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A MULTI-INTEGRAL METHOD FOR A CLASS OF SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

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This paper aims to investigate a class of (weakly) singular two-point boundary value problems $x^{-\alpha}(x^{\alpha}y')' = f(x, y), 0 < x \le 1$, and $\alpha \in \mathcal{R}$. We apply multi-integral methods to solve these problems. Bounds of sup $|f_y|$ which provide the rate of convergence of the iterative sequence are obtained for various values of α. Some numerical results are presented to display the performance of these methods.

Key Words : Singular Two-Point Boundary Value Problems; Multi-Integral Methods; Picard Type of Implicit Methods

1. INTRODUCTION

Singular two-point boundary value problems arise in many applications of engineering and physics[1&6]. In this paper, we consider the (weakly) singular two-point boundary value problem :

$$x^{-\alpha}(x^{\alpha}y')' = f(x, y), x \in (0, 1], \quad \dots (1.1)$$

under the boundary conditions

$$y(0) = A, \; y(1) = B, \quad \dots (1.2)$$

and
$$y(0) = A, \; y'(1) = B. \quad \dots (1.3)$$

Here, A and B are finite constants and $\alpha \in \mathcal{R}$. Problem (1.1) is a cylindrical or spherical problem when $\alpha = 1$ or $\alpha = 2$, respectively. We assume that $f(x, y)$ is continuous real valued function for every $x \in [0, 1], y \in \mathcal{R}$ and differentiable for every y.

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Intensive research has been developed for solving singular nonlinear differential equations[1,4&9]. In [3], a second order spline finite difference method has been introduced for solving problem (1.1). The linear case $f(x, y) = ky + g(x)$ of problem (1.1), where $g \in C[0, 1]$ and $0 < x < \pi^2$ has been studied in [5]. Chawla and Katti[2] have proposed on $O(h^2)$-convergent finite difference method to solve problem have proposed an $O(h^2)$-convergent finite difference method to solve problem (1.1) under quite general conditions on f' and f'', where $\alpha \in (0, 1)$. In [8], multi-integral iterative methods have been applied to compute the solutions of nonlinear fourth order differential equations under different boundary conditions. Moreover, existence and uniqueness of the iterative solution and the rates of convergence of the generated sequences have been proved. When $\alpha = 2$, problem (1.1) has been studied and the rate of convergence has been obtained[7].

The purpose of this paper is to use Picard type implicit method to solve problem (1.1) and to investigate the convergence of this method.

This paper is organized as follows. In Section 2, we derive the solutions of problem (1.1) using multi-integral methods. Section 3 involves the convergence analysis of the introduced multi-integral methods. Some numerical experiments that illustrates the obtained convergence rate are presented in Section 4. Concluding remarks are given in the last section.

2. Derivation of Solutions

This section is devoted to introducing multi-integral techniques to find the solutions of problem (1.1) under the boundary conditions (1.2) and (1.3).

The following theorem describes a method for obtaining an explicit formulation to the solutions of problem (1.1).

***Theorem* 2.1** — *If $y(x)$ is the solution of problem (1.1) under the boundary conditions (1.2) and the real valued function $f(x, y)$ is continuous for every $x \in (0, 1], y \in \mathcal{R}$ are differentiable for every y, then*

$$xy(x) = B + A(\alpha - 1)(x - 1) + (\alpha - 2)\int_x^1 y(\eta)\, d\eta - (1 - x)\int_0^1 \eta f(\eta, y(\eta))\, d\eta$$

$$+ \int_x^1 \int_\xi^1 \eta f(\eta, y(\eta))\, d\eta\, d\xi, \text{ or} \qquad \text{... (2.1)}$$

$$xy(x) = xA(\alpha - 1) - (\alpha - 2)\int_0^x y(\eta)\, d\eta + \int_0^x \eta(x - \eta) f(\eta, y(\eta))\, d\eta. \qquad \text{... (2.2)}$$

For the sake of simplicity, we shall drop the argument η of $y(\eta)$.

