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COMMON FIXED POINT THEOREMS FOR WEAK COMPATIBLE MAPPINGS

JEONG SHEOK UME

Department of Applied Mathematics, Changwon National University,
Changwon 641 773, Korea
E-mail: jsume@sarim.changwon.ac.kr

AND

TAE HWA KIM

Department of Applied Mathematics, Pukyong National University
Pusan 608 737, Korea
E-mail: taehwa@dolphin.pknu.ac.kr

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This paper introduces a p-weakly compatible and improve Jungck's fixed point theorem, Meir and Keeler's fixed point theorem and Park and Base's fixed point theorem by using the concept of a w-distance p and a p-weakly compatible mapping.

Key Words and Phrases : w-Distance; p-Weak Compatibility

1. INTRODUCTION

Recently, Kada-Suzuki-Takahashi[3] introduced the concept of w-distance on a metric space as follows: Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied :

(1) $p(x, z) \le p(x, y) + p(y, z)$ for any $x, y, z \in X$;

(2) for any $x \in X, p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous; and

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

In [2], the concept of compatible mappings was introduced as a generalization of commuting mappings. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park and Bae[5].

In this paper, we introduce the concept of p-weakly compatible mappings and improve Jungck's fixed point theorem[1], Meir and Keeler's fixed point theorem[4] and Park and Bae's fixed point theorem[5].

PRELIMINARIES

Before proving the main theorems, we will introduce some definitions and lemmas.

Definition 2.1[6] — Let (X, d) be a metric space and let f and g be self-mappings of X. The mappings f and g will be called weakly commuting if and only if

$$d(fgx, gfx) \leq d(fx, gx)$$

for each $x \in X$.

Definition 2.2 — Let (X, d) be a metric space with a w-distance p and let f and g be self-mappings of X. The mappings f and g will be called a p-weakly commuting if and only if

$$\max\ [p\,(fgx, gx), p(gfx, fgx)] \leq d(fx, gx)$$

for each $x \in X$. Indeed, if a w-distance p is a metric, we say metric p-weakly commuting instead of p-weakly commuting. Clearly, metric p-weakly commuting implies a p-weakly commuting, but the converse is false (see Example 2).

Definition 2.3[2] — Let (X, d) be a metric space and $f, g : X \rightarrow X$. The mappings f and g will be called compatible mappings if and only if for every sequence $\{x_n\}$ such that $\lim_{n \to \infty} f x_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$, it implies

$$\lim_{n \to \infty} d(fgx_n, gfx_n) = 0.$$

Definition 2.4 — Let (X, d) be a metric space with a w-distance p, and $f, g : X \rightarrow X$. The mappings f and g will be called a p-weakly compatible mappings if and only if for every sequence $\{x_n\}$ such that $\lim fx_n = \lim gx_n = t$ for some $t \in X$ it implies

$$\lim_{n \to \infty} \max\ [p(fgx_n, g fx_n), p(g f x_n, f gx_n)] = 0.$$

Indeed, if a w-distance p is a metric, we say metric p-weakly compatible instead of p-weakly compatible. Clearly, metric p-weakly compatible mappings are p-weakly compatible, but the converse is false (see Example 1).

Lemma 2.5[3] — Let X be a metric space with metric d and p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold :

(i) if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;

(ii) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to x;

(iii) if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence; and

(iv) if $p(y, x_n) \leq \alpha_n$ for any $n \in N$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.6 — Let (X, d) be a metric space with a w-distance p and let f and g be self-mappings of X and $\left\{x_n\right\}_{n=0}^{\infty}$ be a sequence in X satisfying $f x_n = gx_{n+1}$ for n = 0, 1, 2, ... Assume that there exists a continuous self mapping γ of $[0, \infty)$ such that

$$p(fx, fy) \leq Y(p(gx, gy)) \quad \text{... (2.1)}$$

for all $x, y \in X$, and

$$\gamma(t) < t \text{ for each } t > 0. \quad \text{... (2.2)}$$

Then

(I) For an arbitrary $\varepsilon > 0$, there exists a positive integer M such that $M \leq n < S$ implies $p(fx_n, fx_s) < \varepsilon$.

