Indian J. pure appl. Math., 32(4) : 573-580, April 2001

GENERALISED CR-SUBMAMIFOLDS OF A *TRANS*-SASAKIAN MANIFOLD

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(*Received* 4 *April* 2000; *accepted* 17 *October* 2000)

The purpose of the present paper is to study the generalised CR-submanifolds of a *trans*-Sasakian manifold and obtain their basic properties. The condition of integrability of the distributions of generalised CR-submanifolds are obtained.

Key Words and Phrases : Generalised CR-Submanifold; *Trans*-Sasakian Manifold

1. INTRODUCTION

Bejancu[1 & 2] defined and studied CR-submanifolds of Kaehlerian manifolds. CR-submanifolds of Sasakian manifold were studied by Kobayashi[6] and Hasan Shahid[10]. Chen[5] introduced the notion of a generic submanifold of a Kaehler manifold. Generic submanifolds of Sasakian manifolds were studied by Hasan Shahid[11] and Verheyen[15]. Benjancu and Papaghiuc[3] defined almost semi-invariant submanifold of Sasakian manifold. In 1985, Oubina[9] studied a new class of almost contant Riemannian manifold known as *Trans*-Sasakian manifold which generalises both α-Sasakian and β-Kenmotsu structure. Hasan Shahid studied CR-submanifolds[12], generic submanifolds[13] and almost semi-invariant manifolds[14] of *trans*-Sasakian manifold. Ion Mihai[7] introduced a new class of submanifolds called "Generalised CR-submanifolds" of a Kaehler manifold. This class contains both CR-submanifolds and slant submanifolds. Mihai[8] also studied generalised CR-submanifold of a Sasakian manifold.

The purpose of the present paper is to study the generalised CR-submanifold of a trans-Sasakian manifold.

2. PRELIMINARIES

Let $\overline{M}$ be a $(2n + 1)$ - dimensional almost contact metric manifold with (φ, ξ, η, g) as the almost contact metric structure, where φ is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\overline{M}$.

$$\varphi^2 = -I + \eta \otimes \xi;\ \varphi(\xi) = 0;\ \eta(\xi) = 1;\ \eta\, 0\, \varphi = 0 \qquad \text{... (2.1)}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y) \qquad \text{... (2.2)}$$

for vector fields X, Y tangent to $\overline{M}$.

An almost contact metric structure (φ, ξ, η, g) on $\overline{M}$ is called *trans*-Sasakian[4] if and only if

$$(\overline{\nabla} \times \varphi)\, Y = \alpha\ \{g(X, Y)\, \xi - \eta\,(Y)X\} \ + \ \beta\ \{g(\varphi X, Y)\, \xi - \eta\,(Y)\, \varphi X\} \qquad \text{... (2.3)}$$

for all X, Y tangent to $\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection with respect to g.

From (2.3) it follows that

$$\overline{\nabla} \times \xi = -\,\alpha\,\varphi X + \beta\,(X - \eta\,(X)\,\xi) \qquad \text{... (2.4)}$$

for any vector field X tangent to $\overline{M}$.

Now let M be an m-dimensional submanifold isometrically immersed in a *trans*-Sasakian manifold $\overline{M}$ such that the structure vector field ξ of $\overline{M}$ is tangent to the submanifold M. We denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and by $\{\xi\}^{\perp}$ the complementary orthogonal distribution to $\{\xi\}$ in $T(M)$.

For any $X \in T(M)$ we have $g(\varphi X, \xi) = 0$. Then we put

$$\varphi X = bX + cX, \qquad \text{... (2.5)}$$

where $bX \in \{\xi\}^{\perp}$ and $cX \in T^{\perp}(M)$. Thus $X \rightarrow bX$ is an endomorphism of the tangent bundle $T(M)$ and $X \rightarrow cX$ is a normal bundle valued 1-form on M.

Definition — A submanifold M of an almost contact metric manifold $\overline{M}$ with almost contact metric structure (φ, ξ, η, g) is said to be a generalised CR-submanifold if

$$D_x^{\perp} = T_x(M) \cap \varphi T_x^{\perp}(M);\ x \in M$$

defines a differentiable subbundle of $T_x(M)$. Thus for $X \in D^{\perp}$ one has $bX = 0$.

