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A CONFORMALLY FLAT CONTACT RIEMANNIAN (κ, μ)-SPACE

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In this paper, we determine a conformally flat (κ, μ)-space and determine a (κ, μ)-space with vanishing C-Bochner curvature tensor.

Key Words : Contact Riemannian (κ, μ)-Space; Conformally Flat Space; *C*-Bochner Curvature Tensor

1. INTRODUCTION

In [8] and [9] Tanno showed that a conformally flat K-contact Riemannian manifold is of constant curvature +1, which generalizes the corresponding result for a Sasakian manifold due to Okumura[7]. And Blair and Koufogiorgos[3] proved that the standard contact Riemannian structure of the unit tangent sphere bundle is conformally flat if and only if the base manifold is two-dimensional and of constant Gaussian curvature 0 or + 1.

Recently, in [4] the authors introduced the class of contact Riemannian manifolds $(M; \eta, \phi, g)$ which is determined by the equation :

$$R(X, Y)\,\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where κ, μ are real numbers and $2h$ is the Lie derivative of ϕ in the characteristic direction ξ. A contact Riemannian manifold belonging to this class is called a *(κ, μ)-space*[5]. This class contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. It was proved in [4] the standard contact Riemannian structure of the unit tangent sphere bundle is a (κ, μ)-space if and only if the base manifold is of constant curvature c with $\kappa = c(2 - c)$ and $\mu = -2c$. (By virtue of the result of Tashiro[11], we know that when $c \neq 1$, the unit tangent sphere bundle is non-Sasakian.) In the 3-dimensional case (κ, μ) spaces include the unimodular Lie groups $SU(2)$, $SL(2, \mathbb{R})$, $E(2)$, $E(1, 1)$ with a (non-Sasakian) left invariant metric. Also, it is remarkable that this class of spaces is invariant under D-homothetic transformation (see Section 2). Very recently, Boeckx[5] presented explicit examples for all possible dimensions and all possible (κ, μ). In his classification of this space we find new examples up to D-homothetic transformation, that is, Lie groups with left-invariant contact Riemannian structures with dimension ≥ 5.

These circumstances lead us to consider a conformally flat (κ, μ)-spaces. In section 3, we prove following theorem A.

Theorem A — *Let M be a conformally flat (κ, μ)-space. Then M is a 3-dimensional flat manifold or a Sasakian manifold of constant curvature + 1.*

On the other hand, as a Kählerian analogue to the Weyl conformal curvature tensor the Bochner curvature tensor in a Kählerian manifold is known. By using the Boothby-Wang fibration, the *C-Bochner curvature tensor*[6] is constructed from the Bochner curvature tensor. In section 4, we prove following theorem *B*.

Theorem B — *Let M be a (κ, μ)-space with vanishing C-Bochner curvature tensor. Then M is a Sasakian manifold.*

We denote by $T_1 M(c)$ the unit tangent sphere bundle of a space of constant curvature c with standard contact Riemannian structure. Applying Theorem *B* for $T_1 M(c)$, we have

Corollary C — If $T_1 M(c)$ is of vanishing *C*-Bochner curvature tensor, then $c = 1$.

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. A $(2n + 1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists an associated Riemannian metric g and $(1, 1)$-type tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \qquad \dots (2.1)$$

where X and Y are vector fields on M. From (2.1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \qquad \dots (2.2)$$

A Riemannian manifold M equipped with structure tensors (η, ϕ, g) satisfying (2.1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, \phi, g)$. Given a contact Riemannian manifold M, we define a $(1, 1)$-type tensor field h by $h = \frac{1}{2} L_\xi \phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0 \text{ and } h\phi = -\phi h, \qquad \dots (2.3)$$

and

$$\nabla_X \xi = -\phi X - \phi hX, \qquad \dots (2.4)$$

where ∇ is Levi-Civita connection.

