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ON SUFFICIENT CONDITIONS FOR STARLIKENESS

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In a recent paper the present author in collaboration with two others studied sufficient conditions for a function f on the unit disk to be starlike. Nunokowa *et al.*[2] refined those results subsequently. The aim of this paper is to improve the earlier results in a different direction.

Key Words : Starlikeness; Sufficiency; Subordination; Univalent Functions

1. INTRODUCTION

Let A denote the class of functions f in the unit disk $E = \{z : |z| < 1\}$ with $f(0) = 0 = f'(0) - 1$, and S denote the subclass of A consisting of univalent functions. Let S^* denote the subclass of S consisting of starlike functions f, that is functions f which satisfy $Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ for $z \in E$. Let $H(\alpha)$ denote the class of functions $f \in A$ satisfying the condition, $\frac{f(z)}{z} \neq 0$,

$$Re\left\{\alpha\frac{z^2 f'(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > 0, \alpha \geq 0, z \in E. \qquad \text{... (1)}$$

Eq. (1) is equivalent to

$$\alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \alpha \geq 0, z \in E \qquad \text{... (2)}$$

in the language of subordination. We say that f is subordinate to g, when f and g are analytic in E, $f(0) = g(0)$, g is univalent in E and write symbolically $f(z) \prec g(z)$ in E if there exists an analytic function ω in E with $\omega(0) = 0, |\omega(z)| < 1, z \in E$ such that $f(z) = g(\omega(z))$, for $z \in E$. 'g' is called the superordinate function. In [3] it was proved that if $f \in H(1)$, then f is strongly starlike of order $\frac{1}{2}$. A function $f \in S$ is called strongly starlike of order $\alpha, 0 < \alpha, \leq 1$ if and only if $\left| arg\frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2}, z \in E$ and we denote this class of functions by $\tilde{S^*}(\alpha)$. Nunokowa *et al.*[2] proved that $f \in H(1)$ implies $f \in \tilde{S^*}(\beta)$, for a certain $\beta < \frac{1}{2}$. We introduce the following

Definition 1 — Let $H(\alpha, \phi)$ denote the class of functions $f \in A$ with $\frac{f(z)}{z} \neq 0, z \in E$, such that

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \phi(z), z \in E, \qquad \text{...(3)}$$

where ϕ is an analytic univalent function in E with $\phi(0) = 1$, for $z \in E$.

For $\alpha = 1, \phi(z) = \frac{1+z}{1-z}$, $H(\alpha, \phi)$ coincides with $H(1)$. In [3], we considered the class $H\left(\alpha, \frac{1-z(\alpha+1)}{1+z}\right)$, choosing $\phi(z) = \frac{1-z(\alpha+1)}{1+z}$ and proved that $f \in H(\alpha, \phi) \Rightarrow f \in S^*$ for any $\alpha \geq 0$. Note that $f \in H\left(\alpha, \frac{1-z(\alpha+1)}{1+z}\right)$, is equivalent to the requirements $Re\left\{\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > -\frac{\alpha}{2}$, $z \in E$. Nunokowa *et al.*[2] improved our result by replacing $-\frac{\alpha}{2}$ by $-\frac{\alpha}{2}\left[1 + 3\left(\text{Im}\frac{zf'(z)}{f(z)}\right)^2\right]$ and arriving at the same conclusion. We propose to improve our earlier result by refining the superordinate function ϕ and making an assumption less restrictive, to obtain the same results.

We need the following results due to Miller and Mocanu[1] in the sequel.

Theorem A — *(Theorem 3)*[1] — *Let q be univalent in the unit disk E and let θ and ϕ be analytic in a domain D containing $q(E)$ with $\phi(\omega) \neq 0$ when $\omega \in q(E)$. Set $Q(z) = zq'(z)\,\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is starlike univalent in E, and*

(ii) *$Re \frac{zh''(z)}{Q(z)} = \text{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right] > 0, z \in E.$*

If p is analytic in E, with $p(0) = q(0)$, $p(E) \in D$ and

$$\theta(p(z)) + zp'(z)\,\phi(p(z)) \prec \theta(q(z)) + zq'(z)\,\phi(q(z)) = h(z),$$

then $p \prec q$ and q is the best dominant of the subordination.

