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ON A HYPERSURFACE OF A MATSUMOTO SPACE

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In 1985, Matsumoto[6] discussed the properties of special hypersurface of Randers space with $b_i(x) = \partial_i b$ being the gradient of a scalar function $b(x)$. He has considered a hypersurface which is given by $b(x)$ = constant. In this paper, we have considered the hypersurface of Matsumoto space with the same equation $b(x)$ = constant. The conditions under which this hypersurface be a hyperplane of the first or second kinds have been obtained. This hypersurface is not a hyper-plane of third kind.

Key Words : Hypersurface; Matsumoto Space; Randers space; Finsler Space

1. MATSUMOTO SPACE

A slope of a mountain is represented as the graph S of a differentiable function $X^3 = f(x^1, x^2)$, where (x^1, x^2, x^3) is a rectangular coordinate system in a three-dimensional Euclidean space. If we put $y^i = \dot{x}^i$ and $\partial_i = \dfrac{\partial}{\partial x^i}$, then a Riemannian metric α is included on S by

$$\alpha(x, y) = [(y^1)^2 + (y^2)^2 + (b_1 y^1 + b_2 y^2)^2]^{\frac{1}{2}} \qquad \text{... (1.1)}$$

where $x = (x^i), y = (y^i)$ and $b_i = \partial_i f$

$$\beta(x, y) = b_1 y^1 + b_2 y^2. \qquad \text{... (1.2)}$$

when a man can walk v meters per minute on a horizontal plane, how many minutes does it take him to walk along a road on S ?

Recently, Matsumoto[7] showed that the man will walk in $S = \int_0^t L(x, t), y(t))\, dt$ minutes along a road $x(t)$ on S, by taking L as

$$L = \frac{\alpha^2}{v\alpha - w\beta}, \qquad \text{... (1.3)}$$

where $2w$ is the gravitational constant and thus slope of a mountain is regarded as a Finsler space with such a time measure L.

It is to be noted that (1.3) is an (α, β) metric. The notion of (α, β) metric was introduced by Matsumoto[5] and has been studied in detail. As well-known examples, there are Randers metric $\alpha + \beta$[8], Kropina metric $\frac{\alpha^2}{\beta}$[3-4] and generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m}$ $(m \neq 0, -1)$ whose studies[2] have greatly contributed to the growth of Finsler geometry. Therefore, the metric of type (1.3) seems to be an interesting new example of (α, β) metric.

Since $L = \frac{\alpha^2}{v\,\alpha - w\,\beta} = \frac{(\alpha/v)^2}{\alpha/v - \frac{w\beta}{v}}$, we shall normalize (1.3) as $L = \frac{\alpha^2}{\alpha - \beta}$. ... (1.4)

Hence, by taking a general Riemannian metric α and a general non-zero 1-form β on a general differentiable manifold M^n, Aikou, Hashiguchi and Yamaguchi[1] gave the following :

Definition 1.1 — On an n-dimensional differentiable manifold M^n, an (α, β)-metric L of type (1.4) is called a Matsumoto metric and the Finsler space (M^n, L) is called a Matsumoto space.

The derivatives of Matsumoto metric L with respect to α and β are given by —

$$L_{\alpha} = \alpha\,(\alpha - 2\,\beta)/(\alpha - \beta)^2, \; L_{\beta} = \alpha^2/(\alpha - \beta)^2 \qquad \text{... (1.5)}$$

$$L_{\alpha\,\alpha} = 2\,\beta^2/(\alpha - \beta)^3, \; L_{\beta\,\beta} = 2\,\alpha^2/(\alpha - \beta)^3 \qquad \text{... (1.6)}$$

and $$L_{\alpha\,\beta} = -2\,\alpha\,\beta/(\alpha - \beta)^3,$$

where $$L_{\alpha} = \frac{\partial L}{\partial \alpha}, L_{\beta} = \frac{\partial L}{\partial \beta}, L_{\alpha\,\alpha} = \frac{\partial L_{\alpha}}{\partial \alpha}, L_{\beta\,\beta} = \frac{\partial L_{\beta}}{\partial \beta}$$