We denote by D the complementary orthogonal subbundle to $D^{\perp} \oplus \{\xi\}$ in $T(M)$. For any $X \in D$, $bX \neq 0$. Also we have $bD = D$.

Thus for a generalised CR-submanifold M we have the orthogonal decomposition

$$\mathrm{T}(M) = D \oplus D^{\perp} \oplus \{\xi\}. \qquad \text{... (2.6)}$$

3. Basic Lemmas

Let M be a generalised CR-submanifold of the *trans*-Sasakian manifold $\overline{M}$. We denote by g both the Riemannian metrics on $\overline{M}$ and M.

For each $X \in T(M)$ we put

$$X = PX + QX + \eta\,(X)\,\xi, \qquad \text{... (3.1)}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.

For any $N \in T_x^{\perp}(M)$ we put

$$\varphi N = tN + fN, \qquad \text{... (3.2)}$$

where tN is the tangential part of φN and fN is the normal part of φN.

By using (2.2) we have

$$g(\varphi X, cY) = g(\varphi X, bY + cY) = g(\varphi X, \varphi Y) = g(X, Y) = 0, \text{ for } X \in D_x^{\perp} \text{ and } Y \in D_x$$

Therefore, $$g(\varphi D_x^{\perp}, cD_x) = 0. \qquad \text{... (3.3)}$$

We denote by ν the orthogonal complementary vector bundle to $\varphi D^{\perp} \oplus cD$ in $T^{\perp}(M)$. Thus we have

$$T^{\perp}(M) = \varphi D^{\perp} \oplus cD \oplus \nu \qquad \text{... (3.4)}$$

Lemma 3.1 — The morphism t and f satisfy

$$t(\varphi D^{\perp}) = D^{\perp} \qquad \text{... (3.5)}$$

and $$t(cD) \subset D. \qquad \text{... (3.6)}$$

PROOF : For $X \in D^{\perp}$ and $Y \in D$,

$$g(t\varphi X, Y) = g(t\varphi X + f\varphi X, Y) = g(\varphi^2 X, Y) = -g(X, Y) = 0.$$

Also $$g(t\varphi X, \xi) = g(\varphi^2 X, \xi) = -g(\varphi X, \varphi \xi) = 0.$$

Therefore $$t(\varphi D^{\perp}) \subset D^{\perp}.$$

For $X \in D^{\perp}$, we have

$$-X = \varphi^2 X = t\varphi X + f\varphi X \text{ which implies } -X = t\varphi X.$$

Consequently, $D^{\perp} \subset t(\varphi D^{\perp})$. Hence the relation (3.5) follows. The relation (3.6) is trivial.

Now we denote by $\overline{\nabla}$ (resp. ∇) the Riemannian connection on $\overline{M}$ (resp. M) with respect to the Riemannian metric g. The linear connection induced by $\overline{\nabla}$ on the normal bundle $T^{\perp}(M)$ is denoted by $\nabla^{\perp}$. Then the equations of Gauss and Weingarten are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \text{... (3.7)}$$

and $$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \qquad \text{... (3.8)}$$

for $X, Y \in T(M)$ and $N \in T^{\perp}(M)$, where h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. These tensor fields are related by

$$g(h(X, Y), N) = g(A_N X, Y) \qquad \text{... (3.9)}$$

for $$X, Y \in T(M) \text{ and } N \in T^{\perp}(M).$$

We denote

$$u(X, Y) = \nabla_X bPY - A_{cPY} X - A_{\varphi QY} X.$$

Lemma 3.2 — Let M be a generalised CR-submanifold of the *trans*-Sasakian manifold $\overline{M}$. Then we have

$$P(u(X, Y)) - bP\nabla_X Y - Pth(X, Y) = -\alpha\eta(Y)PX - \beta\eta(Y)PbX, \quad \text{... (3.10)}$$

$$Q(u(X, Y)) - Qth(X, Y) = -\alpha\eta(Y)QX - \beta\eta(Y)QbX, \quad \text{... (3.11)}$$

$$\eta(u(X, Y)) = \alpha g(\varphi X, \varphi Y) + \beta g(\varphi bX, \varphi Y) \quad \text{... (3.12)}$$

and

$$h(X, bPY) + \nabla_X^{\perp} cPY + \nabla_X^{\perp} \varphi QY - cP\nabla_X Y - \varphi Q\nabla_X Y - fh(X, Y)$$

$$= -\beta\eta(Y)cX, \quad \text{... (3.13)}$$

for $X, Y \in T(M)$.