Theorem 1 — *Let $f \in H(\alpha; h_1)$ where*

$$h_1(z) = \frac{2\alpha(z^2 + 2z) + 1 - z^2}{(1-z)^2}, 1 \geq \alpha > 0, z \in E.$$

Then $f \in S^*$.

PROOF : By assumption

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec h_1(z), z \in E. \quad \text{... (4)}$$

Setting $p(z) = \frac{zf'(s)}{f(z)}$, we obtain $\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - p(z)$,

$$\frac{z^2 f'(z)}{f'(z)} = \frac{zf''(z)}{f'(z)} \cdot \frac{zf'(z)}{f(z)} = \left(\frac{zp'(z)}{p(z)} - 1 + p(z) \right) p(z) = zp'(z) - p(z) + p^2(z). \quad \text{... (5)}$$

Using (5), we can rewrite (4), $\alpha(zp'(z) - p(z) + p^2(z)) + p(z) = p(z) \prec h_1$, that is,

$$\alpha(zp'(z) + p^2(z)) + (1 - \alpha)\, p(z) \prec h_1. \quad \text{... (6)}$$

We chose $q(z) - \frac{1+z}{1-z}$, $\phi(z) = \alpha$, $\theta(z) = \alpha z^2 + (1 - \alpha) z$. The function q is evidently univalent in E, $q(E)$ is the right half-plane; θ and ϕ satisfy the conditions mentioned in Theorem A. Choosing Q, h as in Theorem A, we have

$$Q(z) = zq'(z)\, \phi(q(z)) = \alpha zq'(z) \quad \text{... (7)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha q^2(z) + (1 - \alpha)\, q(z) + \alpha zq'(z). \quad \text{... (8)}$$

Substituting the values of $q(z), q'(z)$ in (8) we obtain

$$h(z) = h_1(z).$$

Again from (7) we get $Re \frac{zQ'(z)}{Q(z)} = 1 + Re \frac{zq''(z)}{q'(z)} > 0$, since $q(z) = \frac{1+z}{1-z}$ is convex univalent. So Q is starlike univalent in E.

Further an easy computation yields

$$Re \frac{zh'(z)}{Q(z)} = Re \left\{ \frac{1-\alpha}{\alpha} + 2q(z) + \frac{zQ'(z)}{Q(z)} \right\}$$

$$= \frac{1-\alpha}{\alpha} + 2\, Re \left(\frac{1+z}{1-z} \right) + Re \left(\frac{zQ'(z)}{Q(z)} \right) > 0, \text{ since } 0 < \alpha \leq 1. \quad \text{... (9)}$$

Eq. (9) shows h is close-to-convex and hence univalent in E.

Also p is analytic in E, since $\frac{f(z)}{z} \neq 0$ in E, $p(0) = q(0) = 1$.

$$\theta(p(z)) + zp'(z)\, \phi(p(z)) = \alpha p^2(z) + (1 - \alpha)\, p(z) + \alpha zp'(z) \prec h_1(z)$$

from (6), where $h_1(z) = h(z) = \theta(q(z)) + zq'(z)\, \phi(q(z))$ from (8).

All conditions of Theorem A are satisfied and we, therefore, conclude from Theorem A that $p \prec q$, that is $\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$, which means $f \in S^*$. Theorem A also ensures that $q(z) = \frac{1+z}{1-z}$, is the best dominant. We now investigate the mapping properties of the function $h(z) = \frac{2\alpha(z^2+2z)+1-z^2}{(1-z)^2}$, $z \in E$.