and $$L_{\alpha\,\beta} = \frac{\partial L_{\alpha}}{\partial \beta}.$$

If in Matsumoto space $F^n = (M^n, L)$, where $L = \frac{\alpha^2}{\alpha - \beta}$, we put

$$\alpha = (a_{ij}\,(x)\,y^i y^j)^{\frac{1}{2}}, \; \beta = b_i(x)\,y^i,$$

then the normalized element of support $l_i = \dot{\partial}_i L$ is given by

$$l_i = \alpha^{-1}\,L_{\alpha}\,y_i + L_{\beta}\,b_i, \qquad \text{... (1.7)}$$

where $y_i = a_{ij}\,y^j$. The angular metric tensor $h_{ij} = L^{-1}\,\dot{\partial}_i\,\dot{\partial}_j L$ is given by

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_1 (b_i\,Y_j + b_j\,Y_i) + q_2 Y_1 Y_j, \qquad \text{... (1.8)}$$

where

$$\left.\begin{aligned} p &= LL_{\alpha}\alpha^{-1} = \alpha^2(\alpha - 2\beta)/(\alpha-\beta)^3 \\ q_0 &= LL_{\beta\beta} = 2\alpha^4/(\alpha-\beta)^4 \\ q_1 &= LL_{\alpha\beta}\alpha^{-1} = -2\alpha^2\beta/(\alpha-\beta)^4 \\ q_2 &= L\alpha^{-2}(L_{\alpha\alpha} - L_{\alpha}\alpha^{-1}) = -\alpha(\alpha-3\beta)/(\alpha-\beta)^4. \end{aligned}\right\} \qquad \text{... (1.9)}$$

The fundamental tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ is given by

$$g_{ij} = p\,a_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + p_2 Y_i Y_j \qquad \text{... (1.10)}$$

where

$$\left.\begin{aligned} p_0 &= q_0 + L_{\beta}^2 = 3\alpha^4/(\alpha-\beta)^4 \\ p_1 &= q_1 + L^{-1}pL_{\beta} = \alpha^2(\alpha-4\beta)/(\alpha-\beta)^4 \\ p_2 &= q_2 + p^2L^{-2} = -\beta(\alpha-4\beta)/(\alpha-\beta)^4 \end{aligned}\right\} \qquad \text{... (1.11)}$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1}a^{ij} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j \qquad \text{... (1.12)}$$

where

$$\left.\begin{aligned} & b^i = a^{ij}b_j,\ S_0 = (pp_0 + (p_0p_2 - p_1^2)\,\alpha^2)/\zeta p \\ & S_1 = (pp_1 - (p_0p_2 - p_1^2)\,\beta)/\zeta p \\ & S_2 = (pp_2 + (p_0p_2 - p_1^2)\,b^2)/\zeta p,\ b^2 = a_{ij}b^i b^j \\ & \zeta = p(p + p_0B^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2 b^2 - \beta^2). \end{aligned}\right\} \qquad \text{... (1.13)}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by[9]

$$2p\,C_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k \qquad \text{... (1.14)}$$

where $$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1q_0,\ m_i = b_i - \alpha^{-2}\beta Y_i. \qquad \text{... (1.15)}$$

It is noted that the covariant vector m_i is a non-vanishing one, and is orthogonal to the element of support y^i.

