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# A SECOND ORDER NECESSARY CONDITION FOR FRACTIONAL PROGRAMMING

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A treatment for multiobjective fractional programming problem has been given where use has been made for second-order nonsmooth derivative. Optimality conditions involving second-order derivatives has been introduced. The present results are in fact extensions of the previous such results of Castellani for nonlinear programming.

**Key Words : Multiobjective Fractional Programming; Castellani Results; Nonlinear Programming; Nonsmooth Optimization**

## 1. INTRODUCTION

In this paper, we shall consider the following nonsmooth fractional programming original problem as follows :

$$(FP) \text{ Minimize } \frac{f(x)}{h(x)}$$

$$\text{subject to } g(x) \leq 0, x \in X,$$

where $X$ is a convex set and $f$, $g$ are real-valued functions defined on $X \in R^n$ with $h > 0$ (or $h \neq 0$) on $X$ and $g$ is a vector function defined on $X$. In recent times, problems of this type are known in the area of mathematical programming problems as generalized fractional programming problems and have been the subject of immense interest in the past few years. For example, a number of optimality criteria, duality relations and computational algorithms for various classes of generalized fractional programming problems have appeared in the literature[2, 3&5].

Now-a-days, this interest has shifted towards higher order necessary optimality conditions and some kinds of higher order directional derivatives have been introduced. The notions of second order directional derivatives were first introduced by Ben-Tal and Zowe[2]. These results later on were generalized by Penot[10]. In the sequel, Studniarski[11] stated new second-order necessary optimality condition for a nonsmooth nonlinear optimization problem with a finite number of inequality and equality constraints. To derive these optimality conditions, Studniarski used the results of Ref. [2]. Finally, Castellani[5] improved Studniarski's result in the presence of inequality constraints only and he showed the close connection between the second-order directional derivatives and the second order variational sets in such treatments[3, 6&7].

In the present study, we shall focus our attention to derive a necessary optimality condition of an auxiliary problem in relation to the original fractional programming problem. After collecting some preliminaries in section 2, from Ref. [5], which are useful later for our purpose we have given main result in section 4.

Next we consider for a given $\mu$ an auxiliary problem in relation to the original fractional programming problem as follows :

$$(FP)_{\text{aux}} \text{ Minimize } f(x) - \mu\, h(x)$$

$$\text{subject to } g(x) \leq 0, x \in X.$$

If $x^0$ is optimal for $(FP)$ then $\exists$ a parameter $\lambda^*$ such that $x^0$ is optimal for $(FP)_{\text{aux}}$ with $\mu = \lambda^*$.

## 2. Preliminary Notations and Definitions

The following variational sets from Ref. [5] will be needed in the sequel :

*Definition* 2.1 — Let $Q \subseteq R^n$ be a set and $x_0, v \in R^n$; the following second-order variational sets of $Q$ at $x_0$ with respect to the direction $v$ are defined by

$$IT^2(Q; x_0, v) = \{y \in R^n \mid \exists \varepsilon > 0 \text{ s. t. } \forall\, t \in \,]0, \delta[,$$

$$\forall\, y' \in B(y, \delta), x_0 + tv + t^2 y' \in Q\}$$

$$= \{y \in R^n \mid \forall\, \{t_n\} \downarrow 0, \forall\, \{y_n\} \to y, x_0 + t_n v + t_n^2 y_n \in Q\} \qquad \text{... (1)}$$

$$T^2(Q; x_0, v) = \{Y \in R^n \mid \forall\, \delta > 0, \exists t \in \delta[, \exists y' \in B(y, \delta) \text{ s.t. } x_0 \mid tv + t^2 y' \in Q\}$$

$$= \{y \in R^n \mid \exists\, \{t_n\} \downarrow 0, \exists\, \{y_n\} \to y \text{.s.t. } x_0 + t_n v + t_n y_n \in Q\} \qquad \text{... (2)}$$

*Lemma* 2.1 (Ref. [5]) — For each $Q_1, Q_2 \subseteq R^n$ such that $Q_1 \cap Q_2 = \phi$ we have

$$T^2(Q_1; x_0, v) \cap IT^2(Q_2; x_0, v) = \phi$$

$$\forall\, x_0 \in R^n, \ \forall\, v \in R^n.$$

## 3. Second Order Directional Derivatives

In general, it will be difficult to define second-order directional derivatives for the auxiliary problem $(FP)_{\text{aux}}$.