We can rewrite $h(z)$ in the form

$$h(z) = (2\alpha - 1) + \frac{2(1-\alpha+z(4\alpha-1))}{(1-z)^2}. \qquad \text{... (10)}$$

Clearly $h(0) = 1$, $h(-1) = -\frac{\alpha}{2}$, h takes real values for real z and $h(E)$ is symmetric with respect to the real axis. The function h maps the unit circle onto a close-to-convex curve symmetric with respect to the real axis, passing through the point $-\frac{\alpha}{2}$ with $Re\ h$ and $Im\ h$ going to infinity in the left half plane. The curve lies actually in the left half plane and meets the real axis only at $-\frac{\alpha}{2}$. Indeed $h(e^{i\theta}) = (2\alpha - 1) + \frac{2(1-\alpha)+2(4\alpha-1)e^{i\theta}}{(1-e^{i\theta})^2}$ from (10). On simplification this gives $h(e^{i\theta}) = u(e^{i\theta}) + iv(e^{i\theta})$ where

$$u(e^{i\theta}) = Re\,h(e^{i\theta}) = \frac{-\alpha(2+\cos\theta)}{(1-\cos\theta)};\ v(e^{i\theta}) = Im\,h(e^{i\theta}) = \frac{(1-\alpha)\sin\theta}{1-\cos\theta}.$$

Eliminating θ, we get the equation to the curve $h(e^{i\theta})$ as

$$v^2 = -\frac{2(1-\alpha)^2}{3\alpha}\left(u + \frac{\alpha}{2}\right) \qquad \text{... (11)}$$

in the ω-plane. This shows that h maps the unit circle onto the parabola whose vertex is $\left(-\frac{\alpha}{2}, 0\right)$, focus is $\left(-\frac{(1-\alpha)^2}{6\alpha}, 0\right)$ and axis is the negative real axis. And $h(E)$ is the region outside this parabola and includes the half plane, $Re\ \omega > -\frac{\alpha}{2}$. Therefore, Theorem 1 definitely improves the earlier result, namely[3], Theorem 1) for $0 \le \alpha \le 1$ and is best possible in the sense that the assumption cannot be made less restrictive. Setting $\alpha = 1$ in Theorem 1 we obtain the following result.

Corollary 1 — If $f \in A, \frac{f(z)}{z} \neq 0$ and $\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1+4z+z^2}{(1-z)^2}$ for $z \in E$, then $f \in S^*$.

The functions $\frac{1+4+z^2}{(1-z)^2}$ maps E onto the ω-plane slit along the negative real axis from $\omega = -\frac{1}{2}$. The parabola (11) degenerate into a straight line for $\alpha = 1$. It actually becomes the slit from $-\frac{1}{2}$ to ∞ in the negative real axis.

Theorem 2 — *Let* $f \in A$ *and satisfy in* $E, \frac{f(z)}{z} \neq 0$,

$$\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \left\{ \frac{2\beta z + (1+z)^{1+\beta}(1-z)^{1-\beta}}{(1-z)^{1+\beta}(1+z)^{1-\beta}} \right\}, 0 < \beta \leq 1;$$

then $f \in S^*(\beta)$.

Remark 1 : For $\beta = 1$, the theorem reduces to corollary 1.

Remark 2 : For $\beta = \frac{1}{2}$, the theorem says that if

$$\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \left\{ \frac{1+z}{1-z} + \frac{z}{(1-z)(1-z^2)^{\frac{1}{2}}} \right\},$$

then $f \in \tilde{S}^*$. Setting $h(z) = \left\{ \frac{1+z}{1-z} + \frac{z}{(1-z)(1-z^2)^{\frac{1}{2}}} \right\}$, we observe that $h(E)$ is a region that properly contains the right half plane.

Thus for $\beta = \frac{1}{2}$, Theorem 2 improves an earlier result ([3], Theorem 2).

PROOF OF THEOREM 2 : Setting $p(z) = \frac{zf'(z)}{f(z)}$, the assumption in Theorem 2 can be expressed as

$$zp'(z) + p^2(z) \prec h_\beta(z), z \in E, \quad \dots (12)$$

where
$$h_\beta(z) = \left\{ \frac{2\beta z + (1+z)^{1+\beta}(1-z)^{1-\beta}}{(1-z)^{1+\beta}(1+z)^{1-\beta}} \right\}. \quad \dots (13)$$

We again use Theorem *A* by choosing $q(z) = \left(\frac{1+z}{1-z} \right)^\beta, 0 < \beta < 1, \phi(z) = 1$ and $\theta(z) = z^2 \cdot q(E)$ is the angular region in the right halfplane, $|\arg q(z)| < \frac{\pi}{2}\beta$, q is univalent. As in Theorem *A*,

$$Q(z) = zq'(z)\,\phi(q(z)) = \frac{2\beta z}{(1-z)^{1+\beta}(1+z)^{1-\beta}}$$

and
$$h(z) = \theta(q(z)) + Q(z) = \left(\frac{1+z}{1-z} \right)^{2\beta} + Q(z) = h_\beta(z).$$

Since $Q = zq'$, we have $\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} > 0$, q being convex.

