_\infty\,S^n_{\gamma,q}(g)+\|g^{(n)}\|_\infty\,J[\gamma,q]\,S^n_{\gamma,q}(f)])^q$$

$$+\ (\Omega\,[\|f^{(n)}\|_\infty\,J[\gamma,q]\,S^n_{\gamma,q}(g)+(b-a)^n\,\|g^{(n)}\|_\infty\,S^n_{\gamma,q}(f)]\,)^q).$$

It is clear from this inequality that there exists a constant K depending of a, b, γ, q and n such that

$$\left(\|fg\|_{(B^\infty_{\gamma,q})^{(n)}}\right)^q\le K\left(\|f\|_{(B^\infty_{\gamma,q})^{(n)}}\right)^q\left(\|g\|_{(B^\infty_{\gamma,q})^{(n)}}\right)^q,$$

for all f and $g\in(B^\infty_{\gamma,q})^{(n)}$.

Then

$$\|fg\|_{(B^\infty_{\gamma,q})^{(n)}}\le K^{1/q}\,\|f\|_{(B^\infty_{\gamma,q})^{(n)}}\,\|g\|_{(B^\infty_{\gamma,q})^{(n)}} \qquad \text{... (3.3)}$$

for all f and $g\in(B^\infty_{\gamma,q})^{(n)}$.

Therefore $(B^\infty_{\gamma,q})^{(n)}$ is a Banach algebra with the product

$$f\circ g=fg\Lambda_n^{-1},\ f,g\in(B^\infty_{\gamma,q})^{(n)},$$

where $$\Lambda_n=\operatorname{Inf}\left(K>0:\|fg\|_{(B^\infty_{\gamma,q})^{(n)}}\le K\,\|f\|_{(b^\infty_{\gamma,q})^{(n)}}\,\|g\|_{(B^\infty_{\gamma,q})^{(n)}},\right. \qquad \text{... (3.4)}$$

for all $$\left.f,g\in(B^\infty_{\gamma,q})^{(n)}\right)$$

4. THE QUASIPRODUCTS

The classical inner product in $L^2[a,b]$ is given by

$$\langle f,g\rangle=\int_a^b f(t)\,g(t)dt,\ f,g\in L^2[a,b].$$

These products and generalizations play an important role in several variables, Hilbert spaces and several applications in orthogonality, approximation theory and finite element method.

Let us consider the so called quasiproducts

$$>f,g<_n=(fg)^{(n)}(b)-(fg)^{(n)}(a),\ f,g\in(B^\infty_{\gamma,q})^{(n)}.$$

These quasiproducts are not inner products, (the condition $>f,f<\,\ge 0$ is not satisfied), however they satisfy inequalities Cauchy Schwartz type with respect to the norms of the algebras of n fixed points and Besov norm of derivative which can be useful in certain applications.

Theorem 2 —

(i) $| >f, g<_{n+1} | \le 2\Lambda_{n+1} \|f\|_{(B^{\infty}_{\gamma,q})^{(n+1)}} \|g\|_{(B^{\infty}_{\gamma,q})^{(n+1)}}$, *for all* $f, g \in (B^{\infty}_{\gamma,q})^{(n+1)}$.

(ii) $| >f, g<_{n+1} \le (b-a)\,\Lambda_n \|f\|_{(B^{\infty}_{\gamma,q})^{(n)}} \|g\|_{(B^{\infty}_{\gamma,q})^{(n)}}$, *for all* $f, g \in (B^{\infty}_{\gamma,q})^{(n)}$.

PROOF : (i) From (3.3) and (3.4),

$$| >f, g<_{n+1} | = | (fg)^{(n+1)}(b) - (fg)^{(n+1)}(a) | \le 2 \| (fg)^{(n+1)} \|_{\infty}$$

$$\le 2 \|fg\|_{(B^{\infty}_{\gamma,q})^{(n+1)}} \le 2\Lambda_{n+1} \|f\|_{(B^{\infty}_{\gamma,q})^{(n+1)}} \|g\|_{(B^{\infty}_{\gamma,q})^{(n+1)}}.$$

(ii) Also using (3.3) and (3.4),

$$| >f, g<_{n+1} | = | (fg)^{(n+1)}(b) - (fg)^{(n+1)}(a) | = \left| \int_a^b (fg)^{(n)}(t)dt \right|$$

$$\le (b-a) \| (fg)^{(n)} \|_{\infty} \le (b-a) \|fg\|_{(B^{\infty}_{\gamma,q})^{(n)}}$$

$$\le (b-a)\,\Lambda_n \|f\|_{(B^{\infty}_{\gamma,q})^{(n)}} \|g\|_{(B^{\infty}_{\gamma,q})^{(n)}}$$

for all $f, g \in (B^{\infty}_{\gamma,q})^{(n)}$.

4. EXAMPLES

Example 1 — (The Banach Algebras of n + 1 Fixed Points)

Let us consider $n + 1$ fixed points in $[a, b]$, $x_0, x_1, \dots, x_n$. We define the space

$$F_n = \left\{ f \in C^{(n)}[a, b] : f(x_0) = f(x_1) = \dots = f(x_n) = 0 \text{ and } S_{\gamma,q}(f^{(n)}) < \infty \right\}.$$

Clearly F_n is a vector space and $S^n_{\gamma,q}(f)$, $(f \in F_n)$ define a norm in F_n.