Let $\left\{ {i \atop jk} \right\}$ be the components of Christoffel's symbol of the associated Riemannian space R^n and ∇_k be covariant differentiation with respect to x^k relative to this Christoffel's symbol. We shall use the following tensors.

$$2\,E_{ij} = b_{ij} + b_{ji},\ 2F_{ij} = b_{ij} - b_{ji}. \qquad \text{... (1.16)}$$

where $\quad b_{ij} = \nabla_j\, b_i.$

If we denote the Cartan's connection $C\Gamma$ as $\left(\Gamma^{*i}_{jk}\, \Gamma^{*i}_{ok}\, C^i_{jk} \right)$ then the difference tensor $D^j_{jk} = \Gamma^{*i}_{jk} - \left\{ {i \atop jk} \right\}$ of Matsumoto space is given by[9].

$$\begin{aligned} D^i_{jk} &= B^i\, E_{jk} + F^i_k\, B_j + F^i_j\, B_k + B^i_j\, b_{ok} + B^i_k\, b_{oj} \\ &\quad - b_{0m}\, g^{im}\, B_{jk} - C^i_{jm}\, A^m_k - C^i_{km}\, A^m_j + C_{jkm} A^m_s\, g^{is} \\ &\quad + \lambda^s \left(C^i_{jm}\, C^m_{sk} + C_{km} sp^i\, C^m_{sj} - C^m_{jk}\, C^i_{ms} \right) \end{aligned} \qquad \text{... (1.17)}$$

where

$$\left.\begin{aligned} & B_k = p_0 b_k + p_1 Y_k,\ B^i = g^{ij}\, B_j,\ F^k_i = g^{kj}\, F_{ji}, \\ & B_{ij} = \left\{ p_1\, (a_{ij} - \alpha^{-2}\, Y_i\, Y_j) + \frac{\partial p_0}{\partial \beta}\, m_i\, m_j \right\} / 2, \\ & B^k_i = g^{kj}\, B_{ji}, \\ & A^m_k = B^m_k\, E_{00} + B^m\, E_{k_0} + B_k\, F^m_0 + B_0\, F^m_k, \\ & \lambda^m = B^m\, E_{00} + 2B_0\, F^m_0,\ B_0 = B_i\, y^i, \end{aligned}\right] \qquad \text{... (1.18)}$$

Here and in the following we denote 0 as contraction with y^i except for the quantities p_0, q_0 and s_0.

2. Induced Cartan Connection

Let F^{n-1} be a hypersurface, of F^n, given by the equations $x^i = x^i\,(u^\alpha)$. Suppose that the matrix of the projection factor $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ is of rank $n - 1$. The element of support y^i of F^n is to be taken tongential to F^{n-1} i.e.,

$$y^i = B^i_\alpha\,(u)\, v^\alpha. \qquad \text{... (2.1)}$$

Thus v^α is the element of support of F^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and HV-torsion tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij}\, B^i_\alpha\, B^j_\beta,\ C_{\alpha\beta\gamma} = C_{ijk}\, B^i_\alpha\, B^j_\beta\, B^j_\gamma \qquad \text{... (2.2)}$$

At each point u^{α} of F^{n-1}, a unit normal vector $N^i(u, v)$, is defined by

$$g_{ij}(x(u), y(u, v))\, B^i_{\alpha} N^j = 0,\; g_{ij}(x(y), y(u, v))\, N^i N^j = 1. \qquad ...(2.3)$$

As for the angular metric tensor h_{ij} we have

$$h_{\alpha\beta} = h_{ij} B^i_{\alpha} B^j_{\beta},\; h_{ij} B^i_{\alpha} N^j = 0,\; h_{ij} N^i N^j = 1. \qquad ...(2.4)$$

If (B^{α}_i, N_i) denote the inverse of (B^i_{α}, N^i), then we have

and
$$\left.\begin{array}{l} B^{\alpha}_i = g^{\alpha\beta} g_{ij} B^j_{\beta},\; B^i_{\alpha} B^{\beta}_i = \delta^{\beta}_{\alpha} \\ B^{\alpha}_i N^i = 0,\; B^i_{\alpha} N_i = 0,\; N_i = g_{ij} N^j \\ B^i_{\alpha} B^{\alpha}_j + N^i N_j = \delta^i_j. \end{array}\right] \qquad ...(2.5)$$