Thus Q is starlike univalent in E and for $z \in E$

$$Re\,\frac{zh'}{Q} = Re\left\{ z\frac{\theta'(q(z))\,q'(z)}{zq'(z)} + \frac{zQ'(z)}{Q(z)} \right\} \Rightarrow Re\,2q(z) + \frac{Re\,zQ'(z)}{Qz} > 0,$$

Since, $\left| arg\left(\frac{1+z}{1-z}\right)^{\beta} \right| = \left| \beta\, arg\left(\frac{1+z}{1-z}\right)\right| < \beta\frac{\pi}{2} \le \frac{\pi}{2}$ for $z \in E$. D is the whole complex plane, where $\theta(z) = z^2$ and $\phi(z) \equiv 1$ are analytic.

We further see that

$$zp'(z) + p^2(z) = \theta(p(z)) + zp'(z)\,\phi(p(z)) \qquad \text{... (14)}$$

and
$$h(z) = h_\beta = zq'(z) + q^2(z) = \theta(q(z)) + zq'(z)\,\phi(q(z)).$$

From (12) and (14) we find

$$\theta(p(z)) + zp'(z)\,\phi(p(z)) \prec \theta(q(z)) + zq'(z)\,\phi\, q(z)).$$

The conditions for application of Theorem A are all fulfilled and we, therefore, conclude that $\frac{zf'}{f} = p \prec q = \left(\frac{1+z}{1-z}\right)^{\beta}$ and that q is the best dominant. This proves $f \in \tilde{S}^*$. We now investigate the image $h(E)$ of the unit disk to see how our assumptions in Theorem 2 are less restrictive.

To this end we express $h(e^{i\theta}) = u + i\,v$, where u and v are real.

$$u = u(\theta) = \frac{\beta \cos\frac{\pi}{2}(1+\beta)\sin^{1+\beta}\theta\,(1-\cos\theta)^{\beta-1} + \sin^{2\beta}\theta\cos(\pi\beta)(1+\cos\theta)}{(1-\cos\theta)^{2\beta}(1+\cos\theta)} \qquad \text{... (15)}$$

and
$$v = v(\theta) = \frac{\beta \sin\frac{\pi}{2}(1+\beta)\sin^{1+\beta}\theta\,(1-\cos\theta)^{\beta-1} + \sin^{2\beta}\theta\sin(\pi\beta)(1+\cos\theta)}{(1-\cos\theta)^{2\beta}(1+\cos\theta)}$$

so that, after simplification, we obtain

$$u\cos\frac{\pi\beta}{2} + v\sin\frac{\pi\beta}{2} = \frac{\sin^{2\beta}\theta}{(1-\cos\theta)^{2\beta}}\cos\frac{\pi\beta}{2} = \left(\cot\frac{\theta}{2}\right)^{2\beta}\cot\frac{\pi\beta}{2}$$

$$u\sin\pi\beta - v\cos\pi\beta = \frac{-\beta\left(\cot\frac{\theta}{2}\right)^{1+\beta}\cos\frac{\pi\beta}{2}}{2\cos^2\left(\frac{\theta}{2}\right)} \qquad \text{... (16)}$$