(II) The sequence $\{fx_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

PROOF : From hypothesis, we have

$$p(fx_n, fx_{n+1}) \leq \gamma(p(gx_n, gx_{n+1})) = \gamma(p(fx_{n-1}, fx_n)) < p(fx_{n-1}, fx_n)$$

for $n = 1, 2, \ldots$. Thus $\{p(fx_n, fx_{n+1})\}$ is a decreasing sequence of nonnegative real numbers and there exists nonnegative real numbers L such that $\lim_{n \to \infty} p(fx_n, fx_{n+1}) = L$. We claim that $L = 0$. For, if $L > 0$, then the inequality

$$p(fx_n, fx_{n+1}) \leq \gamma(p(fx_{n-1}, fx_n))$$

on passing the limit as $n \to \infty$ and in view of continuity of the function γ yields $L \leq \gamma(L) < L$, which is a contradiction. Therefore, we have $L = 0$ and so $p(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$. To prove (I), suppose that (I) does not hold. Then, there exists an $\varepsilon > 0$ such that for all sufficiently large positive integers k, there exist positive integers s_k, n_k with $k \leq n_k < s_k$ such that

$$\varepsilon \leq p(fx_{n_k}, fx_{s_k}) \text{ and } p(fx_{n_k}, fx_{n_k - 1}) < \varepsilon. \quad \text{... (2.3)}$$

From the above result and (2.3), we have

$$p(fx_{n_k}, fx_{s_k}) \to \varepsilon \text{ and } p(fx_{n_k}, fx_{n_k - 1}) \to 0 \text{as } k \to \infty \text{ and}$$

$$p(fx_{n_k}, fx_{s_k}) \leq p(fx_{n_k}, fx_{n_k + 1}) + p(fx_{n_k + 1}, fx_{s_k})$$

$$\leq p(fx_{n_k}, fx_{n_k + 1}) + \gamma(p(gx_{n_k + 1}), gx_{s_k})) = p(fx_{n_k}, fx_{n_k + 1}) + \gamma(p(fx_{n_k}, fx_{n_k - 1})) \ldots \quad \text{(2.4)}$$

By hypothesis and (2.4), we obtain $\varepsilon \leq \gamma(\varepsilon) < \varepsilon$. This is a contradiction. Therefore (I) holds.

(II) From (3) of definition of a w-distance p and (I), we have that $\{fx_n\}$ is a Cauchy sequence.

Lemma 2.7 — Let (X, d) be a complete metric space with a w-distance p, and let f and g be selfmappings of X and $\{x_n\}$ be a sequence in X satisfying $fx_n = gx_{n+1}$ for $n = 0, 1, 2, \ldots$, and the following conditions : for given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq p(gx, gy) < \varepsilon + \delta \Rightarrow p(fx, fy) < \varepsilon, \quad \text{... (2.5)}$$

and $$p(gx, gy) < \varepsilon \;\Rightarrow\; p(fx, fy) \leq \frac{1}{2} p(gx, gy). \qquad \text{... (2.6)}$$

Then

(III) For an arbitray $\varepsilon > 0$, there exists a positive integer M such that $M \leq n < s$ implies $p(fx_n, fx_s) < \varepsilon$.

(IV) The sequence $\{fx_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

PROOF : By (2.5) and (2.6), we have

$$p(fx, fy) \leq p(gx, gy) \text{ for all } x, y \in X.$$

Thus $\{p(fx_n, fx_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Then there is a nonnegative real number r such that $\lim_{n \to \infty} p(fx_n, fx_{n+1}) = r$. We show that $r = 0$. Suppose that $r > 0$. Then, for given $\delta > 0$, there exists a positive integer M such that for each $m \geq M$, we have

$$r \leq p(fx_m, fx_{m+1}) = p(gx_{m+1}, gx_{m+2}) < r + \delta. \qquad \text{... (2.7)}$$

