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SHORT-TIME ASYMPTOTICS OF THE TRACE OF THE WAVE OPERATOR FOR A GENERAL ANNULAR DRUM IN R^2 WITH ROBIN BOUNDARY CONDITIONS

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Short-time asymptotics of the trace of the wave operator $\hat{\mu}(t) = \sum_{j=1}^{\infty} \exp(-it\mu_j^{1/2})$, where $\{\mu_j\}_{j=1}^{\infty}$ are the eigenvalues of the negative Laplacian in R^2, is studied for a variety of domains, where $-\infty < t < \infty$ and $i = \sqrt{-1}$. The dependance of $\hat{\mu}(t)$ on the connectivity of bounded drums and the Robin boundary conditions are analyzed. Particular attention is given to a general annular drum in R^2 together with the Robin boundary conditions on its boundaries.

Key Words : Short Time Asymptotics; Wave Operator; Laplacian in R^2; Robin Boundary Conditions; Heat Equation; Kernel; Wave Equation; Annular Drum; Inverse Problem; Eigenvalues

1. INTRODUCTION

The underlying inverse problem is to deduce some geometric quantities associated with a bounded domain from complete knowledge of the eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ for the negative Laplace operator $-\Delta = -\sum_{n=1}^{2} \left(\frac{\partial}{\partial x^n}\right)^2$ in the (x^1, x^2)-plane.

Let $\Omega \subseteq R^2$ be a simply connected bounded domain with a smooth boundary $\partial\Omega$. Consider the Robin problem

$$(\Delta + \mu)\, u = 0 \text{ in } \Omega, \left(\frac{\partial}{\partial n} + \gamma\right) u = 0 \text{ on } \partial\Omega, \quad \text{... (1.1)}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial\Omega$, γ is a positive constant and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Denote its eigenvalues, counted according to multiplicity, by

$$0 \le \mu_1 \le \mu_2 \le \mu_3 \le \ldots \le \mu_j \le \ldots \to \infty \text{ as } j \to \infty. \quad \text{... (1.2)}$$

Zayed *et al.*[14, 16] have recently discussed the problem (1.1) and have determined some geometrical properties associated with Ω, by using the asymptotic expansion of the trace of the wave operator

$$\hat{\mu}(t) = tr\,(e^{-it\Delta^{1/2}}) = \sum_{j=1}^{\infty} e^{-it\mu_j^{1/2}} \qquad \dots (1.3)$$

which represents a tempered distribution for $-\infty < t < \infty$ and $i = \sqrt{-1}$.

Zayed[17] has shown that if $\gamma = 0$ (the Neumann problem), then

$$\hat{\mu}(t) = \frac{|\Omega|}{2\pi t} H(|t|) + \frac{|\partial\Omega|}{8} sign\, t + a_0 |t| + O(t^2\, sign\, t) \text{ as } |t| \to 0^+, \qquad \dots (1.4)$$

while if $\gamma \to \infty$ (the Dirichlet problem), then

$$\hat{\mu}(t) = \frac{|\Omega|}{2\pi t} H(|t|) - \frac{|\partial\Omega|}{8} sign\, t + a_0 |t| + O(t^2\, sign\, t) \text{ as } |t| \to 0^+, \qquad \dots (1.5)$$

where $H(|t|)$ is the Heaviside's unit function, and

$$sign\, t = \begin{cases} 1 & t \geq 0, \\ 0 & t = 0, \\ -1 & t < 0. \end{cases}$$

In these formulae, $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is the length of $\partial\Omega$ and the sign $\pm$ of the second term in (1.4) and (1.5) determines whether we have the Neumann or the Dirichlet problem. The coefficient a_0 has geometric significance, e.g., if Ω is smooth and convex, then $a_0 = \frac{1}{6}$, and if Ω is permitted to have a finite number of smooth convex holes "h", then $a_0 = (1-h)\frac{1}{6}$. Further, the order term $O\,(t^2\, sign\, t)$ in (1.4) and (1.5) is yet undetermined. So, in the present paper, we discuss what geometric quantities are contained in this order term.