PROOF : For $X, Y \in T(M)$ by using (2.5), (3.1), (3.2) and (3.7), (3.8) in (2.3), we have

$$\nabla_X bPY + h(X, bPY) - A_{cPY}X + \nabla_X^{\perp} cPY - A_{\varphi QY}X + \nabla_X^{\perp} \varphi QY - bP\nabla_X Y$$

$$- cP\nabla_X Y - \varphi Q\nabla_X Y - Pth(X, Y) - Qth(X, Y) - fh(X, Y)$$

$$= \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$$

Then (3.10), (3.11), (3.12) and (3.13) follow by taking components on each of the vector bundles D, $D^{\perp}$, $\{\xi\}$ and $T^{\perp}(M)$ respectively.

Lemma 3.3 — Let M be a generalised CR-submanifold of the *trans*-Sasakian manifold $\overline{M}$. Then we have

$$P(t\nabla_X^{\perp} N + A_{fN}X - \nabla_X tN) = bPA_N X, \quad \text{... (3.14)}$$

$$Q(t\nabla_X^{\perp} N + A_{fN}X - \nabla_X tN) = 0, \quad \text{... (3.15)}$$

$$\eta(A_{fN}X - \nabla_X tN) = -\beta g(cX, N) \quad \text{... (3.16)}$$

and

$$h(X, tN) + \varphi QA_N X + \nabla_X^{\perp} fN + cPA_N X = f\nabla_X^{\perp} N \quad \text{... (3.17)}$$

for $X \in T(M)$ and $N \in T^{\perp}(M)$.

PROOF : For $X \in T(M)$ and $N \in T^{\perp}(M)$ by using (2.5), (3.1), (3.2) and (3.7), (3.8) in (2.3) we get

$$P\nabla_X tN + Q\nabla_X tN + \eta(\nabla_X tN)\xi + h(X, tN) - PA_{fN}X - QA_{fN}X - \eta(A_{fN}X)\xi$$

$$+ \nabla_X^{\perp} fN + bPA_N X + cPA_N X + \varphi QA_N X - Pt\nabla_X^{\perp} N - Qt\nabla_X^{\perp} N$$

$$- f\nabla_X^{\perp} N = \beta g(cX, N)\xi$$

Then (3.14), (3.15), (3.16) and (3.17) follow by taking components on each of the vector bundles D, $D^{\perp}$, $\{\xi\}$ and $T^{\perp}(M)$ respectively.

Lemma 3.4 — Let M be a generalised CR-submanifold of the *trans*-Sasakian manifold $\overline{M}$. Then we have

$$\nabla_X \xi = -\alpha bX + \beta X;\ h(X, \xi) = -\alpha cX, \text{ for } X \in D \qquad \ldots (3.18)$$

$$\nabla_Y \xi = \beta Y;\ h(Y, \xi) = -\alpha \varphi Y, \text{ for } Y \in D^{\perp} \qquad \ldots (3.19)$$

$$\nabla_\xi \xi = 0;\ h(\xi, \xi) = 0 \qquad \ldots (3.20)$$

PROOF : The lemma follows from (2.4) by using (2.5), (3.1) and (3.7).

Lemma 3.5 — Let M be a generalised CR-manifold of the *trans*-Sasakian manifold $\overline{M}$. Then we have

$$A_{\varphi X} Y = A_{\varphi Y} X \qquad \ldots (3.21)$$

for $X, Y \in D^{\perp}$.

PROOF : By using (2.2), (2.3), (3.7) and (3.9) we get

$$g(A_{\varphi X} Y, Z) = g(h, Y, Z), \varphi X) = g(\overline{\nabla}_X Y, \varphi X) = -g(\varphi \overline{\nabla}_z Y, X)$$

$$= -g(\overline{\nabla}_{z\varphi} Y, X) = g(\varphi Y, \overline{\nabla}_Z X) = g(h(Z, X), \varphi Y) = g(h(X, Z), \varphi Y)$$

$$= g(A_{\varphi Y} X, Z),$$

for $X, Y \in D^{\perp}$ and $Z \in T(M)$.

Hence the Lemma follows.