The induced connection $IC\Gamma = \left(\Gamma^{*\alpha}_{\beta\gamma}, G^{\alpha}_{\beta}, C^{\alpha}_{\beta\gamma}\right)$ of F^{n-1} induced from the Cartan's connection $C\Gamma = \left(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk}\right)$ is given by[6]

$$\Gamma^{*\gamma}_{\beta\gamma} = B^{\alpha}_i \left(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_{\beta} B^k_{\gamma}\right) + M^{\alpha}_{\beta} H_{\gamma} \qquad ...(2.6)$$

$$G^{\alpha}_{\beta} = B^{\alpha}_i \left(B^i_{0\beta} + \Gamma^{*i}_{0j} B^j_{\beta}\right) \qquad ...(2.7)$$

$$C^{\alpha}_{\beta\gamma} = B^{\alpha}_i C^i_{jk} B^j_{\beta} B^k_{\gamma}, \qquad ...(2.8)$$

where
$$M_{\beta\gamma} = N_i C^i_{jk} B^j_{\beta} B^j_{\gamma},\; M^{\alpha}_{\beta} = g^{\alpha\gamma} M_{\beta\gamma} \qquad ...(2.9)$$

$$H_{\beta} = N_i \left(B^i_{0\beta} + \Gamma^{*i}_{0j} B^j_{\beta}\right) \qquad ...(2.10)$$

and $B^i_{\beta\gamma} = \dfrac{\partial B^i_{\beta}}{\partial u^{\gamma}}$, $B^i_{0\beta} = B^i_{\gamma\beta} v^{\alpha}$. The quantities $M_{\beta\gamma}$ and H_{β} are called second fundamental v-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_i \left(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_{\beta} B^k_{\gamma}\right) + M_{\beta} H_{\gamma} \qquad ...(2.11)$$

where
$$M_{\beta} = N_i c^i_{jk} B^j_{\beta} N^k. \qquad ...(2.12)$$

The relative *h*- and *v*-covariant derivatives of projection factor B^i_{α} with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \; B^i_\alpha|_\beta = M_{\alpha\beta} N^i. \quad \text{... (2.13)}$$

The equation (2.11) shows that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta \quad \text{... (2.14)}$$

Furthermore (2.10), (2.11) and (2.12) yield

$$H_{0\gamma} = H_\gamma, \; H_{\gamma 0} = H_\gamma + M_\gamma H_0. \quad \text{... (2.15)}$$

We quote the following lemma which is due to Matsumoto[7].

Lemma 2.1 — The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.

The hyperplanes of first, second and third kinds are defined in [7] and we only quote the following :

Lemma 2.2 — A hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_\alpha = 0$.

Lemma 2.3 — A hypersurface F^{n-1} is a hyperplane of the second kind with respect to the connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

Lemma 2.4 — A hypersurface F^{n-1} is a hyperplane of the third kind with respect to the connection $C\Gamma$ if and only if $H_\alpha = 0$, $M_{\alpha\beta} = H_{\alpha\beta} = 0$.

3. Hypersurface F^{n-1} (C) of Matsumoto Space

Let us consider a special Matsumoto metric with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and consider a hypersurface F^{n-1} (c) which is given by the equation $b(x) = c$ (constant). From parametric equations $x^i = x^i(u^\alpha)$ of F^{n-1} (c) we get $\partial_\alpha b(x(u)) = 0 = b_i B^i_\alpha$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of F^{n-1} (c). Therefore, along the F^{n-1} (c) we have

$$b_i B^i_\alpha = 0 \text{ and } b_i y^i = 0. \quad \text{... (3.1)}$$

In general, the induced metric $L(u, v)$ from the Matsumoto metric is given by

$$L(u, v) = \frac{a_{ij}(x(u)) B^i_\alpha B^j_\beta v^\alpha v^\beta}{\sqrt{a_{ij}(x(u)) B^i_\alpha B^j_\beta v^\alpha v^\beta} - b_i(x(u)) B^i_\alpha v^\alpha}.$$

Therefore, the induced metric of the F^{n-1} (c) becomes

$$L(u, v) = \sqrt{a_{\alpha\beta}(u) v^\alpha v^\beta}, \; a_{\alpha\beta} = a_{ij}(x(u)) B^i_\alpha B^j_\beta \quad \text{... (3.2)}$$

which is the Riemannian metric.

