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GENERALISED PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM AND THE LOCATIONS AND STABILITY OF COLLINEAR POINTS

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The Robe's restricted problem of three bodies is generalised in the sense that the rigid spherical shell, filled with homogeneous, incompressible fluid is taken as an oblate spheroid and the second body is radiating outside the shell while the third infinitesimal mass is supposed to be moving inside the shell subject to the attraction of the second body and buoyancy force due to the fluid inside the shell of the first body. The equation of motion of the problem has been found, which is different from Robe's. The locations and stability of collinear points of the photogravitational RTBP have further been studied.

Key Words : Restricted Three-Body Problems; Stability; Collinear Points

1. INTRODUCTION

Robe[3] has considered a new kind of restricted three body problem, in which one body m_1 is a rigid spherical shell, filled with homogeneous, incompressible fluid of density ρ_1. The second one, m_2 is a mass point outside the shell and m_3 is a small solid sphere of density ρ_3 moving inside the shell and is subject to the attraction of m_2 and the buoyancy force due to the fluid ρ_1. He obtained an equilibrium point and studied the linear stability of the point corresponding to the particular solution of the rotating system.

Shrivastava and Gorain[6] considered the effect of perturbation due to coriolis and centrifugal forces on the location of libration points in Robe's circular restricted problem of three bodies.

Here in the present paper we have considered m_1 of density ρ_1 as an oblate spheroid and m_2 as a radiating mass point in Robe's circular RTBP.

As an example from the physical world, the Sun and Earth can be assumed as massive bodies, the fluid in the earth crust is taken to be minor body.

It is of interest now to study the effect of perturbation due to oblateness and radiation of the respective primaries on the location of libration point in the Robe's restricted problem of three bodies.

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The following forces are acting on m_3 :

(*i*) Attraction of m_2; (*ii*) Oblateness effect of m_1; (*iii*) Effect of radiation of m_2;

(*iv*) The gravitational force, exerted by the fluid density ρ_1 i.e., $F_A = -(4/3)\ \pi \rho_1 (i\, m_3 M_1 M_3)$; and

(*v*) The buoyancy force, $F_B = (4/3)\, \pi G \rho_1^2 m_3 M_1 M_3/\rho_3$; where $r_1 = M_1M_3, r_2 = M_2M_3$ (x, y, z) are the coordinates of the infinitesimal mass m_3 and the line joining m_1 and m_2 is the x-axis. The total potential acting at m_3 is

$$-\frac{Gm_2 q}{r_2} + \frac{4}{3}\pi G \rho_1 \left(1 - \frac{\rho_1}{\rho_3}\right) r_1^2 - \frac{Gm_1}{r_1} - \frac{Gm_1 A_1}{2r_1^3}.$$

2. Equations of Motion

$$\left.\begin{aligned} \ddot{x} - 2n\,\dot{y} &= \partial\frac{\Omega}{\partial x} \\ \ddot{y} - 2n\,\dot{x} &= \partial\frac{\Omega}{\partial y} \\ \text{and} \quad \ddot{z} &= \partial\frac{\Omega}{\partial z} \end{aligned}\right\}, \qquad \dots (1)$$

where

$$\Omega = \frac{u^2}{2}(x^2 + y^2) - kr_1^2 + \frac{\mu q}{r_2} + \frac{1-\mu}{r_1} + \frac{(1-\mu)A_1}{2\, r_1^3}. \qquad \dots (2)$$

Here,

$$k = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right); \mu = \frac{m_2}{m_1 + m_2}, 0 < \mu < 1,$$

$n^2 = 1 + \frac{3}{2}A_1;\ q = 1 - \frac{F_p}{F_g}$ where A_1 stands for oblateness effect and q is the mass reduction factor due to radiation effect.

$$r_1^2 = (x - x_1)^2 + y^2 + z^2 \ \ \&\ \ r_2^2 = (x - x_2)^2 + y^2 + z^2$$

and

$$x_1 = -\mu, x_2 = 1 - \mu, 0 < \mu < 1.$$

Equilibrium exists when

$$\Omega_x = \Omega_y = \Omega_z = 0.$$

We consider the case where $\rho_1 = \rho_3$ i.e., $k = 0$; then eq. (2) reduces to

$$\Omega = \frac{u^2}{2}(x^2 + y^2) + \frac{\mu q}{r_2} + \frac{1-\mu}{r_1} + \frac{(1-\mu)\, A_1}{2\, r_1^3}. \qquad \dots (3)$$

$$\therefore\ \Omega_x = n^2 x - \mu q \frac{(x-1+\mu)}{r_2^3} - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3}{2}\frac{(1-\mu)A_1(x+\mu)}{r_1^5} = 0. \quad \dots (4)$$

$$\Omega_y = y\left[n^2 - \frac{\mu q}{r_2^3} - \frac{(1-\mu)}{r_1^3} - \frac{3A_1(1-\mu)}{2r_1^5}\right] = 0. \quad \dots (5)$$

$$\Omega_z = z\left[\frac{\mu q}{r_2^3} - \frac{(1-\mu)}{r_1^3} - \frac{3A_1(1-\mu)}{2r_1^5}\right] = 0. \quad \dots (6)$$

From (5) and (6) we see that $y = 0$ and $z = 0$, and the collinear libration points exist.