*Definition* 3.1 — Let $f, h : R^n \to R$ and $x_0, v \in R^n$. The upper regularized Dini derivative of $(f - \lambda^* h)$ at $x_0$ in the direction $v$ by

$$\bar{d}\,(f-\lambda^* h)(x; v) = \limsup_{(v,\,t)\to(v,\,0^+)} \frac{(f-\lambda^* h)(x_0+tv)-(f-\lambda^* h)(x_0)}{t} \quad \text{... (3)}$$

In a similar way, lower regularized Dini derivative of $d\,(f-\lambda^* h)$ at $x_0$ in the direction $v$ by

$$\underline{d}\,(f-\lambda^* h)\ (x_0;\ v) = \liminf_{(v,\,t)\to(v,\,0^+)} \frac{(f-\lambda^* h)(x_0+tv)-(f-\lambda^* h)(x_0)}{t} \quad \text{... (4)}$$

*Definition* 3.2 — Let $f, h : R^n \to R$ and $x_0$, $v\, w \in R^n$. The second-order upper regularized Dini derivative of upper regularized Dini derivative of $(f-\lambda^* h)$ at $x_0$ in the directions $v$ and $w$ by

$$d^2\,(f-\lambda^* h)(x_0; v, w) = \limsup_{(w,\,t)\to(w,\,0^+)}$$

$$\frac{(f-\lambda^* h)(x_0+tv+t^2 w)-(f-\lambda^* h)(x_0)-td\,(f-\lambda^* h)(x_0; v)}{t^2}. \quad \text{... (5)}$$

In a similar way the second-order lower regularized Dini derivative of $(f-\lambda^* h)$ at $x_0$ in the directions $v$ and $w$ by

$$d^2\,(f-\lambda^* h)(x_0; v, w) = \limsup_{(w,\,t)\to(w,\,0)}$$

$$\frac{(f-\lambda^* h)(x_0+tv+t^2 w)-(f-\lambda^*\ )(x_0)-td\,(f-\lambda^* h)(x_0; v)}{t^2}. \quad \text{... (6)}$$

Next, we define second-order variational sets of the strict level set $L_{(f-\lambda^* h)}(x_0)$ as follows

*Definition* 3.3 — Define

$$L(f-\lambda^* h)(x_0) = \{x \in R^n \mid (f-\lambda^* h)(\dot{x}) < (f-\lambda^* h)(x_0)\}.$$

*Lemma* 3.1 — Let $(f-\lambda^* h) : R^n \to R$ and $x_0 \in R^n$; then

(*a*) for every $v \in R^n$ such that

$$\bar{d}^2\,(f-\lambda^* h)(x_0; v) < 0,$$

We have

$$IT^2\,(L_{(f-\lambda^* h)}(x_0)\,x_0; V) = R^n;$$

(*b*) for every $v \in R^n$ such that

$$d^2 (f - \lambda^* h) (x_0; V) \le 0 \text{ and } d^2 (f - \lambda^* h) (x_0; v, w) < 0,$$

we have $w \in T^2 (L_{(f - \lambda^* h)} (x_0); x_0; Q)$.

Analogously for $d^2$ and $IT^2$.

If $f$ and $h$ are Lipschitzian at $x_0$, then $w \in$ int

$$T^2 (L_{(f - \lambda^* h)} (x_0); x_0; V)$$

(c) if $\exists\, v \in R^n$ such that

$$\bar{d}_{(f - \lambda^* h)} (x_0; V) = 0 \text{ and } \bar{d}^2_{(f - \lambda^* h)} (x_0; V, w) \le 0,$$

we have $w \in IT^2 (L_{(f - \lambda^* h)} (x_0); x_0; V)$.

Analogously for $d^2$ and $IT^2$

PROOF : Proof follows from Ref. [5].

## 4. NECESSARY OPTIMALITY CONDITIONS

In the present exposition, we will state and prove necessary second order optimality condition for a nonsmooth optimization problem for fractional programming problem with a finite number of inequality constraints. Before stating necessary conditions we will state a lemma which will be needed in the sequel.

*Lemma* 4.1 — Let $x_0 \in R^n$ and $f_i, h_i \in R^n \to R$ with $i \in I = \{1, \ldots, p\}$ such that $(f_i - \lambda_i^* h_i) (x_0) = 0$ for each $i \in I$. Let

$$(f - \lambda^* h) (x) = \max_{i \in I} (f_i - \lambda_i^* h_i) (x); \text{ then}$$

$$\bar{d} (f - \lambda^* h) (x_0; V) = \max_{i \in I} \bar{d} (f_i - \lambda_i^* h_i) (x_0; V), \ \forall\, v \in R^n. \quad \ldots (7)$$

Moreover, if $\bar{d} (f - \lambda_i^* h_i) (x_0; V) = 0$ for each $i \in I$, then

$$d^2 (f - \lambda^* h) (x_0; V, w) = \max_{i \in I} \bar{d}^2 (f_i - \lambda_i^* h_i) (x_0; V, w), \ \forall\, w \in R^n \quad \ldots (8)$$

PROOF : Proof is similar as in Ref. [5].