At the point of $F^{n-1}(c)$, from (1.9), (1.11) and (1.13), we have

$$p = 1, q_0 + 2, q_1 = 0, q_2 = -\alpha^{-2}, p_0 = 3, p_1 = \alpha^{-1}, p_2 = 0,$$

$$\zeta = 1 + 2b^2, S_0 = 2/(1 + 2b^2), S_1 = \{\alpha(1 + 2b^2)\}$$

$$S_2 = -b^2/\{\alpha^2(1 + 2b^2)\}. \quad \dots (3.3)$$

Therefore, from (1.12) we get

$$g^{ij} = a^{ij} \frac{2}{1 + 2b^2} b^i b^j - \frac{1}{\alpha(1 + 2b^2)}(b^i y^j + b^j y^i) + \frac{b^2}{\alpha^2(1 + 2b^2)} y^i y^j. \quad \dots (3.4)$$

Thus along F^{n-1}, (3.4) and (3.1) lead to $g^{ij} b_i b_j = \frac{b^2}{1 + 2b^2}$.

Therefore, we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{1 + 2b^2}} N_i, b^2 = a^{ij} b_i b_j. \quad \dots (3.5)$$

Again from (3.4) and (3.5) we get

$$b^i = a^{ij} b_j = \sqrt{b^2(1 + 2b^2)} N^i + b^2 \alpha^{-1} y^i. \quad \dots (3.6)$$

Hence, we have the following :

Theorem 3.1 — *Let F^n be a Matsumoto space with a gradient $b_i(x) = \partial_i b(x)$ and let $F^{n-1}(c)$ be a hypersurface of F^n which is given by b(x) = c (constant). Suppose the Riemannian metric $a_{ij}(x)\, dx^i dx^j$ be positive definite and b_i be non-zero field. Then the induced metric on $F^{n-1}(c)$ is a Riemannian metric given by (3.2) and relations (3.5) and (3.6) hold.*

Along $F^{n-1}(c)$, the angular metric tensor and metric tensor are given by

$$h_{ij} = a_{ij} + 2b_i b_j - \frac{Y_i Y_j}{\alpha^2} \quad \dots (3.7)$$

and

$$g_{ij} = a_{ij} + 3b_i b_j + \frac{1}{\alpha}(b_i y_j + b_j y_i). \quad \dots (3.8)$$

From (3.1), (3.7) and (2.4) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then along $F^{n-1}(c)$ $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$ = From (1.11), we get $\frac{\partial p_0}{\partial \beta} = 12\, \alpha^4/(\alpha - \beta)^5$. Thus along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = \frac{12}{\alpha}$ and therefore (1.15) gives $r_1 = 6/\alpha, m_i = b_i$. Therefore, the *hv*-torsion tensor becomes

$$C_{ijk}=\frac{1}{2\alpha}(h_{ij}b_k+h_{jk}b_i+h_{ki}b_j)+\frac{3}{\alpha}b_i b_j b_k \quad \dots (3.9)$$

Therefore, (2.4), (2.9), (2.12), (3.1) and (3.9) give

$$M_{\alpha\beta}=\frac{1}{2\alpha}\sqrt{\frac{b^2}{1+2b^2}}\,h_{\alpha\beta},\, M_\alpha=0. \quad \dots (3.10)$$

Hence, from (2.14) it follows that $H_{\alpha\beta}$ is symmetric.