The object of this paper is to discuss the following more general inverse problem: Let Ω be a general annular drum in R^2 consisting of a simply connected bounded inner domain Ω_1 with a smooth boundary $\partial\Omega_1$ and a simply connected bounded outer domain $\Omega_2 \supset \overline{\Omega}_1$ with a smooth boundary $\partial\Omega_2$ where $\overline{\Omega}_1 = \Omega_1 \cup \partial\Omega_1$. Suppose that the eigenvalues (1.2) are given exactly for the Helmholtz equation

$$(\Delta + \mu)\, u = 0 \text{ in } \Omega, \qquad \dots (1.6)$$

together with the following Robin boundary conditions

$$\left(\frac{\partial}{\partial n_1} + \gamma_1\right) u = 0 \text{ on } \partial\Omega_1, \quad \left(\frac{\partial}{\partial n_2} + \gamma_2\right) u = 0 \text{ on } \partial\Omega_2, \qquad \dots (1.7)$$

where $\frac{\partial}{\partial n_1}$ and $\frac{\partial}{\partial n_2}$ denote differentiations along the inward pointing normal to $\partial\Omega_1$ and $\partial\Omega_2$ respectively, while γ_1 and γ_2 are positive constants.

The basic problem is to determine some geometric quantities associated with the general annular drum Ω in R^2, from the asymptotic expansion of the spectral function

$$\hat{\mu}(t) = \sum_{j=1}^{\infty} \exp(-it\mu_j^{1/2}) \text{ as } |t| \to 0^+. \qquad \text{... (1.8)}$$

Note that the problem (1.6)-(1.7) has been discussed recently by Zayed[18] in the case where $\Omega = \{(r, \theta) : a \le r \le b, 0 \le \theta \le 2\pi\}$ is a circular annulus.

We close this section with the remark that an alternative to (1.8) is to study the trace of the heat kernel $\Theta(t) = \sum_{j=1}^{\infty} \exp(-t\mu_j)$ as $t \to 0^+$, (see for example [1-5], [7-13], [15], [19-21]). But, it is well known that the wave equation methods have given very strong results; the definitive one is that of Hörmander[6], who has studied the distribution $\hat{\mu}(t) = tr(e^{-itP})$ near $t = 0$ for an elliptic positive semidefinite pseudodifferential operator P in R^n of order m.

2. Formulation of the Mathematical Problem

With reference to the articles[14, 16 — 18], it can be easily seen that the spectral function $\hat{\mu}(t)$ associated with the problem (1.6)-(1.7) is given by

$$\hat{\mu}(t) = \int\int_{\Omega} G(\underset{\sim}{x}, \underset{\sim}{x}; t)\, d\underset{\sim}{x}, \qquad \text{... (2.1)}$$

where $G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ is the Green's function for the wave equation

$$\left(\Delta - \frac{\partial^2}{\partial t^2}\right) G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) = 0 \text{ in } \Omega \times \{-\infty < t < \infty\}, \qquad \text{... (2.2)}$$

subject to the Robin boundary conditions (1.7) and the initial conditions

$$\lim_{t \to 0} G(\underset{\sim}{x}_1 \underset{\sim}{x}_2; t) = 0, \quad \lim_{t \to 0} \frac{\partial G(\underset{\sim}{x}_1 \underset{\sim}{x}_2; t)}{\partial t} = \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2), \qquad \text{... (2.3)}$$

where $\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$ is Dirac delta function located at the source point $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$.

Let us write

$$G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) = G_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) + \chi(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t), \qquad \text{... (2.4)}$$

where $$G_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) = \frac{H(|t| - |\underset{\sim}{x}_1 \underset{\sim}{x}_2|)}{2\pi\sqrt{t^2 - |\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^2}}, \qquad \text{... (2.5)}$$

is the "fundamental solution" of the wave eq. (2.2) while $\chi(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ is the "regular solution" chosen in such a way that $G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ satisfies the Robin boundary conditions (1.7). On setting $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 = \underset{\sim}{x}$ we find that

$$\hat{\mu}(t) = \frac{|\Omega|}{2\pi t} H(|t|) + K(t), \qquad \text{... (2.6)}$$

where

$$K(t) = \iint_{\Omega} \chi(\underset{\sim}{x}, \underset{\sim}{x}; t)\, d\underset{\sim}{x}. \qquad \text{... (2.7)}$$

The problem now is to determine the asymptotic expansion of $K(t)$ for small $|t|$. In what follows we shall use Fourier transforms with respect to $-\infty < t < \infty$ and use $-\infty < \eta < \infty$ as the Fourier transform parameter. Thus, we define

$$\hat{G}\, \underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = \int_{-\infty}^{+\infty} e^{-2\pi i \eta t} G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)\, dt. \qquad \text{... (2.8)}$$

An application of the Fourier transform to the wave eq. (2.2) shows that $\hat{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta)$ satisfies the reduced wave equation

$$(\Delta + 4\pi^2\eta^2)\, \hat{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = -\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) \text{ in } \Omega, \qquad \text{... (2.9)}$$

together with the Robin boundary conditions (1.7).