A contact Riemannian manifold for which ξ is Killing is called a *K*-contact Riemannian manifold. It is easy to see that a contact Riemannian manifold is *K*-contact if and only if $h = 0$. For a contact Riemannian manifold M one may define naturally an almost complex structure J on $M \times \mathbb{R}$;

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\phi, \phi] + 2d\,\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is characterized by the condition

$$(\nabla_X \phi)\, Y = g\,(X, Y)\,\xi - \eta\,(Y)\,X \qquad \text{... (2.5)}$$

for all vector fields X and Y on the manifold. We denote by R the Riemannian curvature tensor of M defined by

$$R\,(X, Y)\,Z = \nabla_X (\nabla_Y Z) - \nabla_Y (\nabla_X Z) - \nabla_{[X, Y]} Z$$

for all vector fields X, Y, Z on M. It is well-known that M is Sasakian if and only if

$$R\,(X, Y)\,\xi = \eta\,(Y)\,X - \eta\,(X)\,Y \qquad \text{... (2.6)}$$

for all vector fields X and Y. For a contact Riemannian manifold M, the tangent space $T_p M$ of M at each point $p \in M$ is decomposed as $T_p M = D_p \oplus \{\xi\}_p$ (direct sum), where $D_p = \{v \in T_p M \mid \eta\,(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. The $2n$-dimensional distribution D is called the contact distribution. A D-homothetic transformation[10] is defined by a change of structure tensors of the form :

$$\bar{\eta} = a\,\eta, \quad \bar{\xi} = \left(\frac{1}{a}\right)\xi, \; \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\,\eta \otimes \eta,$$

where a is a positive constant. For more details about a contact Riemannian manifold, we refer to [1].

In [2], Blair proved that a contact Riemannian manifold M satisfies $R(X, Y)\,\xi = 0$ for all vector fields X, Y on M if and only if M is locally the product of an $(n + 1)$ -dimensional flat manifold and an n-dimensional manifold of positive constant curvature 4. Recently, Blair, Koufogiorgos and Papantoniou[4] introduced a class of contact Riemannian manifolds which are defined by the equation :

$$R\,(X, Y)\,\xi = \kappa\,(\eta\,(Y)\,X - \eta\,(X)\,Y) + \mu\,(\eta\,(Y)\,hX - \eta\,(X)\,hY) \qquad \text{... (2.7)}$$

where κ, μ are constants h already introduced. We call such a space a (κ, μ)-space. It was shown that[4] a contact Riemannian manifold satisfying (2.7) is obtained by applying a D-homothetic transformation on a contact Riemannian manifold with $R\,(X, Y)\,\xi = 0$. It is well-known that the unit tangent sphere bundle of a flat Riemannian manifold admits a contact Riemannian structure satisfying $R(X, Y)\,\xi = 0$. (Cf. [1, 137 pp.]). In [4], the authors classified the 3-dimensional case and showed that this class contains the unit tangent sphere bundles of Riemannian manifolds of constant sectional curvature. Furthermore in the same paper it is shown that a contact Riemannian manifold M in this class satisfies

$$(\nabla_X h)\,Y - (\nabla_Y h)\,X = (1-\kappa)\;\{2g\,(X, \phi Y)\,\xi + \eta\,(X)\,\phi Y - \eta\,(Y)\,\phi X\}$$

$$+\;(1-\mu)\;\{\eta\,(X)\,\phi h Y - \eta\,(Y)\,\phi h X\}, \qquad \text{... (2.8)}$$

where X, Y are vector fields on M. Here, we state two useful results in proving our Theorems A and B.

Theorem 2.1[4] — *Let $M = (M; \eta, \phi, g)$ be a (κ, μ)-space, then $\kappa \leq 1$. If $\kappa = 1$, then $h = 0$ and M is a Sasakian manifold. If $k < 1$, then M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$. Moreover*

$$R(X_\lambda, Y_\lambda) Z_{-\lambda} = (\kappa - \mu)\{g(\phi Y_\lambda, Z_{-\lambda}) \phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda}) \phi Y_\lambda$$

$$R(X_{-\lambda}, Y_{-\lambda}) Z_\lambda = (\kappa - \mu)\{g(\phi Y_{-\lambda}, Z_\lambda) \phi X_{-\lambda} - g(\phi X_{-\lambda}, Z_\lambda) \phi Y_{-\lambda}\},$$