Lemma 3.6 — Let M be a generalised CR-submanifold of the *trans*-Sasakian manifold $\overline{M}$. Then we have

$$\nabla_\xi V \in D^{\perp}, \text{ for } V \in D^{\perp} \qquad \ldots (3.22)$$

and

$$\nabla_\xi W \in D, \text{ for } W \in D. \qquad \ldots (3.23)$$

PROOF : Let us take $X = \xi$ and $V = \varphi N$ in (3.14) where $N \in \varphi D$. Taking account that $tN = \varphi N, fN = 0$ we get

$$P \nabla_\xi V = Pt \nabla_\xi^{\perp} N - bPA_N \xi. \qquad \ldots (3.24)$$

The second relation of (3.18) gives

$$g(PN_N \xi, W) = g(A_N \xi, W) = g(h(W, \xi), N) = -\alpha g(cW, N) = 0$$

for $W \in D$.

Hence, (3.24) becomes

$$P \nabla_\xi V = Pt \nabla_\xi^{\perp} N. \qquad \ldots (3.25)$$

On the other hand (3.17) implies

$$h(\xi, V) = f\nabla_\xi^\perp N - \varphi QA_N \xi. \quad \ldots (3.26)$$

For $V \in D^\perp$, $h(\xi, V) = h(V, \xi) = -\alpha\varphi V \in \varphi D^\perp$, by (3.19).

Now for $X \in D^\perp$ by virtue of Lemma 3.5 and of (3.9) we have

$$g(h(\xi, V), \varphi X) = g(h(V, \xi), \varphi X) = g(A_{\varphi X} V, \xi) = g(A_{\varphi V} X, \xi)$$

$$= g(h(X, \xi), \varphi V) = g(h(X, \xi), -N) = -g(A_N \xi, X) = -g(\varphi A_N \xi, \varphi X)$$

$$= -g(\varphi PA_N \xi, \varphi X) - g(\varphi QA_N \xi, \varphi X) = -g(\varphi QA_N \xi, \varphi X), \text{ since } cD^\perp \; \varphi\Delta\perp.$$

Therefore, $h(\xi, V) = -\varphi QA_N \xi$, which together with (3.26) implies $f\nabla_\xi^\perp N = 0$.

Hence $\nabla_\xi^\perp N \;\; \varphi D^\perp$, since f is an automorphism of $cD \oplus \nu$. Thus, $t\nabla_\xi^\perp N \in D^\perp$ and from (3.25) it follows that

$$P\nabla_\xi V = 0, \text{ for all } V \in D^\perp. \quad \ldots (3.27)$$

Next from (3.16) we have

$$\eta(\nabla_\xi V) = 0 \quad \ldots (3.28)$$

for all $V = \varphi N \in D^\perp$, where $N \in \varphi D^\perp$.

Hence (3.22) follows from (3.27) and (3.28).

Finally by using (3.1), (3.20) and (3.22), we have

$$g(\nabla_\xi W, X) = g(\nabla_\xi W, PX), \text{ for } X \in T(M) \text{ and } W \in D.$$

Thus we have $\nabla_\xi W \in D$, for $W \in D$ and this completes the proof.

Corollary 3.1 — Let M be a generalised CR-submanifold of the trans-Sasakian manifold $\overline{M}$. Then we have

$$[Y, \xi] \in D^\perp, \text{ for } Y \in D^\perp \quad \ldots (3.29)$$

and

$$[X, \xi] \in D, \text{ for } X \in D. \quad \ldots (3.30)$$

The above corollary follows immediately from the Lemma 3.4 and 3.6.

4. Integrability of Distributions

Theorem 4.1 — *Let M be a generalised CR-submanifold of the trans-Sasakian manifold $\overline{M}$. Then the distribution $D^\perp$ is always involutive.*

Proof : For $X, Y \in D^\perp$ by using (3.19) we get

$$g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) = g(X, \nabla_Y \xi) - g(Y, \nabla_X \xi) = 0. \quad \ldots (4.1)$$

On the other hand, from (3.10), we have

$$bP\nabla_X Y = -PA_{\varphi Y}X - Pth(X, Y), \text{ for } X, Y \in D^{\perp} \quad \text{... (4.2)}$$

and then by using Lemma 3.5 we get from (4.2)

$$bP[X, Y] = 0, \text{ for } X, Y \in D^{\perp}. \quad \text{... (4.3)}$$

As b is an automorphism of D, the theorem follows from (4.1) and (4.3).