From (16), we note that $u + v\tan\frac{\pi\beta}{2} > 0$, for $\beta < 1$; $v = 0$ for $\beta = 1$. We first consider the case $\beta \ge \frac{1}{2}$. From (15), for $0 < \theta < \pi$, $v > 0$, and $u < 0$ for $0 < \theta < \pi$ and $\beta \ge \frac{1}{2}$. Also from the second of eqs. (16), $u \sin \pi\beta - v \cos \pi\beta < 0$. Since $\cos \pi\beta < 0$ for $\frac{1}{2} < \beta < 1$, we note that $v < u \tan(\pi\beta)$. $v = u \tan \pi\beta$ is the equation to the straight line through the origin with negative slope for $\beta > \frac{1}{2}$ and $v < \tan \pi\beta$ implies that the curve $h(e^{i\theta})$ for $0 < \theta < \pi$ lies below this st. line in the second quadrant since $u < 0$ for $\beta \geq \frac{1}{2}$. Similarly, $u + v \tan \frac{\pi\beta}{2} > 0$ shows that the curve lies above the straight line whose equations is $v = -\cot \frac{\pi\beta}{2} = u \tan\left(\frac{\pi}{2} + \frac{\pi\beta}{2}\right)$, in the second quadrant for $0 < \theta < \pi$. In fact $v = -u \cot \frac{\pi\beta}{2}$ is an asymptote to the curve and thus the branch of the curve $h(e^{i\theta}), 0 < \theta < \pi$, lies in the angle between these straight lines in the second quadrant. The branch of the curve $h(e^{i\theta}), \pi < \theta < 2\pi$, is symmetric with respect to the real axis, in view of the fact $h(\bar{z}) = \overline{h(z)}, z \in E$, as is easily seen from (13). Thus $h(z)$ maps E onto the domain lying outside these two branches of the curve $h(e^{i\theta})$ lying in the second and third quadrants and $h(E)$ properly contains the right half plane. Next we consider the case $0 < \beta < \frac{1}{2}$. Now $Re\ h(e^{i\theta})$ assumes both positive and negative values depending on θ. For example $Re\ h(e^{i\frac{\pi}{2}}) = \mathrm{Re}\, h(i) = \cos(\pi\beta) - \beta \sin \frac{\pi\beta}{2}$ and $\beta = \frac{1}{3}, h(i) = \frac{1}{3} > 0$, whereas $h(e^{i\theta})$ goes to infinity in the second quadrant for $\theta = 0$ & $\theta = \pi$. Again in view of the fact $\cos(\pi\beta) > 0$ for $\beta < \frac{1}{2}$, from (16) we note that $v > u \tan(\pi\beta)$ and therefore the curve $h(e^{i\theta})$ for $0 < \theta < \pi$ lies above the straight line $v = u \tan(\pi\beta)$, which has now a positive slope. It also lies above the straight line $v = -\cot\left(\frac{\pi\beta}{2}\right) = u \tan\left(\frac{\pi}{2}(1+\beta)\right)$. For $\pi < \theta < 2\pi$, the branch of the curve traced by $h(e^{i\theta})$ is the reflection of the branch corresponding to $0 < \theta < \pi$ on the real axis]. Thus for $\beta < \frac{1}{2}$, h maps E onto the region lying in all the quadrants, outside the branches of $h(e^{i\theta})$. Before we conclude we prove the following result.

***Theorem* 3** — *Let $f \in A$ and satisfy $\frac{f(z)}{z} \neq 0$ and $\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} < \frac{1+z}{(1-z)^2}$ for $z \in E$. Then $Re\ \frac{zf'}{f} > \frac{1}{2}$ in E.*

PROOF : Setting as before $p = \frac{zf'(z)}{f(z)}$, the hypothesis takes the form

$$zp'(z) + p^2(z) \prec h_1(z), \qquad \dots (17)$$

where $h_1(z) = \frac{1+z}{(1-z)^2}$. Choosing $q(z) = \frac{1}{1-z}$, $\phi(z) \equiv 1$, $\theta(z) = z^2$, we proceed as in the proof of the previous theorem. $Q(z) = zq'(z)\,\phi(q(z)) = \frac{z}{(1-z)^2}$, $h(z) = \theta(q(z)) + Q(z) = q^2(z) + Q(z) = \frac{1+z}{(1-z)^2} = h_1(z)$, Q is evidently univalently starlike in E. $Re\, \frac{z h'}{Q} = Re \left\{ \frac{1+z}{1-z} + \frac{2}{1-z} \right\} > 1$ for $|z| < 1$. Applying Theorem A, we conclude $\frac{zf'(z)}{f(z)} = p \prec q = \frac{1}{1-z}$. This proves $Re\, \{zf'(z)/f(z)\} > 1/2$, $z \in E$. By Theorem A, the function $1/(1-z)$ is the best dominant for the subordination.

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