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OSCILLATION OF FUNCTIONAL PARABOLIC DIFFERENTIAL EQUATIONS UNDER THE ROBIN BOUNDARY CONDITION†

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This paper obtains some oscillatory results for a class of nonlinear functional parabolic equations with the Robin boundary condition by use of a new method.

Key Words : Nonlinear Delay; Parabolic Equation; Robin Boundary Condition; Oscillation

1. INTRODUCTION

Recently, Fu[1], Yu and Fu[2], Wang[3& 4] have considered a class of ordinary differential equation and parabolic equation and hyperbolic equation with continuous deviating argument, respectively. In this paper, we studied the oscillation of the following functional parabolic equation by use of a new method

$$\frac{\partial u}{\partial t} = a(t)\, \Delta u + a_1(t)\, \Delta u(x, \rho(t)) - p(x, t)\, u$$

$$- \int_a^b q(x, t, \zeta) f(u[x, g(t, \zeta)]\, d\sigma(\zeta) + h(x, t) \qquad \ldots (E)$$

i.e., using the following Robin eigenvalue problem

$$\Delta \Phi + \lambda \Phi = 0 \quad x \in \Omega \qquad \ldots (1)$$

and

$$\frac{\partial \Phi}{\partial n} + \beta(x)\, \Phi = 0 \quad x \in \partial\Omega \qquad \ldots (2)$$

to study the oscillation of the eq. (*E*) satisfying the following Robin boundary condition

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$$\frac{\partial u}{\partial n} + \beta(x)u(x, t) = 0, \; x \in \partial \Omega, \quad \text{... } (B)$$

where Δu is the Laplacian in $\boldsymbol{R}^n, \boldsymbol{R}_+ = [0, +\infty), u = u(x, t), \Omega$ is a bounded domain in $\boldsymbol{R}^n$ with a piecewise smooth boundary $\partial \Omega$, n denotes the unit exterior vector normal to $\partial \Omega$.

We assume throughout thi paper that the following conditions (H) hold:

(H1) $a(t), a_1(t), \tau(t), \rho(t) \in C(\boldsymbol{R}_+, \boldsymbol{R}_+)$; $\tau(t) \le t, \rho(t) \le t, \tau(t), \rho(t)$

are nondecreasing, and

$$\lim_{t \to +\infty} \tau(t) = \lim_{t \to +\infty} \rho(t) = +\infty,$$

(H2) $p(x, t) \in C(\bar{\Omega} \times \boldsymbol{R}_+, R_+)$, $q(x, t, \zeta) \in C(\bar{\Omega} \times \boldsymbol{R}_+ \times [a, b], \boldsymbol{R}_+)$,

$$h(x, t) \in (\bar{\Omega} \times \boldsymbol{R}_+, \boldsymbol{R}_+)$$

$$P(t) = \min_{x \in \Omega} \{p(x, t)\}, \; Q(t, \zeta) = \min_{x \in \Omega} \{q(x, t, \zeta)\}.$$

(H3) $g(t, \zeta) \in C(\boldsymbol{R}_+ \times [a, b] \boldsymbol{R})$; $g(t, \zeta) \le t, \zeta \in [a, b]$; $g(t, \zeta)$ is nondecreasing with respect to t and ζ; and $\lim_{t \to +\infty} \min \{g(t, \zeta)\} = +\infty$.

(H4) $f(u) \in C(\boldsymbol{R}, \boldsymbol{R})$ is a positive and convex function in $\boldsymbol{R}_+$, and $f(-u) = -f(u)$, $\sigma(\zeta) \in C([a, b], \boldsymbol{R})$ is nondecreasing, integral of eq. (E) is Stieltjes integral.

(H5) $\beta(x) \in C(\partial \Omega, (0, +\infty))$.

Definition — A solution $u(x, t)$ of the problem $(E,)$, (B) is called oscillatory in the domain G if for each positive number T there exists a point $(x_0, t_0) \in \Omega \times [T, +\infty)$ such that the condition $u(x_0, t_0) = 0$ holds.