Theorem 4.2 — *Let M be a generalised CR-submanifold of the trans-Sasakian manifold $\overline{M}$. Then the distribution D is never involutive.*

PROOF : For $X, Y \in D$ by using (3.18), we have

$$g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) = g(X, \nabla_Y \xi) - g(Y, \nabla_X \xi)$$

$$= -\alpha g(X, bY) + \alpha g(Y, bX) = 2\alpha g(Y, bX). \quad \text{... (4.4)}$$

Taking $X \neq 0$ and $Y = bX$ in (4.4), it follows that D is not involutive.

Next we have the following theorem :-

Theorem 4.3 — *Let M be a generalised CR-submanifold of the trans-Sasakian manifold $\overline{M}$. Then the distribution $D \oplus \{\xi\}$ is involutive if and only if*

$$h(bX, Y) - h(X, bY) + \nabla_Y^{\perp} cX - \nabla_X^{\perp} cY \in cD \oplus \nu. \quad \text{... (4.5)}$$

PROOF : Applying φ to (3.13) and then taking component in $D^{\perp}$ we have

$$Q\nabla_X Y = -Qt(h(X, bY) + \nabla_X^{\perp} cPY - fh(X, Y)),$$

for $\quad X, Y \in D$

and thus

$$Q[X, Y] = Qt(h(Y, bX) - h(X, bY) + \nabla_Y^{\perp} cX - \nabla_x^{\perp} cY) \quad \text{... (4.6)}$$

for $\quad X, Y \in D.$

Hence, the theorem follows from (4.6) and (3.30).

5. Geometry of Leaves

Theorem 5.1 — *Let M be a generalised CR-submanifold of the trans-Sasakian manifold $\overline{M}$. Then the leaves of distribution $D^{\perp}$ are totally geodesic in M if and only if*

$$h(X, bZ) + \nabla_X^{\perp} cZ \in cD \oplus \nu \quad \text{... (5.1)}$$

for $\quad X \in D^{\perp}$ and $Z \in D \oplus \{\xi\}.$

PROOF : For $X\, Y \in D^{\perp}$ and $Z \in D \oplus \{\xi\}$ by using (2.2) (2.3), (3.7) and (3.8) we get

$$
\begin{aligned}
g(\nabla_X Y, Z) &= -g(Y, \overline{\nabla}_X Z) = -g(\overline{\nabla}_X Z, Y) = -g(\varphi\, \overline{\nabla}_X Z, \varphi Y) \\
&= g((\overline{\nabla}_{X\, \phi}) Z, \varphi Y) - g(\overline{\nabla}_{X\, \phi}\, Z, \varphi Y) = -g(\overline{\nabla}_X bZ + \overline{\nabla}_X bZ + \overline{\nabla}_{Xc} Z, \varphi Y) \\
&= -g(\nabla_X bZ + h(X, bZ) - A_{CZ} X + \nabla_X^{\perp}\, cZ, \varphi Y) \\
&= -g(h(X, bZ) + \nabla_X^{\perp}\, cZ, \varphi Y). \qquad \text{... (5.2)}
\end{aligned}
$$

Hence the theorem follows from (5.2).

Theorem 5.2 — *Let M be a generalised CR-submanifold of the trans-Sasakian manifold* $\overline{M}$. *Then the distribution* $D \oplus \{\xi\}$ *is involutive and its leaves are totally geodesic in M if and only if*

$$h(X, bY) + \nabla_X^{\perp}\, cY \in cD \oplus \nu \qquad \text{... (5.3)}$$

for $\qquad X, Y \in D \oplus \{\xi\}$.

PROOF : For $X, Y \in D \oplus \{\xi\}$ and $Z \in D^{\perp}$ by using (2.2) (2.3), (2.5), (3.7) and (3.8) we get

$$
\begin{aligned}
g(\nabla_X Y, Z) &= g(\overline{\nabla}_X Y, Z) = g(\varphi\, \overline{\nabla}_X Y, \varphi Z) = g(\overline{\nabla}_{X\phi}\, Y, \varphi Z) \\
&= g(\nabla_X bY + h(X, bY) - A_{cY} X + \nabla_X^{\perp}\, cY, \varphi Z). \qquad \text{... (5.4)}
\end{aligned}
$$

Hence, the theorem follows from (5.4).

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