The asymptotic expansion of $K(t)$, for small $|t|$, may then be deduced directly from the asymptotic expansion of $\hat{K}(\eta)$, for large $|\eta|$, where

$$\hat{K}(\eta) = \iint_{\Omega} \hat{\chi}(\underset{\sim}{x}, \underset{\sim}{x}; \eta)\, d\underset{\sim}{x}. \qquad \text{... (2.10)}$$

3. DERIVATION OF OUR RESULTS

It is well known (see for example [13, 16, 17, 18]) that the reduced wave eq. (2.9) has the fundamental solution

$$\hat{G}_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = -\frac{1}{2} Y_0(2\pi\eta\, r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2}), \qquad \text{... (3.1)}$$

where $r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2} = |\underset{\sim}{x}_1 - \underset{\sim}{x}_2|$ is the distance between the points $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ of the general annualar drum Ω, while Y_0 is the Bessel function of the second kind and of zero order. The existence of (3.1) enables us to construct an integral equation for $\hat{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta)$ satisfying the Robin boundary conditions (1.7).

Therefore, if we consider the main problem (1.6)-(1.7) with the case $0 < \gamma_1, \gamma_2 << 1$, then Green's theorem gives the following integral equation :

$$\hat{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = -\frac{1}{2} Y_0\left(2\pi\eta\, r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2}\right)$$

$$-\frac{1}{2}\int_{\partial\Omega_1} \hat{G}(\underset{\sim}{x}_1, \underset{\sim}{y}; \eta)\left[\left(\frac{\partial}{\partial n_{1\underset{\sim}{y}}} + \gamma_2\right) Y_0(2\pi\eta\, r_{\underset{\sim}{y}\underset{\sim}{x}_2})\right] d\underset{\sim}{y}.$$

$$+\frac{1}{2}\int_{\partial\Omega_2} \hat{G}(x_1, y; \eta)\left[\left(\frac{\partial}{\partial n_{2y}} + \gamma_2\right) Y_0(2\pi\eta\, r_{y x_2})\right] dy. \qquad \text{... (3.2)}$$

On applying the iteration methods (see [14], [19], [20]) to the integral eq. (3.2), we obtain the Green's function $\hat{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta)$ which has the following regular part :

$$\hat{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = \frac{1}{4}\int_{\partial\Omega_1} Y_0\left(2\pi\eta\, r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)\left[\left(\frac{\partial}{\partial n_{1,\underset{\sim}{y}}} + \gamma_1\right) Y_0(2\pi\eta\, r_{\underset{\sim}{y}\underset{\sim}{x}_2})\right] d\underset{\sim}{y}$$

$$-\frac{1}{4}\int_{\partial\Omega_2} Y_0\left(2\pi\eta\, r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)\left[\left(\frac{\partial}{\partial n_{2,\underset{\sim}{y}}} + \gamma_2\right) Y_0(2\pi\eta\, r_{\underset{\sim}{y}\underset{\sim}{x}_2})\right] d\underset{\sim}{y}$$

$$-\frac{1}{4}\int_{\partial\Omega_1}\int_{\partial\Omega_1} Y_0\left(2\pi\eta\, r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right) M_1(\underset{\sim}{y}, \underset{\sim}{y}')\left[\left(\frac{\partial}{\partial n_{1,\underset{\sim}{y}'}} + \gamma_1\right) Y_0(2\pi\eta\, r_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right] d\underset{\sim}{y}\, d\underset{\sim}{y}'$$