$$R(X_\lambda, Y_{-\lambda}) Z_{-\lambda} = \kappa g(\phi X_\lambda, Z_{-\lambda}) \phi Y_{-\lambda} + \mu g(\phi X_\lambda, Y_{-\lambda} \phi Z_{-\lambda},$$

$$R(X_\lambda, Y_{-\lambda}) Z_\lambda = -\kappa g(\phi Y_{-\lambda}, Z_\lambda) \phi X_\lambda - \mu g(\phi Y_{-\lambda}, X_\lambda) \phi Z_\lambda$$

$$R(X_\lambda, Y_\lambda) Z_\lambda = \{2(1+\lambda) - \mu\} \{g(Y_\lambda, Z_\lambda) X_\lambda - g(X_\lambda, Z_\lambda) Y_\lambda\},$$

$$R(X_{-\lambda}, Y_{-\lambda}) Z_{-\lambda} = \{2(1-\lambda) - \mu\} \{g(Y_{-\lambda}, Z_{-\lambda}) X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda}) Y_{-\lambda}\},$$

where $X_\lambda, Y_\lambda, Z_\lambda \in D(\lambda)$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in D(-\lambda)$.

Theorem 2.2[4] — *For (κ, μ)-space with $\kappa < 1$, the Ricci operator Q is given by*

$$Q = \{2(n-1) - n\mu\} I + \{2(n-1) + \mu\} h + \{2(1-n) + n(2\kappa + \mu)\} \eta \otimes \xi. \quad \text{... (2.9)}$$

3. Conformally Flat (κ, μ)-Space

Let $M = (M^{2n+1}; \eta, \phi, g)$ be a $(2n+1)$-dimensional $(n > 1)$ conformally flat (κ, μ)-space. In case that M is Sasakian, it was proved in [8] and [9] that M is a constant curvature + 1. We now suppose that M is non-Sasakian ($\kappa \neq 1$). Then the well-known Weyl's theorem gives

$$R(X, Y) Z = 1/(2n-1) \{g(Y, Z) QX - g(X, Z) QY + g(QY, Z) X - g(QX, Z) Y\}$$

$$- (r/2n(2n-1)) \{g(Y, Z) X - g(X, Z) Y\}, \quad \text{... (3.1)}$$

where r denotes the scalar curvature. From (3.1) it follows that

$$R(X, Y) \xi = 1/(2n-1) \{\eta(Y) QX - \eta(X) QY + \eta(QY) X - \eta(QX) Y\}$$

$$- (r/2n(2n-1)) \{\eta(Y) X - \eta(X) Y\}. \quad \text{... (3.2)}$$

But from (2.9) we get

$$Q\xi = 2n\kappa\xi.$$

And thus (3.2) becomes

$$R(X, Y) \xi = 1/(2n-1) \{\eta(Y) QX - \eta(X) QY + 2n\kappa\eta(Y) X - 2n\kappa\eta(X) Y\}$$

$$- (r/2n(2n-1)) \{\eta(Y) X - \eta(X) Y\}.$$

Together with (2.7), we have

$$\kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

$$= 1/(2n-1)\ \{\eta(Y)QX - \eta(X)QY + 2n\kappa\eta(Y)X - 2n\kappa\eta(X)Y\}$$

$$- (r/2n\ (2n-1))\ \{\eta(Y)X - \eta(X)Y\} \qquad \text{... (3.3)}$$

for any vector fields X, Y on M. If we put $X = \xi$ in (3.3), then we obtain

$$\kappa(\eta(Y)\xi - Y) - \mu hY$$

$$= 1/(2n-1)\ \{4n\kappa\eta(Y)\xi - \{[2(n-1) - n\mu]Y + [2(n-1) + \mu]hY\} - 2n\kappa Y\}$$

$$- (r/2n\ (2n-1))\ \{\eta(Y)\xi - Y\}. \qquad \text{... (3.4)}$$

Applying h in (3.4), then we get

$$\kappa h + \mu h^2 = 1/(2n-1)\ \{[2(n-1) - n\mu]h + [2(n-1) + \mu]h^2 + 2n\kappa h\}$$

$$- (1/2n\ (2n-1))\ rh.$$

If we take the trace of both sides, then since the trance of h is zero we have

$$2(n-1)(\mu-1)\cdot trh^2 = 0,$$

which yields that $\mu = 1$. Hence, the Ricci operator Q is given by

$$Q = (n-2)I + (2n-1)h + [(2-n) + 2n\kappa]\,\eta\otimes\xi. \qquad \text{... (3.5)}$$