PROOF : Since $y(x)$ is a solution of problem (1.1), then we have

$$y'' + \frac{\alpha}{x} y' = f(x, y). \qquad \text{... (2.3)}$$

Multiplying eq. (2.3) by x and integrating it from x to 1, we obtain

$$y'(1) - xy'(x) - (\alpha - 1)\, y(x) + (\alpha - 1)\, y(1) = \int_x^1 \eta f(\eta, y)\, d\eta. \qquad \ldots (2.4)$$

Setting $x = 0$ in eq. (2.4) and using the boundary conditions (1.2) at $x = 0$ and $x = 1$, we obtain

$$y'(1) = (\alpha - 1)\, A - (\alpha - 1)\, B + \int_0^1 \eta f(\eta, y)\, d\eta. \qquad \ldots (2.5)$$

Combining eqs. (2.4) and (2.5), we obtain

$$[xy(x)]' = A(\alpha - 1) - (\alpha - 2)\, y(x) + \int_0^1 \eta f(\eta, y)\, d\eta - \int_x^1 \eta f(\eta, y)\, d\eta. \qquad \ldots (2.6)$$

Integrating eq. (2.6) from a to x (for $a > 0$), we obtain

$$xy(x) - ay(a) = A\,(\alpha - 1)\,(x - a) - (\alpha - 2) \int_a^x y\, d\eta + (x - a) \int_0^1 \eta f(\eta, y)\, d\eta$$

$$- \int_a^x \int_\xi^1 \eta f(\eta, y)\, d\eta\, d\xi. \qquad \ldots (2.7)$$

Setting $x = 1$ in eq. (2.7) and using the boundary conditions (1.2), we obtain

$$a\, y(a) = B - A\,(\alpha - 1)\,(1 - a) - (1 - a) \int_0^1 \eta f(\eta, y)\, d\eta$$

$$+ (\alpha - 2) \int_a^1 y\, d\eta + \int_a^1 \int_\xi^1 \eta f(\eta, y)\, d\eta\, d\xi. \qquad \ldots (2.8)$$

The proof of part (2.1) is completed by combining eqs. (2.7) and (2.8).

To proving part (2.2), integrating eq. (2.6) from 0 to x, we obtain

$$x\, y(x) = Ax\,(\alpha - 1) - (\alpha - 2) \int_0^x y\, d\eta + \int_0^x \int_0^\xi \eta f(\eta, y)\, d\eta\, d\xi. \qquad \ldots (2.9)$$

The proof of part (2.2) is completed by applying the identity $\int_0^t \int_0^s g(\eta)\, d\eta\, ds = \int_0^t (t - s)\, g(s)\, ds$ on eq. (2.9). ■

Theorem 2.2 — *Let the hypotheses of Theorem (2.1) hold. Then*

$$y(x) = A + x^{1-\alpha} \int_0^x \int_0^t s\, t^{\alpha-2} f(s, y)\, ds\, dt, \text{ for } \alpha > 1, \text{ or} \quad \text{... (2.10)}$$

$$y(x) = x^{1-\alpha} \left[A\,(x^{\alpha-1} - 1) + B - \int_x^1 \int_0^t s\, t^{\alpha-2} f(s, y)\, ds\, dt \right]. \quad \text{... (2.11)}$$

PROOF : Substituting from eq. (2.5) into formula (2.4), we obtain

$$[x^{\alpha-1}\, y(x)]' = x^{\alpha-2} \left[A\,(\alpha - 1) + \int_0^x s f(s, y)\, ds \right]. \quad \text{... (2.12)}$$

Integrating both sides of eq. (2.12) from 0 to x ($\alpha > 1$), we obtain eq. (2.10). The proof of eq. (2.11) can be completed by integrating eq. (2.12) from a to x (for $a > 0$) and setting $x = 1$. ■

Theorem 2.3 — *Let the hypotheses of theorem (2.1) hold under the boundary conditions (1.3). Then the eqs. (2.1) and (2.2) are satisfied.*

PROOF : Setting $x = 0$ in eq. (2.4) and using boundary conditions (1.3) at $x = 0$ and $x = 1$, we obtain

$$(\alpha - 1)\, y(1) = (\alpha - 1) A - B + \int_0^1 \eta f(\eta, y)\, d\eta. \quad \text{... (2.13)}$$

Combining eqs. (2.4) and (2.13), we obtain

$$[xy(x)]' = A(\alpha - 1) - (\alpha - 2)\, y(x) + \int_0^1 \eta f(\eta, y)\, d\eta - \int_x^1 \eta f(\eta, y)]\, d\eta .$$

To complete the proof, we follow the same steps of the proof of Theorem (2.1). ■

3. Convergence Analysis

This section involves bounds for sup $|f_y|$ for different values of α. These bounds guarantee convergence to a unique solution of problem (1.1).

In the following theorem, we shall prove that the iterative sequence $\{y^k\}$ of solutions y^k is (Cauchy sequence) convergent.