$$-\frac{1}{4}\int_{\partial\Omega_2}\int_{\partial\Omega_2} Y_0\left(2\pi\eta\, r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right) M_2(\underset{\sim}{y}, \underset{\sim}{y}')\left[\left(\frac{\partial}{\partial n_{1,\underset{\sim}{y}'}} + \gamma_2\right) Y_0(2\pi\eta\, r_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right] d\underset{\sim}{y}\, d\underset{\sim}{y}'$$

$$+\frac{1}{4}\int_{\partial\Omega_1}\left\{\int_{\partial\Omega_2} Y_0\left(2\pi\eta\, r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right) M_2(\underset{\sim}{y}, \underset{\sim}{y}')\, d\underset{\sim}{y}\right\}\left\{\left(\frac{\partial}{\partial n_{1,\underset{\sim}{y}'}} + \gamma_2\right) Y_0(2\pi\eta\, r_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right\} d\underset{\sim}{y}'$$

$$+\frac{1}{4}\int_{\partial\Omega_2}\left\{\int_{\partial\Omega_1} Y_0\left(2\pi\eta\, r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right) M_1(\underset{\sim}{y}, \underset{\sim}{y}')\, d\underset{\sim}{y}\right\}\left\{\left(\frac{\partial}{\partial n_{2,\underset{\sim}{y}'}} + \gamma_2\right) Y_0(2\pi\eta\, r_{\underset{\sim}{y}'\underset{\sim}{x}_2})\right\} d\underset{\sim}{y}'$$

... (3.3)

where $$M_1(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\upsilon=0}^{\infty} (-1)^{\nu} K_1^{(\upsilon)}(\underset{\sim}{y}', \underset{\sim}{y}),$$

$$M_2(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\upsilon=0}^{\infty} (-1)^{\nu} K_2^{(\upsilon)}(\underset{\sim}{y}', \underset{\sim}{y}),$$

$$K_1(\underset{\sim}{y}, \underset{\sim}{y}') = \frac{1}{2}\left(\frac{\partial}{\partial n_{1,\underset{\sim}{y}}} + \gamma_1\right) Y_0\left(2\pi\eta\, r_{\underset{\sim}{y}\underset{\sim}{y}'}\right)$$

and
$$K_2(\underset{\sim}{y}', \underset{\sim}{y}) = \frac{1}{2}\left(\frac{\partial}{\partial n_{2,\underset{\sim}{y}}} + \gamma_2\right) Y_0\left(2\pi\eta\, r_{\underset{\sim}{y}\underset{\sim}{y}'}\right)$$

On using argument similar to that obtained in [14, 16, 19], we deduce after some mathematical analysis, that the asymptotic expansion of $\hat{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta)$ has the following form :

$$\hat{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = \hat{\chi}_1(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) + \hat{\chi}_2(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta), \qquad \text{... (3.4)}$$

where (a) : if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_1)$ then

$$\hat{\chi}_1(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = \frac{1}{4}\left\{1 - \gamma_1\left(\frac{\partial}{\partial \xi_1^2}\right)^{-1}\right\} Y_0(2\pi\eta\,\rho_{12})$$

$$+ O\{\eta^{-1}\exp(-A_1\eta\,\rho_{12})\}, \text{ as } |\eta| \to \infty. \qquad \text{... (3.5)}$$

(b) : if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_2)$ then

$$\hat{\chi}_2(\underset{\sim}{x}_1, \underset{\sim}{x}_2; \eta) = -\frac{1}{4}\left\{1 - \gamma_2\left(\frac{\partial}{\partial \xi_1^2}\right)^{-1}\right\} Y_0(2\pi\eta\,\rho_{12})$$

$$+ O\{\eta^{-1}\exp(-A_2\eta\,\rho_{12})\}, \text{ as } |\eta| \to \infty, \qquad \text{... (3.6)}$$

where A_1 and A_2 are positive constants, while ρ_{12} is the distance between the points

$\underset{\sim}{\xi}_1 = (\xi_1^1, \xi_1^2)$ and $\underset{\sim}{\xi}_2 = (\xi_2^1, -\xi_2^2)$ of the upper half plane $\xi^2 > 0$ (see [14]).