And thus together with (3.1) we get

$$R(X, Y)Z$$

$$= 1/(2n-1)\ \{(n-2)g(Y,Z)X + (2n-1)g(hY,Z)X + [(2-n) + 2n\kappa]\eta(Y)\eta(Z)X$$

$$- (n-2)g(X,Z)Y - (2n-1)g(hX,Z)Y - [(2-n) + 2n\kappa]\eta(X)\eta(Z)Y$$

$$+ g(Y,Z)\ \{(n-2)X + (2n-1)hX + [(2-n) + 2n\kappa]\eta(X)\xi\}$$

$$- g(X,Z)\ \{(n-2)Y + (2n-1)hY + [(2-n) + 2n\kappa]\eta(Y)\xi\}\}$$

$$- r/2n\ (2n-1))\ \{g(Y,Z)X - g(X,Z)Y\},$$

where the scalar curvature $r = 2n\,(n-2+\kappa)$. For X_λ, $Y_\lambda \in D(\lambda), Z_{-\lambda} \in D(-\lambda)$, then we have

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0. \qquad \text{... (3.6)}$$

But, Theorem 2.1 yields that

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = (\kappa - \mu)\ \{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\}, \qquad \text{... (3.7)}$$

where $X_\lambda, Y_\lambda \in D(\lambda), Z_{-\lambda} \in D(-\lambda)$ and $\lambda = \sqrt{1-\kappa}$. Thus, from (3.6) and (3.7) we have

$$\kappa = \mu = 1,$$

which is a contradiction to $\kappa < 1$. And hence, we conclude that M must be Sasakian and is of constant sectional curvature + 1, when $n > 1$.

Next, let M be a 3-dimensional (k, μ)-space. Suppose that M is non-Sasakian and conformally flat. Then the Wely's theorem gives

$$(\nabla_Z Q) X - (\nabla_X Q) Z = 1/4 \ \{(Zr) X - (X r) Z\}.$$

From (2.9) $r = 2(\kappa - \mu)$ and is therefore constant. Thus it follows that

$$(\nabla_Z Q) X - (\nabla_X Q) Z = 0.$$

Now differentiating (2.9) and using (2.8), $(\nabla_Z Q) X - (\nabla_X Q) Z = 0$ yields

$$\begin{aligned} &\mu \ \{(1-\kappa) \ \{2g(Z, \phi X) \xi + \eta(Z) \phi X - \eta(X) \phi Z\} + (1-\mu) \\ &\qquad \{\eta(Z) \phi h X - \eta(X) \phi h Z\}\} \\ &= (2\kappa + \mu) \ \{g((\phi + \phi h) Z, X) \xi + \eta(X)(\phi + \phi h) Z - g((\phi + \phi h) X, Z) \xi \\ &\qquad - \eta(Z)(\phi + \phi h) X\}, \end{aligned} \qquad \text{... (3.8)}$$

where X, Z are vector fields on M. If we put $Z = \xi$ in (3.8), then by using (2.2) and (2.3) we get

$$\mu\{(1-\kappa)\phi + (1-\mu)\phi h\} = -(2\kappa + \mu)(\phi + \phi h). \qquad \text{... (3.9)}$$

If we take the component of ξ-direction in (3.8), then we get

$$(2\mu + 2\kappa - \mu\kappa)\phi = 0. \qquad \text{... (3.10)}$$

and thus together with (3.9) we get also

$$(2\mu + 2\kappa - \mu\kappa)\phi = 0. \qquad \text{... (3.11)}$$

Since M is non-Sasakian, from (3.10) and (3.11) we obtain

$$2\mu + 2\kappa - \mu\kappa = 0 \text{ and } 2\mu - \mu^2 + 2\kappa = 0.$$

Thus, we have

$$\mu(\mu - \kappa) = 0,$$

which yields $\kappa = \mu = 0$, where we have used $\kappa < 1$.Thus, M satisfies $R(X, Y)\xi = 0$, and hence M is flat (cf [2]). Therefore, by virtue of the result in [8], [9] we have Theorem *A*.