Theorem 3.4 — *If the generating sequence $\{y^k\}$ from the iteraction process*

$$x\, y^{k+1}(x) = B + A\,(\alpha - 1)\,(x - 1) + (\alpha - 2) \int_x^1 y^k\, d\eta \quad \text{... (3.1)}$$

$$+ (x-1) \int_0^1 \eta f(\eta, y^k)\, d\eta + \int_x^1 \int_\xi^1 \eta f(\eta, y^k)\, d\eta\, d\xi,$$

is bounded by a constant C_1 and the sup $|f_y| = 6(\alpha - 2)$ for $2 < \alpha < 3$, then the sequence $\{y^k\}$ is convergent.

PROOF : The main idea of the proof is to show that the sequence $\{y^k\}$ is a Cauchy sequence. Given $\varepsilon > 0$, there exists an integer N such that for any $n, m \geq N$, we have

$$(y^n - y^m) = \frac{\alpha - 2}{x} \int_x^1 [y^{n-1} - y^{m-1}]\, dt + \left(1 - \frac{1}{x}\right) \int_0^1 s\, [f(s, y^{n-1}) \quad \ldots (3.2)$$

$$- f(s, y^{m-1})]\, ds + \frac{1}{x} \int_x^1 \int_t^1 s\, [f(s, y^{n-1}) - f(s, y^{m-1})]\, ds\, dt.$$

Taking the norm of both sides of eq. (3.2) and applying the mean value theorem, we obtain

$$\| y^n - y^m \| \leq \frac{\alpha - 2}{x} \| y^{n-1} - y^{m-1} \| \int_x^1 dt + \sup |f_y|\, \| y^{n-1} - y^{m-1} \| \left| 1 - \frac{1}{x} \right| \int_0^1 s ds$$

$$+ \frac{1}{x} \sup |f_y|\, \| y^{n-1} - y^{m-1} \| \int_x^1 \int_t^1 sd\, sdt$$

$$= \| y^{n-1} - y^{m-1} \| \left| \frac{(\alpha - 2)(1 - x)}{x} + \sup |f_y| \left[\frac{x^3 - 1}{6x} \right] \right|$$

$$= \| y^{n-1} - y^{m-1} \| \left| \frac{1}{x} \left[\alpha - 2 - \frac{\sup |f_y|}{6} \right] + 2 - \alpha + \frac{x^2 \sup |f_y|}{6} \right| .$$

Setting sup $|f_y| = 6(\alpha - 2)$, we obtain

$$\| y^n - y^m \| \leq \| y^{n-1} - y^{m-1} \|\, |2 - \alpha + x^2 (\alpha - 2)|$$

$$= \| y^{n-1} - y^{m-1} \|\, |(2 - \alpha)(1 - x^2)|$$

$$\leq \| y^{n-1} - y^{m-1} \|\, |(2 - \alpha) \max (1 - x^2)| = \| y^{n-1} - y^{m-1} \|\, |2 - \alpha|.$$

For $2 < \alpha < 3$, we obtain

$$\| y^n - y^m \| \leq \| y^{n-1} - y^{m-1} \|\, |2 - \alpha| = K_1 \| y^{n-1} - y^{m-1} \| \leq K_1^m\, | y^{n-m} - y^0 | \cdot$$

$$\leq K_1^m\, [| y^{n-m} - y^{n-m-1} | + | y^{n-m-1} - y^{n-m-2} | + \ldots + | y^1 - y^0 |]$$

$$\leq K_1^m [K_1^{n-m-1} | y^1 - y^0 | + K_1^{n-m-2} | y^1 - y^0 | + \ldots + | y^1 - y^0 |]$$

$$= K_1^m [K_1^{n-m-1} + K_1^{n-m-2} + \ldots + 1] | y^1 - y^0 |$$

$$\leq C_1 \frac{K_1^m}{1 - K_1} = \hat{C}_1 \frac{K_1^m}{1 - K_1} \leq \varepsilon_1,$$

where $K_1 = | 2 - \alpha | < 1$. This completes the proof. ■

Theorem 3.5 — *If the generating sequence $\{y^k\}$ from the iteration process*

$$y^k (x) = A (\alpha - 1) - \frac{(\alpha - 2)}{x} \int_0^x y^{k-1} \, d\eta + \frac{1}{x} \int_0^x \eta (x - \eta) f(\eta, y^{k-1}) \, d\eta. \qquad \ldots (3.3)$$

is bounded by a constant C_2 and sup $| f_y | \leq 6 (\alpha - 2)$ for $2 < \alpha < 3$, then the sequence $\{y^k\}$ is convergent.