From (2.5) and (2.7), we obtain $p(fx_{m+1}, fx_{m+2}) < r$, which contradicts (2.7). Therefore we have $\lim_{n \to \infty} p(fx_n, fx_{n+1}) = \lim_{n \to \infty} p(gx_{n+1}, gx_{n+2}) = 0$. Similarly we obtain $\lim_{n \to \infty} p(fx_{n+1}, fx_n) = \lim_{n \to \infty} p(gx_{n+2}, gx_{n+1}) = 0$. To prove (III), suppose that (III) does not hold. Then, there exists an $\varepsilon > 0$ such that for all sufficiently large positive integers k, there exist positive integers s_k, n_k with $k \leq n_k > s_k$ satisfying (2.3). From above results and (2.3), we have

$$p(fx_{n_k}, fx_{s_k}) \to \varepsilon, p(fx_{n_k}, fx_{s_k - 1}) \to \varepsilon \text{ and}$$

$$p(fx_{n_k - 1}, fx_{s_k}) \to \varepsilon \text{ as } k \to \infty. \qquad \text{... (2.8)}$$

From (2.8), for $\varepsilon > 0$, there exists a positive integer M such that for each $k \geq M$, we have

$$\frac{\varepsilon}{2} < p(fx_{n_k} \cdot fx_{s_k - 1}) = p(gx_{n_k + 1} + gx_{s_k}) < \frac{\varepsilon}{2} + \varepsilon.$$

From (2.5) and (2.9) we obtain

$$p(fx_{n_k + 1}, fx_{s_k}) < \frac{\varepsilon}{2},$$

which contradicts (2.8). Therefore (III) holds. The proof of (IV) is the same as (II) of Lemma 2.6.

3. MAIN RESULTS

Theorem 3.1 — *Let (X, d) be a complete metric space with a w-distance p, and let f and g be a p-weakly compatible selfmappings of X satisfying $f(X) \subset g(X)$, (2.1), (2.2) and for each $z \in X$ with*

$z \neq fz$ or $z \neq gz$,

$$\inf \{p(fx, z) + p(gx, z) + p(fgx, gfx) + p(gfx, fgx : x \in X\} > 0. \quad \text{... (3.1)}$$

Then f and g have a unique common fixed point.

PROOF : By hypotheses, we can get all conditions of Lemma 2.6. Thus, by (II) of Lemma 2.6, it follows that $\{fx_n\}$ is a Cauchy sequence. Since X is a complete metric space and $fx_n = gx_{n+1}$, $\{fx_n\}$ and $\{gx_n\}$ have a limit point z in X. Suppose that $z \neq fz$ or $z \neq gz$. Since

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z,$$

by (I) of Lemma 2.6 and (2) of definition of a w-distance p, we obtain

$$\lim_{n \to \infty} p(fx_n, z) = \lim_{n \to \infty} p(gx_n, z) = 0.$$

Since the pair of mappings f and g is p-weakly compatible, we have

$$0 < \inf \{p(fx, z) + p(gx, z) + p(fgx, gfx) + p(gfx, fgx) : x \in X\}$$

$$\leq \inf \{p(fx_n, z) + p(gx_n, z) + p(fgx_n, gfx_n) + p(gfx_n, fgx_n) : n \in N\} = 0.$$

This is contradiction. Thus z is a common fixed point of f and g. The uniqueness of the common fixed point is clear by (2.1), (2.2) and (i) of Lemma 2.5.

Since p-weakly commuting implies a p-weakly compatible, from Theorem 3.1 we obtain the following corollary.

Corollary 3.2 — Let (X, d) be a complete metric space with a w-distance p, and let f and g be a p-weakly commuting selfmappings of X satisfying $f(X) \subset g(X)$, (2.1), (2.2) and (3.1). Then f and g have a unique common fixed point.

Since a metric d is a w-distance, from Theorem 3.1 and simple calculation, we obtain the following corollary.

Corollary 3.3[1] — A continuous self map of a complete metric space (X, d) has a fixed point iff there exists $r \in (0, 1)$ and a mapping $g : X \to X$ which commutes with $f(gf = fg)$ and satisfies : $g(X) \subset f(X)$ and $d(gx, gy) \leq r d(fx, fy)$ for $x, y \in X$. In fact, f and g have a unique common fixed point.