We consider the auxiliary problem as follows :

$(FP)_{\text{aux}}$ Minimize $f(x) - \lambda^* h(x)$

subject $x \in Q$,

where $Q = \{x \in R^n \mid g_i(x) \leq 0, i \in I\}$ with $f, h, g_i : R^n \to R$ and $I = \{1, ..., p\}$. For every $x_0 \in R^n$ we choose $I(x_0) = \{i \in I \mid g_i(x) = 0\}$.

***Theorem*** 4.1 — *Let $x_0 \in Q$ be a local optimal solution for the auxiliary problem $(FP)_{aux}$ with $f$ and $h$ are Lipschitzian at $x_0$ and $g_i$ are upper semi-continuous at $x_0$ for each $i \in I \setminus I(x_0)$. Suppose that there exists $y \in R^n$ such that $g_i\ (x_0; y) < 0$ for every $i \in I(x_0)$; then for each $v, w \in R^n$ such that*

$$d(f - \lambda^* h)(x_0; v) \leq 0, \qquad \text{... } (A)$$

and for all $i \in I(x_0)$, either

$$\bar{d}\, g_i(x_0; v) < 0 \qquad \text{... } (B)$$

or

$$\bar{d}\, g_i(x_0; v) = 0 \text{ and } \bar{d}^2 g_i(x_0; v, w) \leq 0, \qquad \text{... } (C)$$

we have

$$d^2 (f - \lambda^* h)(x_0; v, w) \geq 0. \qquad \text{... } (D)$$

PROOF : We know $x_0 \in Q$ is a local optimal solution for the auxiliary problem $(FP)_{\text{aux}}$ if and only if $\exists\, \delta > 0$ such that

$$(L_{(f - \lambda^* h)}(x_0) \cap Q \cap B(x_0, \delta)) = \phi.$$

From Lemma 2.1, it follows that

$$T^2 (L_{(f-\lambda^* h)}(x_0); x_0, v) \cap IT^2 (Q \cap B(x_0, \delta); x_0, \forall) = \phi, \ \forall\, v \in R^n.$$

Let us define, for all $i \in I$,

$$Q_i = \{x \in R^n \mid g_i(x) \leq 0\}$$

and we choose $Q' = \bigcap_{i \in I(X_0)} Q_i$ and $Q'' = \bigcap_{i \in I - I(X_0)} Q_i$.

Since $g_i$ are upper semicontinuous at $x_0$ for every $i \in I \setminus I(x_0)$ then $x_0 \in \text{int } Q''$, and using Lemma 2.1 for each, we obtain

$$T^2 (L_{(f-\lambda^* h)}(x_0); X_0\, v) \cap IT^2 (Q; x_0, v) = T^2 (L_{(f-\lambda^* h)}(x_0); x_0, \forall)$$

$$\cap\, IT^2 (Q' \cap B(x_0, \delta); X_0, v) = \phi.$$

Since for every pair of disjoint sets $A$ and $B$, we have $\mathrm{Cl}\, A \cap \mathrm{int}\, B = \phi$, then

$$\mathrm{Int}\, T^2 (L_{(f-\lambda^* h)}(x_0); x_0, \forall) \cap Cl \setminus T^2 (Q'; x_0, v) = \phi, \ \forall\ v \in R^n. \qquad \text{... } (E)$$

We prove it by contradiction, let us assume that there exist $v, w \in R^n$ satisfying the hypotheses $(A)$ and either $(B)$ or $(C)$ and such that $D)$ does not hold.

i.e.,

$$d^2 (f - \lambda^* h)(x_0; v, w) < 0. \qquad \text{... } (F)$$

Let $g(x) = \max_{i \in I(x_0)} g_i(x)$; but from the Lemma 4.1, we find that either

$$\bar{d}\, g(x_0; v) < 0 \ \text{ or } \ dg\,(x_0; v) = 0 \ \text{ and } \ d^2 g\,(x_0; v, w) \le 0.$$

Moreover, by assumption $\exists$ and $L > 0$ such that

$g_i\,(x_0; v, w) \le -L$, for every $i \in I(x_0)$; then $\exists$ a $\delta > 0$ such that

$$\max_{i \in I(x_0)} \frac{g_i\,(x + ty) - g_i(x)}{t} \le -L = 1/2 < 0, \ \forall\ x \in B\,(x_{0,\,\delta})\ \forall\ t \in\, ]\,0, \delta\,[.$$

On using subadditivity of the operator max, finally, we get $g^o\,(x_0; y) < 0$.

From Lemma 3.1 and the isotonicity of the variational set $IT^2$ we have

$$w \in Cl\ IT^2 (L_g\,(x_0); X_0; v) \subseteq Cl\ IT^2\,(Q; x^0, v).$$

Moreover from $(A)$ and $(F)$, we have

$$w \in \mathrm{int}\, T^2 (L_{(f-\lambda^* h)}(x_0); x_0, \forall),$$

which is a contradiction to $(E)$. This completes the proof.

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