PROOF : We follow the proof of theorem (3.4). Thus, we obtain

$$\| y^n (x) - y^m (x) \| \leq \frac{\alpha - 2}{x} \| y^{n-1} - y^{m-1} \|$$

$$\int_0^x d\eta + \frac{\sup | f_y |}{x} \| y^{n-1} - y^{m-1} \| \int_0^x \eta (x - \eta) \, d\eta$$

$$= \| y^{n-1} - y^{m-1} \| \left[\alpha - 2 + \frac{x^2 \sup | f_y |}{6} \right]$$

$$\leq \| y^{n-1} - y^{m-1} \| [\alpha - 2 + x^2 (\alpha - 2)]$$

$$\leq \| y^{n-1} - y^{m-1} \| [(\alpha - 2) \max (x^2 + 1)] = 2K_2 \| y^{n-1} - y^{m-1} \| \leq 2K_2^m \| y^{n-m} - y^0 \|$$

$$\leq 2C_2 \frac{K_2^m}{1 - K_2} C_2 \frac{K_2^m}{1 - K_2} \leq \varepsilon_2,$$

where $K_2 = (\alpha - 2) < 1$. This completes the proof. ■

Theorem 3.6 — *If the generating sequence $\{y^k\}$ from the solution (2.10) is bounded by C_3 and sup $| f_y | < 2 (\alpha + 1)$ for $\alpha > -1$, then $\{y^k\}$ is convergent.*

PROOF : We follow the proof of Theorem (3.4). Thus, we obtain

$$\| y^n (x) - y^m (x) \| \leq \sup | f_y | \| y^{n-1} - y^{m-1} \| x^{1-\alpha} \int_0^x \int_0^t s t^{\alpha - 2} \, ds \, dt$$

$$= \| y^{n-1} - y^{m-1} \| \frac{x^2 \sup |f_y|}{2(\alpha+1)}$$

$$= \| y^{n-1} - y^{m-1} \| \frac{\sup |f_y|}{2(\alpha+1)}$$

$$\leq K_3 \| y^{n-1} - y^{m-1} \| \leq K_3^m \| y^{n-m} - y^0 \|$$

$$\leq 2C_3 \frac{K_3^m}{1-K_3} = C_3 \frac{K_3^m}{1-K_3} \leq \varepsilon_3,$$

where $K_3 = \frac{\sup |f_y|}{2(\alpha+1)} < 1$. This completes the proof. ■

Theorem 3.7 — *If the generating sequence $\{y^k\}$ from the solution (2.11) is bounded by C_4 and sup $|f_y| < 2(\alpha+1) \min\left(\frac{1}{x^{1-\alpha} - x^2}\right)$, $\alpha \in (-\infty, 1)$, then $\{y^k\}$ is convergent.*

PROOF : We follow the proof of theorem (3.4). Thus, we obtain

$$\| y^n(x) - y^m(x) \| \leq \sup |f_y| \, \| y^{n-1} - y^{m-1} \| \, x^{1-\alpha} \int_x^1 \int_0^t s t^{\alpha-2} \, ds \, dt$$

$$= \| y^{n-1} - y^{m-1} \| \frac{\sup |f_y|}{2(\alpha+1)} \max (x^{1-\alpha} - x^2)$$

$$\leq K_4 \| y^{n-1} - y^{m-1} \| \leq K_4^m \| y^{n-m} - y^0 \|$$

$$\leq 2C_4 \frac{K_4^m}{1-K_4} = C_4 \frac{K_4^m}{1-K_4} \leq \varepsilon_4,$$

where $K_4 = \frac{\sup |f_y|}{2(\alpha+1)} \max (x^{1-\alpha} - x^2) < 1$, for $\alpha \in (-\infty, 1)$. This completes the proof. ■

Remark 1 : The following table contains some maximum values of $(x^{1-\alpha} - x^2)$ for different values of α.

α	x	The maximum	α	x	The maximum
– 10	0.83	5.60117E-01	0.0	0.50	2.50000E-01
– 8	0.81	5.06005E-01	0.2	0.47	3.25711E-01
– 6	0.78	4.32744E-01	0.4	0.42	4.17823E-01
– 4	0.74	3.25699E-01	0.6	0.37	5.34963E-01
– 2	0.67	1.48137E-01	0.8	0.28	6.96832E-01

Corollary 3.1 — If ε is the tolerance of convergence, then the number of iterations m of the introduced multi-integral techniques is at least

$$m = \frac{\ln \varepsilon_i - \ln \hat{C}_i + \ln (1 - K_i)}{\ln K_i} + 1, \text{ for } i = 1, 2, 3, 4.$$

The proof is straight forward, when we take $\hat{C}_i \frac{K_i^m}{1 - K_i} \leq \varepsilon_i$.