2. Main Results

Lemma 1 — Assume that (H5) holds, and λ_0 is the smallest eigenvalue of the Robin eigenvalue problem

$$\Delta \Phi + \lambda \Phi = 0 \;\; x \in \Omega \quad \text{... (1)}$$

and

$$\frac{\partial \Phi}{\partial n} + \beta(x) \Phi = 0, \; x \in \partial \Omega, \quad \text{... (2)}$$

and $\Phi(x)$ is the corresponding eigenfunction, then $\lambda_0 > 0$ and $\Phi(x) > 0, (x \in \Omega)$ (see Theorem 3.3.22 of [5]).

Let $G(t) = \left(\int_\Omega \phi(x)\, dx \right)^{-1} \int_\Omega h(x, t)\, \Phi(x) dx$, we have

Lemma 2 — Suppose that the conditions (*H*) hold, and u is a positive (negative) solution of the problem (*E*), (*B*) in $\Omega \times [T, +\infty)$, $T \geq 0$, then the following differential inequality

$$V'(t) + (\lambda_0 a(t) + f(t))\, V(t) + a_1(t)\, V(\rho(t)) + \int_a^b Q(t, \zeta)\,(V[g(t, \zeta)])\, d\sigma(\zeta) \leq G(t) \quad \ldots (3)$$

and

$$(V'(t) + (\lambda_0 a(t) + P(t)) V(t) + a_1(t)\, V(\rho(t)) + \int_a^b Q(t, \zeta) f(V[g(t, \zeta)]\, d\sigma(\zeta) \leq -G(t) \quad \ldots (3'))$$

has eventually positive solution.

PROOF : Let $u(x, t)$ is a positive solution of the Robin boundary value problem (*E*), (*B*) in $\Omega \times [t_0, +\infty)$, for $t_0 \geq 0$, by the condition (H2),

there exists a $t_1 \geq t_0$ such that

$$g(t, \zeta) \geq t_0 \; (t, \zeta) \in [t_1, +\infty) \times [a, b]$$

and $$\tau(t) \geq t_0 \; \rho(t) \geq t_0 \; t \geq t_1$$

then $$u(x, g(t, \zeta)) > 0 \; (x, t, \zeta) \in \Omega \times [t_1, +\infty) \times [a, b]$$

and $$u(x, \tau(t)) > 0 \; u(x, \rho(t)) > 0 \; (x, t) \in \Omega \times [t_1, +\infty).$$

Multiplying both side of eq. (E) by eigenfunction $\Phi(x)$ of the Robin eigenvalue problem (1) and integrating with respect to x over the domain Ω, we have

$$\frac{d}{dt}\left[\int_\Omega u\,\Phi(X)\,dx\right] = a(t)\int_\Omega \Delta\; u\,\Phi(x)\,dx - a_1(t)\int_\Omega \Delta\, u(x, \rho(t))\,\Phi(x)\,dx$$

$$-\int_\Omega p(x, t)\, u\, \Phi'(x) dx$$

$$-\int_\Omega \int_a^b q(x, t, \zeta) f(u[x, g(t, \zeta)])\, \Phi(x)\, d\sigma(\zeta)\, dx + \int_\Omega h(x, t)\,\Phi(x) dx,\; t \geq t_1 \quad \ldots (4)$$

using the Green formula, boundary condition (B) and Robin eigenvalue problem (1), we have

$$\int_\Omega \Delta\, u(x, t)\,\Phi(x) dx = \int_{\partial\Omega}\left(\Phi(s)\frac{\partial u}{\partial n} - u\frac{\partial \Phi(s)}{\partial n}\right) ds + \int_\Omega u\,\Delta\,\Phi(x) dx$$

$$= \int_{\partial\Omega} (-\,\Phi(s)\,\beta(s) u + u\,\beta(s)\,\Phi(s))\, ds - \lambda_0 \int_\Omega u\,\Phi(x) dx$$

$$= -\lambda_0 \int_\Omega u(x, t)\,\Phi(x) dx\; t \geq t_1 \quad \ldots (5)$$

and
$$\int_\Omega \Delta u(x,\rho(t))\,\Phi(x)dx = -\lambda_0 \int_\Omega u(x,\rho(t))\,\Phi(x)dx \quad t \ge t_1. \qquad \dots (6)$$

From Jensen's inequality, we have

$$\int_\Omega \int_a^b q(x,t,\zeta) f(u[x,g(t,\zeta)])\,\Phi(x)\,d\sigma(\zeta)\,dx$$

$$\ge \int_a^b Q(t\,\zeta)\left\{ f\left[\left(\int_\Omega \Phi(x)\,dx\right)^{-1} \int_\Omega u(x,g(t,\zeta))\,\Phi(x)dx\right] \int_\Omega \Phi(x)\,dx\right\} d\sigma(\zeta),$$

$t \ge t_1$.