Remark 1 : $\mathbb{R}^3(x^1, x^2, x^3)$ or T^3 (torus) with $\eta = ½(\cos x^3\, dx^1 + \sin x^3\, dx^2)$ and $g_{ij} = ¼\,\delta_{ij}$ is a flat and hence conformally flat, (non-Sasakian) contact Riemannian manifold (cf. [1]).

4. A (κ, μ)-SPACE WITH VANISHING C-BOCHNER CURVATURE TENSOR

In this section we study a (κ, μ)-space with vanishing contact Bochner curvature tensor. In [6] the authors defined the *contact Bochner curvature tensor*, briefly, C-Bochner curvature tensor C as a Sasakian analogue of the Bochner curvature tensor in a Kählerian manifold, namely,

$$C(X, Y) = R(X, Y) + \frac{1}{2n+1}\{QY \wedge X - QX \wedge Y + Q\phi Y \wedge \phi X - Q\phi X \wedge \phi Y$$

$$+ 2g(Q\phi X, Y)\phi + 2g(\phi X, Y)Q\phi + \eta(Y)(QX \wedge \xi) - \eta(X)(\xi \wedge QY)\}$$

$$- \frac{k+2n}{2n+4}\{\phi Y \wedge \phi X - 2g(\phi X, Y)\phi\} - \frac{k-4}{2n+4}(Y \wedge X)$$

$$+ \frac{k}{2n+4}\{\eta(Y)(\xi \wedge X) - \eta(X)(Y \wedge \xi), \qquad \dots (4.1)$$

where we have put $k = \dfrac{2n+r}{2n+2}$ and $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$.

Let $M = (M; \eta, \phi, g)$ be a (κ, μ)-space with vanishing C-Bochner curvature tensor. Then, since $Q\xi = 2n\kappa\xi$ from (4.1) we have

$$R(X, \xi)\xi = \frac{2n\kappa + 4}{2n+4}(X - \eta(X)\xi).$$

But, from (2.7) we also have

$$R(X, \xi)\xi = \kappa(X - \eta(X)\xi) + \mu hX.$$

From these two equations, we have

$$\frac{2n\kappa + 4}{2n+4}h = \kappa h + \mu h^2. \qquad \dots (4.2)$$

Taking the trace of (4.2), then we obtain

$$\mu h^2 = 0. \qquad \dots (4.3)$$

From (4.2) and (4.3) we conclude that M is Sasakian ($\kappa = 1$ or $h = 0$). Thus we have Theorem B.

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REFERENCES

1. D. E. blair, *Contact manifolds in Riemannian Geometry, Lecture Notes Math.* Springer-Verlag, Berlin-Heidelberg-New York, **509**, 1976.
2. D. E. Blair, *Tôhoku Math. J., Second S.* (1977) 319-24.

3. D. E. Blair and T. Koufogiorgos, *J. Geometry* **49** (1994) 55-66.
4. D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Isreal J. Math.* **91** (1995), 189-214.
5. E. Boeckx, *Illinois J. math., to appear*
6. M. Matsumoto and G. Chuman, *TRU Math.* **5** (1969) 21-30.
7. M. Okumura, *Tôhoku Math. J.* **14** (1962) 135-45.
8. S. Tanno, *Tôhoku Math. J.* **15** (1963) 581-83.
9. S. Tanno, *Proc. Japan. Acad.* **43** (1967) 581-83.
10. S. Tanno, *Illinois J. Math.* **12** (1968) 700-717.
11. Y. Tashiro, *Tôhoku Math. J.* **21** (1969) 117-143.