The existence and uniqueness of the solution of problem (1.1) can be easily proved by using the bounds on sup $|f_y|$ which are mentioned in the previous theorems.

4. Numerical Illustrations

In this section, we illustrate the above results by reporting some numerical experiments which are obtained from solving two problems. The numerical solutions of these problems describe the computational efficiency of the iterations that we have used.[1] We applied Simpson's method on a uniform mesh $x_i = ih$, $i = 0, 1, ..., N$ and $h = \frac{1}{N}$ to compute the integrals on the interval [0, 1]. In the following tables, $\varepsilon_i = \| y(x_j) - y^k(x_j) \|$, $\varepsilon_i^k = \| y^k(x_j) - y^{k-1}(x_j) \|$, where $j = 1, ..., N-1$ and $R_i^k = \ln_2 [\varepsilon_i^N / \varepsilon_i^{2N}]$ (the rate of convergence), $i = 1, 2, 3, 4$.

Let υ be the solution which is obtained by the presented iterations and let m_i be the smallest number s for which $\| \upsilon_s - \upsilon \| \leq \left(2C_i \frac{K_i^m}{1 - K_i} \right)^s \leq 10^{-10}$, $i = 1, 2, 3, 4$.

The indices $i = 1, 2, 3, 4$ refer to the solutions (2.1), (2.2), (2.10) when $\alpha = 2$ and (2.11) when $\alpha = 0.5$, respectively.

The methods have been tested on the following two problems :

Problem 1 — $\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy(x)}{dx} = \beta x^{\beta - 2} y(x) (\beta y(x) + 2\beta - 1), 0 < x < 1,$

$$y(0) = 1, \; y(1) = e \qquad \dots (4.1)$$

The exact solution to problem (4.1) is $y(x) = e^{x^\beta}$, $\beta = 0.4$. We used $y^0 = (1 + e) x + 1$ to be our initial guess.

N	m_1	ε_1	ε_1^k	R_1^k	m_2	ε_2	ε_2^k	R_2^k
32	6	6.5011E-01	2.8421E-03	5.1	5	5.3489E-02	1.1834E-04	4.1
64	7	2.9563E-02	8.4043E-05	5.8	5	7.4663E-03	7.1303E-06	3.8
128	7	3.4320E-03	1.4722E-06	5.9	6	3.1529E-03	5.0942E-07	4.4
256	9	1.1498E-03	2.3059E-08	4.9	7	4.2912E-4	2.3467E-08	4.6
512	9	3.6459E-04	7.8266E-10	-	8	4.1472E-04	9.5299E-10	-

Problem 2 — $\frac{d^2 y(x)}{dx^2} + \frac{\alpha}{x}\frac{dy(x)}{dx} = \frac{g}{x}\cos x - h \sin x - y(x), 0 < x < 1,$

$$y(0) = 0,\ y(1) = \cos 1 + \sin 1. \qquad \text{... (4.2)}$$

We took $g = h = 4$ for $\alpha = 2$ and $g = 1$, $h = 2.5$ for $\alpha = 0.5$. The exact solution of problem (4.2) is $y(x) = x \cos x + \sin x$ and we start the iteration using $y^0 = \cos 1 + x \sin 1 + 1$.

N	m_3	ε_3	ε_3^k	R_3^k	m_4	ε_4	ε_4^k	R_4^k
32	6	7.9540E-02	1.0489E-03	3.5	4	4.4322E-03	8.6039E-03	3.7
64	7	9.4591E-03	9.4988-05	3.9	5	3.2645E-03	6.6409E-05	6.2
128	8	5.1006E-03	6.3321E-06	4.0	5	2.9397E-03	2.1781E-06	6.5
256	8	2.0288E-03	3.9422E-07	4.3	5	2.7226E-03	9.9285E-08	5.1
512	8	6.6391E-04	2.0082E-08	-	6	7.3715E-04	6.8547E-10	-

5. Concluding Remarks

In this paper, we have investigated (weakly) singular nonlinear two-point boundary value problems. This investigation achieved general bounds for sup $|f_y|$ for $\alpha \in \mathcal{R}$. Also, we discussed various iteration techniques to compute the solution numerically.

The above tables show that the accuracy of iteration (2.2) is better than the accuracy of iteration (2.1). More accuracy can be achieved by taking a smaller grid size in Simpson's method or by applying methods that give us more accurate approximations for the integrals. On the other hand our approach can be applied to nonlinear boundary value problems with various boundary conditions. Moreover, the presented results are more accurate than those introduced in references [2 & 7].

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