Therefore, from (4)-(6), and the above last inequality, we have

$$\frac{d}{dt}\left[\int_\Omega u\,\Phi(x)dx\right] = -\lambda_0 a(t) \int_\Omega u(x,t)\,\Phi(x)\,dx - \lambda_0 a_1(t) \int_\Omega u(x,\rho(t))\,\Phi(x)dx$$

$$-\int_a^b Q(x,\zeta)\left\{ f\left[\left(\int_\Omega \Phi(x)dx\right)^{-1} \int_\Omega u(x,g(t,\zeta))\,\Phi(x)dx\right] \int_\Omega \Phi(x)dx\right\} d\sigma(\zeta)$$

$$-P(t) \int_\Omega u(x,t)\,\Phi(x)dx + \int_\Omega h(x,t)\,\Phi(x)dx \quad t \ge t_1,$$

Let
$$V(t) = \left(\int_\Omega \Phi(x)dx\right)^{-1} \int_\Omega u\,\Phi(x)dx \text{ we have}$$

$$V'(t) + (\lambda_0 a(t) + P(t))\,V(t) + \lambda_0 a_1(t)\,V(\rho(t)) + \int_a^b Q(t,\zeta) f[(V(g(t,\zeta))]\,d\sigma(\zeta) \le G(t) \quad t \ge t_1.$$

Therefore $V(t)$ (> 0) is an eventually positive solution of differential inequality (3),

If $u(x,t) < 0$, $(x,t) \in \Omega \times [t_0, +\infty)$, $t_0 \ge 0$, $t_0 \ge 0$, let $v(x,t) = -u(x,t)$, then using the above-mentioned method, we can also get that $\bar{V} = \left(\int_\Omega \Phi(x)dx\right)^{-1} \int_\Omega v(x,t)\,\Phi(x)dx$ is an eventually positive solution of the inequality (3′).

***Theorem* 1** — *Assume that the condition (H) holds, and the following differential inequalities*

$$V'(t) + (\lambda_0 a(t) + P(t))V(t) + a_1(t)\,V(\rho(t)) + \int_0^b Q(t,\zeta) f(V[f(t,\zeta)])\,d\sigma(\zeta) \le G(t) \quad \dots (3)$$

and
$$V'(t) + (\lambda_0 a(t) + P(t))V(t) + a_1(t)V(\rho(t)) + \int_a^b Q(t, \zeta) f(V[g(t, \zeta)])\, d\sigma(\zeta) \le -G(t) \quad \text{... (3')}$$

have no eventually positive solution, then the every solution of the Robin problem (*E*), (*B*) is oscillatory in *G*.

PROOF : Assume that there exists a nonoscillatory solution $u(x, t)$ of the problem (*E*), (*B*). If $u(x, t) > 0$ (< 0) is a positive (negative) solution of the boundary value problem (*E*), (*B*) in $\Omega \times [t_0, +\infty)$, for $t_0 \ge 0$, then from Lemma 2 it follows that $V(t)$ ($\overline{V}(t)$) is an eventually positive solution of the inequality (3) (inequality (3')), which contradicts the condition of Theorem 1.

Analogous to the proof of Theorem 2 in [4], we have

Theorem 2 — *Suppose that the condition (H) holds, and following conditions hold :*

$$\liminf_{t \to +\infty} \int_{t_1}^{t} G(s)ds = -\infty \quad \text{... (7)}$$

and
$$\liminf_{t \to +\infty} \int_{t_1}^{t} G(s)ds = +\infty \quad \text{... (8)}$$

for sufficiently large t_1 (> 0), then every solution of the Robin value problem (*E*), (*B*) is oscillatory in *G*.