Theorem (3.2) — *The second fundamental v-tensor of $F^{n-1}(c)$ is given by (3.10) and the second fundamental h- tensor $H_{\alpha\beta}$ is symmetric.*

Next from (3.1) we get $b_{i|\beta}B^i_\alpha+b_i B^i_{\alpha|\beta}=0$. Therefore, \from (2.13) and the fact that[6] $b_{i|\beta}=b_{i|j}B^j_\beta+b_i|_j N^j H_\beta$, we get

$$b_{i|j}B^i_\alpha B^j_\beta+b_i|_j B^i_\alpha N^j H_\beta+b_i H_{\alpha\beta}N^i=0. \quad \dots (3.11)$$

Since $b_i|_j=-b_h C^h_{ij}$, from (2.12), (3.5) and (3.10) we get

$b_i|_j B^i_\alpha N^j=\sqrt{\frac{b^2}{1+2b^2}}\,M_\alpha=0$. Thus (3.11) gives

$$\sqrt{\frac{b^2}{1+2b^2}}\,H_{\alpha\beta}+b_{i|j}B^i_\alpha B^j_\beta=0. \quad \dots (3.12)$$

It is noted that $b_{i|j}$ is symmetric. Furthermore, contracting (3.12) with v^β and v^α respectively and using (2.1), (2.15) and (3.10) we get

$$\sqrt{\frac{b^2}{1+2b^2}}\,H_\alpha+b_{i|j}B^i_\alpha y^j=0,\ \sqrt{\frac{b^2}{1+2b^2}}\,H_0+b_{i|j}y^i y^j=0. \quad \dots (3.13)$$

In view of Lemmas (2.1), and (2.2), the hypersurface $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $H_0=0$. Thus from (3.13) it follows that $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{i|j}y^i y^j=0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$, of F^n, it may depend on y^i. On the other hand $\nabla_j b_i=b_{ij}$ is the covariant derivative with respect to the Riemannian connection $\left\{ {i \atop jk} \right\}$ constructed from $a_{ij}(x)$, therefore b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j}-b_{ij}$ in the following. The difference tensor $D^i_{jk}=\Gamma^{*i}_{jk}-\left\{ {i \atop jk} \right\}$ is given by (1.17). Since b_i is a {italic grad}ient vector, from (1.16) we have $E_{ij}=b_{ij}$, $F_{ij}=0$, $F^i_j=0$. Thus (1.17) reduces to

$$\left.\begin{aligned} D^i_{jk} &= B^i\, b_{jk} + B^i_j\, b_{0k} + B^i_k\, b_{0j} - b_{0m}\, g^{im}\, B_{jk} \\ &- C^i_{jm}\, A^m_k - C^i_{km}\, A^m_j + C_{jkm}\, A^m_s\, g^{2k} \\ &+ \lambda^s \left(C^i_{jm}\, C^m_{sk} + C^i_{km}\, C^m_{sj} - C^m_{jk}\, C^i_{ms} \right) \end{aligned}\right] \quad \text{... (3.14)}$$

But in view of (3.3) and (3.4), the expressions (1.18) reduce to

$$\left.\begin{aligned} & B_i = 3b_i + \alpha^{-1}\, y_i,\ {}^i = \frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2\,b^2)}, \\ & B_{ij} = \frac{1}{2^{\alpha}}\,(a_{ij} - \alpha^{-2}\, y_i\, y_j + 12\, b_i\, b_j), \\ & B^i_j = \frac{1}{2^{\alpha}}\,(\delta^i_j - \alpha^{-2}\, y_j\, y^i) + \frac{5}{\alpha(1+2b^2)}\, b^i\, b_j - \frac{1+12\,b^2}{2\alpha^2\,(1+2b^2)}\, y^i\, b_j, \\ & A^m_k = B^m_k\, b_{00} + B^m\, b_{k0} \\ \text{and}\quad & \lambda^m = B^m\, b_{00}. \end{aligned}\right\}, \quad \text{... (3.15)}$$