Theorem 3.4 — *Let (X, d) be a complete metric space with a w-distance p, and let f and g be a p-weakly compatible selfmappings of X satisfying $f(X) \subset g(X)$, (2.5), (2.6) and (3.1). Then f and g have a unique common fixed point.*

PROOF : Since $f(X) \subset g(X)$, we obtain a sequence $\{x_n\}$ in X such that $fx_n = gx_{n+1}$. Thus, by Lemma 2.7 it follows that $\{fx_n\}$ is a Cauchy sequence. Since X is complete and $fx_n = gx_{n+1}$, there exists z in X such that $fx_n \to z$ and $gx_n \to z$. Suppose that $z \neq fz$ or $z \neq gz$. Since

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z,$$

by (I) of Lemma 2.6 and (2) of definition of a w-distance p, we obtain

$$\lim_{n \to \infty} p(fx_n, z) = \lim_{n \to \infty} p(gx_n, z) = 0.$$

Since the pair of mappings f and g is p-weakly compatible, we have

$$0 < \inf \{p(fx, z) + p(gx, z) + p(fgx, gfx) + p(gfx, fgx) : x \in X\}$$

$$\leq \inf \{p(fx_n, z) + p(gx_n, z) + p(fgx_n, gfx_n) + p(gfx_n, fgx_n) : n \in N\} = 0.$$

This is contradiction. Thus z is a common fixed point of f and g. The uniqueness of the common fixed point is clear by (2.5), (2.6) and (i) of Lemma 2.5.

Since p-weakly commuting implies a p-weakly compatible, from Theorem 3.4 we obtain the following corollary.

Corollary 3.5 — Let (X, d) be a complete metric space with a w-distance p, and let f and g be a p-weakly commuting selfmappings of X satisfying $f(X) \subset g(X)$, (2.5), (2.6) and (3.1). Then f and g have a unique common fixed point.

Since a metric is a w-distance, from Theorem 3.4 and simple calculation, we obtain the following corollary. This corollary is Park and Bae's fixed point theorem[5] which generalizes a theorem by Meir and Keeler[4].

Corollary 3.6 — If f is a continuous self map of a complete metric space X and g is an (ε, δ)-f-contraction which commutes with f, then f and g have a unique common fixed point in X.

The following examples show that there exist a pair of mappings f and g which is a p-weakly compatible, but which is neither p-weakly commuting nor metric p-weakly compatible mapping and that satisfies all conditions in Theorem 3.1 and Theorem 3.4 and hence have a unique common fixed point.

Example 1 — Let R be the set of all real numbers with the usual metric. Define self-mappings f and g on R by

$$fx = |x| \text{ for all } x \in R$$

and

$$gx = -3x \text{ for all } x \in R.$$

A function $p : R \times R \to [0, \infty)$ defined by

$$p(x, y) = \max \{|3x - y|, 3|x - y|\} \text{ for all } x, y \in R$$

is a w-distance but not metric on R and Definition 2.2 is not satisfied for all $x \in (0, \infty)$. Then for every sequence $\{x_n\}$ in R such that $\lim_{n \to \infty} fx_n = \lim gx_n = 0$, we have

$$\lim_{n \to \infty} \max \{p(fgx_n, gfx_n), p(gfx_n, fgx_n)\} = 0.$$

Thus the pair of mappings f and g is p-weakly compatible. Since

$$p(fx, fy) \leq \frac{1}{3} p(gx, gy) \text{ for all } x, y \in R,$$

f and g satisfy all the conditions of Theorem 3.1 and Theorem 3.4 and hence have the unique common fixed point $x = 0$.

Example 2 — Let $R\_$ be the set of all negative real numbers with the usual metric. Define self mappings f and g on $R\_$ by

$$fx = -|x| \text{ for all } x \in R\_.$$

and

$$gx = \frac{1}{5}x \text{ for all } x \in R\_.$$

Let $p : R \times R \to [0, \infty)$ be a mapping such that

$$p(x, y) = \max\{|3x - y|, 3|x - y|\} \text{ for all } x, y \in R\_.$$

Then

(i) p is a w-distance but it is not a metric on $R\_$ and $p(-1, -2) \neq p(-2, -1)$.

(ii) For all $x \in R\_$,

$$\max\{p(fgx, gfx), p(gfx, fgx)\} \leq d(fx, gx).$$

This means that the pair of mappings f and g is p-weakly commuting but p is not a metric.

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