With reference to two articles[14 & 19], it can be seen that for $\xi^2 \geq h_J > 0$, $(J = 1, 2)$ the functions $\hat{\chi}_J(\underset{\sim}{x}, \underset{\sim}{x}; \eta)$ are of order $O\{\exp(-2\eta A_J h_J)\}$, $(J = 1, 2)$, and the integral of the function $\hat{\chi}(\underset{\sim}{x}, \underset{\sim}{x}; \eta)$ over the general annular drum Ω can be approximated in the following way (see (2.10)):

$$\hat{K}(\eta) = \int_{\xi^2=0}^{h_2} \int_{\xi^1=0}^{L_2} \hat{\chi}_2(\underset{\sim}{x}, \underset{\sim}{x}; \eta)\,[1 - k_2(\xi^1)\,\xi^2]\,d\xi^1\,d\xi^2$$

$$- \int_{\xi^2=0}^{h_1} \int_{\xi^1=0}^{L_1} \hat{\chi}_1(\underset{\sim}{x}, \underset{\sim}{x}; \eta)\,[1 + k_1(\xi^1)\,\xi^2]\,d\xi^1\,d\xi^2$$

$$+ \sum_{J=1}^{2} O\ \{\exp(-2\,\eta A_J h_J)\} \text{ as } |\eta| \to \infty, \qquad \text{... (3.7)}$$

where L_J and $\dot{k}_J$ $(J = 1, 2)$ are respectively the total lengths and the curvatures of the boundaries $\partial\Omega_J$ $(J = 1, 2)$ of the general annular drum Ω.

If the e^{λ}-expansions of $\hat{\chi}_J(\underset{\sim}{x}, \underset{\sim}{x}; \eta)$, (see for example [14]) are introduced into (3.7), one obtains an asymptotic series of the form

$$\hat{K}(\eta) = \sum_{n=1}^{p} a_n \eta^{-n} + O(\eta^{-p-1}) \text{ as } |\eta| \to \infty, \qquad \text{... (3.8)}$$

where the coefficients a_n in (3.8) are calculated from the e^{λ}-expansions with the help of the formula (11.3) of Sec. 11 in [14].

On inverting Fourier transforms to both sides of (3.8) and using (2.6) we deduce, in the case $0 < \gamma_1, \gamma_2 << 1$, that

$$\hat{\mu}(t) = \frac{|\Omega|}{2\pi t} H(|t|) + \frac{1}{8}\left(\sum_{J=1}^{2} L_J\right) sign\, t + \frac{3}{\pi}(\gamma_1 L_1 - \gamma_2 L_2)\frac{|t|}{6}$$

$$+ \frac{7}{512}\left\{\sum_{J=1}^{2} \int_{\partial\Omega_J} \left[k_J^2(\sigma_J) - \frac{64}{7}\left(\frac{\pi\gamma_J}{L_J} - \gamma_J^2\right)\right] d\sigma_J\right\}$$

$$t^2\, sign\, t + O(t^3\, sign\, t) \text{ as } t \to 0^+. \qquad \text{... (3.9)}$$

Note that, if we set $\gamma_1 = \gamma_2 = 0$ in (3.9) we obtain the result of the Neumann boundary conditions on $\partial\Omega_J$ $(J = 1, 2)$.

With reference to the formula (1.4) and to the articles [14], [18], our result (3.9) may be interpreted as follows :

(i) Ω is a general annular drum in R^2 and we have the Robin boundary conditions (1.7) with small impedances γ_1 and γ_2.

(ii) For the first four terms, Ω is a general annular drum in R^2 with the area $|\Omega|$, it has $h = 1 - \frac{3}{\pi}(\gamma_1 L_1 - \gamma_2 L_2)$ holes, the boundaries $\partial\Omega_J (J = 1, 2)$ are of lengths $\sum_{J=1}^{2} L_J$ and of curvatures $\left[k_J^2(\sigma_J) - \frac{64}{7}\left(\frac{\pi\gamma_J}{L_J} - \gamma_J^2\right)\right]^{1/2}$ together with the Neumann boundary conditions, provided "h" is a positive integer.

(iii) The order term $O(t^3 \, sign \, t)$ may contain further information about the geometry of Ω, and its determination is still an open problem.

We close this section with the remark that there are other three cases, where (i) $\gamma_1, \gamma_2 >> 1$ or (ii) $0 < \gamma_1 << 1, \gamma_2 >> 1$ or (iii) $\gamma_1 >> 1, 0 < \gamma_2 << 1$ which can be treated similarly, and have been left for the interested readers.

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