Note that $S^n_{\gamma,q}(f) = 0$ implies that $E_0(f^{(n)}) = 0$, then $f^{(n)} = C$, where C is a certain constant. Then f is a polynomial of degree n at most, and $f(x_0) = f(x_1) = \dots = f(x_n) = 0$ implies $f = 0$.

Theorem 3 — *$(F_n, S^n_{\gamma,q})$ is a Banach space.*

PROOF : Let $\left\{ f_m \right\}_{m=0}^{\infty}$ be a Cauchy sequence in F_n. Then $\left\{ f_m^{(n)} \right\}_{m=0}^{\infty}$ is a Cauchy sequence in $(F_n, S_{\gamma,q})$. Similarly to the proof of *(iii)* in theorem 1, $S_{\gamma,q}$ and $\| \, \|_{B^{\infty}_{\gamma,q}}$ are eqùivalents in F_n. Then $\left\{ f_m^{(n)} \right\}_{m=0}^{\infty}$ is a Cauchy sequence in $B^{\infty}_{\gamma,q}$.

Then there exists a function $g \in \boldsymbol{B}^{\infty}_{\gamma, q}$ such that $\| f_m{}^{(n)} - g \|_{\boldsymbol{B}^{\infty}_{\gamma, q}}$ tends to zero as m tends to infinity.

Let us consider the function

$$f(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)\,!} g(t)dt + p(x),$$

where $p(x)$ is a polynomial of degree n at most such that

$$f(x_0) = f(x_1) = \ldots = f(x_n) = 0.$$

Then

$$f^{(n)} = g + \lambda \text{ and } S_{\gamma, q}(f^{(n)}) = S_{\gamma, q}(g) < \infty.$$

Thus $f \in F_n$ and $S^n_{\gamma, q}(f_m - f) = S_{\gamma, q}(f_m{}^{(n)} - g)$ which tends to zero when tends to infinity.

Proposition 2 — $(F_n, S^n_{\gamma, q})$ is a Banach algebra with the pointwise product by an scalar multiple.

PROOF : Let f and $g \in F_n$. Then from (3.2) :

$$S^n_{\gamma, q}(fg) = S_{\gamma, q}\left(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \right) \leq K[\gamma, q](M[\gamma, q])^n S^n_{\gamma, q}(f)\, {}^{n}_{\gamma, q}(g) \ .$$

Thus if we define

$$\mu_n[\gamma, q] = \text{Inf}\left\{ K > 0 : S^n_{\gamma, q}(fg) \leq K S^n_{\gamma, q}(f)\, S^n_{\gamma, q}(g) \right\}$$

result

$$S^n_{\gamma, q}(fg) \leq \mu_n[\gamma, q]\, S^n_{\gamma, q}(f)\, S^n_{\gamma, q}(g), \text{ for all } f, g \in F_n.$$

Thus, let us define

$$f \circ g = (\mu_n[\gamma, q])^{-1} fg, f, g \in F_n.$$

Then

$$S^n_{\gamma, q}(f \circ g) \leq S^n_{\gamma, q}(f)\, S^n_{\gamma, q}(g), \text{ for all } f, g \in F_n.$$

Example 2 — (The Banach Spaces of n + 1 Possible Outer Points)

Let $x_0, x_1, \ldots, x_n$ be n + 1 points of an interval (c, d) which contains $[a, b]$ and at least one of this points is outer to $[a, b]$.

Let us consider the space

$R_n = f \in C^{(n)}[a, b] : f(x_0) = f(x_1) = \ldots = f(x_n) = 0$ and $S_{\gamma, q}(f^{(n)}) < \infty$ and f is extended to (c, d) in such manner that :

$f \in C^{(n)}(c, d)$ and if f is a polynomial in $[a, b]$ then f is the same polynomial in $(c, d)\}$. The norm is defined by

$$\|f\| = S_{\gamma, q}(f^{(n)}), f \in R_n,$$

and $S_{\gamma, q}$ is considered in the interval $[a, b]$.

Example 3 — (The Banach Spaces of n Possible Outer Points)

Let $x_0, x_1, \ldots, x_{n-1}$ be n points of an interval (c, d) which contains $[a, b]$ and at least one of this points is outer to $[a, b]$.

Let us consider the space

$$S_n = \{f \in C^{(n)}[a, b] : f(x_0) = f(x_1) = \ldots = f(x_{n-1}) = 0, \|f^{(n)}\|_{B^{\infty}_{\gamma, q}} < \infty\}$$

and f is extended to (c, d) in such manner that :

$f \in C^{(n)}(c, d)$ and if f is a polynomial in $[a, b]$ then f is the same polynomial in $(c, d)\}$.

The norm is defined by

$$\|f\| = \|f^{(n)}\|_{B^{\infty}_{\gamma, q}}, f \in S_n,$$

where $\|\cdot\|_{B^{\infty}_{\gamma, q}}$ is considered in the interval $[a, b]$.

The normed spaces considered in the examples 2 and 3 are Banach spaces.

The proof is similar to the proof of theorem 3.

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