Now we concern several particular cases. When $h(x, t) \equiv 0, (x, t) \in G$ eq. (*E*) become the following form :

$$\frac{\partial u}{\partial t} = a(t)\, \Delta u + a_1(t)\, \Delta u(x, \rho(t)) - p(x, t)u$$

$$- \int_a^b q(x, t, \zeta) f(u[x, (t, \zeta)])\, d\sigma(\zeta) \quad (x, t) \in G \quad \text{... (E1)}$$

Analogous to the proof of Lemma 2 and Theorem 1, we have

Theorem 3 — *Assume that (H) holds, and* $h(x, t) \equiv 0, (x, t) \in G$. *If the following differential inequality*

$$V'(t) + (\lambda_0 a(t) + P(t))V(t) + \lambda_0 a_1(t)V(\rho(t)) + \int_a^b Q(t, \zeta) f[V(g(t, \zeta))]\, d\sigma(\zeta) \le 0 \quad \text{... (10)}$$

has no eventually positive solutions, then every solution of the problem (*E*1), (*B*) $h(x, t) \equiv 0, (x, t) \in G$ is oscillatory in *G*.

Analogoues to the proof of Lemma 2 and Theorem 1, we have

Theorem 4 — *Assume that (H) holds, and* $h(x, t) \equiv 0, (x, t) \in G$, *and*

$$f(\alpha)/\alpha \ge c_0 = \text{const} > 0, \; g(t, \zeta) \equiv g_0(t) \; \zeta \in [a, b].$$

If the following differential inequality

$$V'(t) + (\lambda_0 a(t) + P(t))V(t) + \lambda_0 a_1(t)V(\rho(t)) + c_0 V(g_0(t)) \int_a^b Q(t, \zeta)\, d\sigma(\zeta) \le 0 \quad \dots (11)$$

has no eventually positive solutions, then every solution of the problem $(E1)$, (B) $h(x, t) \equiv 0, (x, t) \in G)$ is oscillatory in G.

Lemma 3 (see refs [6]) — Assume $Q(t), Q_i(t) \in C([t_0, +\infty), \boldsymbol{R}_+))$ $i = 1, 2, \dots m$

$g_i(t) \in C([t_0, +\infty), \boldsymbol{R})$, $g_i(t)$ is nondecreasing and $g_i(t) \le t$,

$\lim_{t \to +\infty} g_i(t) = +\infty$, $i = 1, 2, \dots m$ if $\exists\ i \in \{1, 2, \dots m\}$, such that

$$\liminf_{t \to +\infty} \int_{g_i(t)}^{t} \left\{ Q_i(s) \exp \int_{g_i(s)}^{s} Q(r)\, dr \right\} dr > \frac{1}{e} \quad \dots (12)$$

then the following differential inequality

$$y'(t) + Q(t)y(t) + \sum_{i=1}^{m} Q(t)y(g_i(t)) \le 0 \quad t \ge t_0 \quad \dots (13)$$

have no eventually positive solution.

From Lemma 3 and Theorem 4, we have :

Theorem 5 — *Assume that (H) holds,* $h(x, t) \equiv 0, (x, t) \in G$, *and*

$$f(\alpha)/\alpha \ge c_0 = \text{const} > 0,\ g(t, \zeta) \equiv g_0(t)\ \zeta \in [a, b].$$

If the following condition holds

$$\liminf_{t \to +\infty} \int_{\rho(t)}^{t} \left\{ \lambda_0 a_1(s) \exp \int_{\rho(s)}^{s} (\lambda_0 a(r) + P(r))\, dr \right\} ds > \frac{1}{e} \quad \dots (14)$$

or

$$\liminf_{t \to +\infty} \int_{g_0(t)}^{t} \left\{ \int_a^b c_0 Q(s, \zeta)\, d\sigma(\zeta) \right\} \exp \int_{g_0(s)}^{s} (\lambda_0 a(t) + P(r))dr\, ds > \frac{1}{e} \quad \dots (15)$$

then every solution of the Robin boundary value problem $(E1)$, (B) is oscillatory in G.

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