By virtue of (3.1) we have $B^i_0 = 0, B_{i0} = 0$ which gives $A^m_0 = B^m\, b_{00}$.
We, therefore, have

$$D^i_{j0} = B^i\, b_{j0} + B^i_j\, b_{00} - B^m\, C^i_{jm}\, b_{00}, \quad \text{... (3.16)}$$

and

$$D^i_{00} = B^i\, b_{00} = \left[\frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2\,b^2)}\right] b_{00}. \quad \text{... (3.17)}$$

Thus paying attention to (3.1) along the $F^{n-1}(c)$, we finally get

$$b_i\, D^i_{j0} = \frac{2b^2}{1+2\,b^2}\, b_{j0} + \frac{1+12\,b^2}{2\,\alpha(1+2\,b^2)}\, b_{00}\, b_j - 2b^m\, b_i\, C^i_{jm}\, b_{00} \quad \text{... (3.18)}$$

and

$$b_i\, D^i_{00} = \frac{2b^2}{1+2\,b^2}\, b_{00}. \quad \text{... (3.19)}$$

From (2.12), (3.5), (3.6) and (3.10) it follows that

$b^m\, b_i \cdot C^i_{jm}\, B^j_\alpha = b^2\, M_\alpha = 0$. Therefore, the relation

$b_{i\,|\,j} = b_{ij} - b_r\, D^r_{ij}$ and eqs. (3.18), (3.19) give

$$b_{i\,|\,j}\, y^i\, y^j = b_{00} - b_r\, D^r_{00} = \frac{1}{1+2\,b^2}\, b_{00}$$

Consequently, (3.13) may be written as

$$\sqrt{b^2}\, H_\alpha + \frac{1}{\sqrt{1+2b^2}} b_{i0} B^i_\alpha = 0, \sqrt{b^2}\, H_0 + \frac{1}{\sqrt{1+2b^2}} b_{00} = 0. \qquad \text{... (3.20)}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i. Since y^i is to satisfy (3.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i C_j + b_j C_i. \qquad \text{... (3.21)}$$

From (3.1) and (3.21) it follows that $b_{00} = 0, b_{ij} B^i_\alpha B^j_\beta = 0,\ b_{ij} B^i_\alpha y^j = 0$. Hence, (3.20) gives $H_\alpha = 0$. Again from (3.21) and (3.15) we get $b_{i0} b^i = \frac{c_0 b^2}{2}, \lambda^m = 0, A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{2\alpha} h_{\alpha\beta}$. Thus (2.9), (3.4), (3.5), (3.6), (3.10) and (3.14) give

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = \frac{-C_0 b^2}{4\alpha(1+2b^2)^2} h_{\alpha\beta}.$$

Therefore, eq. (3.12) reduces to

$$\sqrt{\frac{b^2}{1+2b^2}}\, H_{\alpha\beta} + \frac{C_0 b^2}{4\alpha(1+2b^2)^2} h_{\alpha\beta} = 0. \qquad \text{... (3.22)}$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

***Theorem* 3.3** — *The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of the first kind is (3. 21) and in this case the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor.*

In view of Lemma (2.3), $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$, and $H_{\alpha\beta} = 0$. Thus from (3.22) we get $C_0 = C_i(x) y^i = 0$. Therefore, there exist a function $e(x)$ such that $c_i(x) = e(x)\, b_i(x)$. Thus (3.21) gives

$$b_{ij} = e b_i b_j. \qquad \text{... (3.23)}$$

***Theorem* 3.4** — *The necessary and sufficient condition for $F^{n-1}(c)$ to be hyperplane of the second kind is (3.23).*

Finally (3.10) and lemma (2.4) show that $F^{n-1}(c)$ does not become a hyperplane of the third kind.

***Theorem* 3.5** — *The hypersurface $F^{n-1}(c)$ is not a hyperplane of the third kind.*

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