3. Location of the Collinear Points

Solution of eqs. (4) and (4) for $y = 0$ results in their collinear points L_1, L_2 and L_3. Since the location of these points depend upon r_1 and r_2 the following there cases can the discussed :

Case I — For the point L_1 i.e., $(x_1, 0, 0)$ a set of substitutions $r_1 = x_1 + \mu$ and $r_2 = x_1 + \mu - 1$ in eq. (4) is done and then by putting $r_2 = \xi_1$, $r_1 = 1 + \xi_1$ and $x_1 = \xi_1 + 1 - \mu$ in it, we obtain on simplification :

$$F_1(\xi) = (2+3A_1)\xi_1^7 + (2+3A_1)(5-\mu)\xi_1^6 + (2+3A_1)(10-4\mu)\xi_1^5$$
$$+ [(2+3A_1)(10-6\mu) - 2\mu q + 2\mu - 2]\xi_1^4 + [(2+3A_1)(5-4\mu) - 8\mu q + 4\mu - 4]\xi_1^3$$
$$- 12\mu q\xi_1^2 - 8\mu q\xi_1 - 2\mu q = 0. \quad \dots (7)$$

Since $\qquad F_1(0) = -2\mu q < 0$

and $\qquad F_1(1) = 56 - 32\mu q - 24\mu + 3A_1(31 - 15\mu) > 0.$

Descarte's sign rule indicates that eq.(7) has one and onlyone positive root for $0 < \mu < 1/2$ and one real root $\xi_1 = 0$ for $\mu = 0$.

Case II — For the point L_2 i.e., $(x_2, 0, 0)$, another set of substitutions $r_1 = \mu - x_2$, $r_2 = x_2 + 1 - \mu$ in eq. (4) and then by putting $r_2 = \xi_2$, $r_1 = 1 - \xi_2$, and $x_2 = \xi_2 - 1 + \mu$, in it we obtain by simplification as

$$F_2(\xi) = (2+3A_1)\xi_2^7 - (2+3A)(5-\mu)\xi_2^6 + (2+3A_1)(10-4\mu)\xi_2^5$$
$$- [(2=3A_1)(10-6\mu) + 2\mu q + 2 - 2\mu]\xi_2^4 + [(2+3A_1)$$
$$(5-4\mu) + 8\mu q + 4(1-\mu)]\xi_2^3 - [2(2+3A_1)(1-\mu)$$

$$+ 12\,\mu q]\,\xi_2^2 + 8\,\mu\,q\,\xi_2 - 2\mu\,q = 0. \qquad \text{... (8)}$$

Since $\qquad F_2\,(0) = -\,2\,\mu\,q < 0$

and $\qquad F_2\,(1)\;(213\;A_1)\;(19\;-\;11\;\mu)\;+\;28\;\mu q\;=\;0$

and $F_2 - (\xi) > 0$ for all ξ_2 greater than 1, there exists only one positive root of eq. (8) for $0 < \mu < 1/2$ and there is one real root, $\xi_2 = 0$ for $\mu = 0$.

Case III — For the point L_3 i.e., $(x_3, 0, 0)$, in view of the substitutions $r_1 = x_3 - \mu$ and $r_2 = 1 + x_3 - \mu$ on eq. (4) and then by replacing $r_1 = \xi_3, r_2 = 1 + \xi_3$ and $x_3 = \mu + \xi_3$ in it and simplifying we obtain

$$F_3\,(\xi) = (2 + 3A_1)\,\xi_3^7 + (2 + 3A)\,(\mu + 2)\,\xi_3^6 + (2 + 3A_1)\,(2\mu + 1)\,\xi_3^5$$

$$+\;[(2 + 3A_1)\,\mu - 2\mu\,q + 2\mu - 2]\,\xi_3^4 - 4\,(1 - \mu)]\,\xi_3^3$$

$$-\;(2 + 3\,A_1)\,(1 - \mu)\,\xi_1^2 - 6(1 - \mu)\,A_1\,\xi_1 - 3\,(1 - \mu)\,A_1 = 0, \qquad \text{... (9)}$$

since $\qquad F_3\,(0) = -\,3\,(1 - \mu)\,A_1 = 0$ and $F_3\,(1) = (16 - 2q + 24A_1)\,\mu > 0.$

Descarte's sign rule indicates one and only one positive root for $0 < \mu \leq 1/2$ and one real root $\xi_3 = 0$ for $\mu = 0$.

After finding the values of ξ_i $(i = 1, 2, 3)$ for eqs. (7), (8) & (9) and substituting them in

$$\left.\begin{aligned} x_1 &= \xi_1 + 1 - \mu \\ x_2 &= \xi_2 - 1 + \mu \\ x_i &= \xi_i + \mu \end{aligned}\right] \qquad \text{... (10)}$$

We get the abscissa for the Lagrangian points, L_1, L_2 and L_3.

Eqs. (10) are solved by Newton-Raphson's method and the shifts $(\Delta\,L)$ in the collinear points due to the oblateness effect versus the mass parameter μ are plotted in Figs. 1 & 2. It is observed that the points L_2 and L_3 move away from the more massive primary while L_1 comes nearer to it with oblateness effect. The shifts increase with the increase in the mass parameter and $\Delta\,L_1 > \Delta\,L_2 > \Delta\,L_3$ for $\mu \neq 0$ at $\mu = 0$ the shifts tend to zero.

In Figs. (1) & (2) we have plotted $\Delta_i\,(i = 1, 2, 3)$, which are the deviations in the values of $x_i\,(i = 1, 2, 3)$ as obtained from eqs. (10) and as obtained from eqs. (7), (8) and (9) for various values of the mass parameter μ.

For $y \neq 0$, eqs. (4) & (5) disclose that

$$r_2^3 = q/n^2,\; r_1 = 1. \qquad \text{... (11)}$$

Eqs. (11) locate the other two points L_4 and L_5. These points forming isoceles triangles with the primaries are known as triangular points. It may be noted that $r_2 \leq 1$.

4.1 Stability of the Libration Points

Putting $x = x_0 + \xi, y = y_0 + \eta$ in eqs. (1) for studying the motion near any of the equilibrium points $L(x_0, y_0)$, we get the first variational equations as

and

$$\left.\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \Omega_{xx}(x_0, y_0)\,\xi + \Omega_{xy}(x_0, y_0)\,\eta \\ \ddot{\eta} + 2n\dot{\xi} &= \Omega_{xy}(x_0, y_0)\,\xi + \Omega_{yy}(x_0, y_0)\,\eta. \end{aligned}\right] \quad \text{... (12)}$$

The characteristic equation of eqs. (12) is

$$\lambda^4 + (4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0)\,\lambda^2 + \Omega_{xx}^0\,\Omega_{yy}^0 - (\Omega_{xy}^0)^2 = 0. \quad \text{... (13)}$$

4.2. Stability of the Collinear Points

At the collinear points, we get

$$\Omega_{xx}^0 = n^2 + \frac{2\mu q}{r_2^3} + \frac{2(1-\mu)}{r_1^3} + \frac{6(1-\mu)A_1}{r_1^5} > 0, \; \Omega_{xy}^0 = 0$$

and

$$\Omega_{yy}^0 = n^2 + \frac{\mu q}{r_2^3} - \frac{(1-\mu)}{r_1^3} - \frac{3A_1(1-\mu)}{2r_1^5}.$$

It can be proved with some afforts that $\Omega_{yy}^0 < 0$ at $L_{1,2,3}$

Consequently, $\Omega_{xx}^0\,\Omega_{yy}^0 - (\Omega_{xy}^0)^2 < 0$.

We can easily find that the roots λ_i $(i = 1, 2, 3, 4)$ of the characteristic eq. (13) are

$$\lambda_{1,2} = \pm[-C_1 + (C_1^2 + C_2^2)^{1/2}]^{1/2} = \pm\lambda$$

and

$$\lambda_{3,4} = \pm[-C_1 - (C_1^2 + C_2^2)^{1/2}]^{1/2} = \pm is,$$

where

$$C_1 = 2n^2 - (\Omega_{xx}^0 + \Omega_{yy}^0)/2 \text{ and } C_2^2 = -\Omega_{xx}^0\,\Omega_{yy}^0 > 0.$$

The general solution of eqs. (12) can be written as

as

$$\xi = \sum_{i=1}^{4} A_i e^{\lambda_i t}, \eta = \sum_{i=1}^{4} B_i e^{\lambda_i t},$$

and

$$(\lambda_i^2 - \Omega_{xn}^0)\,A_i = (2n\,\lambda_i + \Omega_{xy}^0)\,B_i \quad \text{... (14)}$$

It may be noted that $\lambda_{1,2}$ are real and $\lambda_{3,4}$ are purely imaginary. Hence, the collinear equilibria are unstable in the general case. However, as in Szebehely[3] and Sharma and Subba Rao[4]; it is possible to choose the initial conditions (ξ_0, η_0) such that $A_{1,2} = 0$ and the eqs. (14) represent an ellipse, whose eccentricity and semi-major axis are

$$(1 - C_3^{-2})^{1/2} \text{ and } (c_3^2 \xi_0^2 + \eta_0^2)^{1/2}$$

respectively, where $C_3 = (s^2 + \Omega_{xx}^0)/2ns$.

5. Conclusion

(*i*) We find that the position of Libration points are affected by oblateness and radiation of the primaries and this can be compared with the classical Robe-circular restricted problem of three bodies by putting $q = 1$ and $A_1 = 0$ in case where $\rho_1 = \rho_3$ i.e., $k = 0$.

(*ii*) collinear equilibria are unstale in general case but through special choice of initial conditions elliptical periodic orbits exist.

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