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A MODEL FOR A FORESTED GRASSLAND : EFFECT OF OVERGRAZING AND POLLUTION

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In this paper a mathematical model is proposed to study the effects of overgrazing and air pollution on a forested grassland. The effect of diffusion is also incorporated into the model. A model for the conservation of the forested grassland is also proposed and analysed. It is shown that uncontrolled grazing and increasing pollution may lead to eventual extinction of the forested grassland but it can be conserved by using an appropriate effort.

Key words : Forested Grassland; Overgrazing; Pollution; Diffusion; Industrialization; Ecosystem

1. INTRODUCTION

It is well known that uncontrolled pressure of industrialization and associated growth of pollution is causing degradation of forestry biomass at an alarming rate in both industrial and developing countries[1 & 2] leading to undesirable economic, environmental and ecological consequences. Pollutants of different forms affect organisms in several ways. One such example is where the pollutants even at moderate concentrations in the environment may damage the biomass and even destabilize the tropic levels. Similar situations may arise when a plant ecosystem is constantly exposed to air pollutants. Another situation occurs when animal populations are exposed to pollutants for a considerable duration and consequently their physiological systems are affected leading to various kinds of diseases and internal damage to vital organs. A side effect of air pollution is acid rain, which is now of common occurrence. Records high carbon emissions cause among and acid rain and are linked to global warming and climate change and a conservative estimate is that 27,000 unique plant and animal species vanish every year. Acid rain causes direct damage to the leaves of plants. Forests in many parts of the industrialized world are drying because of acid rain.

The damage caused by migrating ungulate populations in forested grassland due to overgrazing, particularly during the peirod of scant rainfall in the region, is of equal concern because it not only depletes the biomass but also creates soil erosion problems in the region. This phenomenon is common in countries such as India where cattle populations are let loose in a given region during the period of drought and migrate into the forested grassland for grazing where there is only a limited effort to control this.

Though some effort (Singh and Saxena)[3] has been made to co-relate the grass biomass with grazing animal populations, very little attention has been given to study this problem systematically using mathematical models (Noy Meir)[4-6]. Noy Meir was probably the first who proposed a mathematical model for grazing of grasslands in terms of grass biomass growth and animal consumption rates which was assumed to be constant. Agarwal, Shukla and Pal[7] in their model took into the account the effect of migrating cattle population from the outside into grasslands in terms of grass biomass growth and animal consumption rates no effects of environmental pollution on the grass biomass growth and animal consumption rates has been taken.

In view of the above, a mathematical model for the degradation of forested grassland due to overgrazing by cattle in the grassland as well as migration from the plains and due to pollutants is proposed and analysed using Liapunov's stability theory to predict the level of depletion of biomass. A model for the conservation of grassland biomass by using an appropriate effort is also proposed and analysed.

2. Mathematical Model

Let us consider a forested grassland in a simple closed region D. For simplicity, we may wish upon occasion to think of D as a one dimensional linear habitat given by $0 \le x \le L$. The biomass of the forested grassland is polluted due to acid rain (H_2SO_4) falling on the grassland from the atmosphere and is also being depleting due to grazing by an overpopulation of ungulates in the grassland and by cattle migrating from outside into the grassland. It is assumed that the dynamics of grass biomass follows logistic growth and its carrying capacity is dependent on pollutant concentration. It is further assumed that the growth rate of grass biomass is dependent on pollutant concentration and the density of cattle population decreases as both of them increase. The growth rate of density of the cattle population is dependent on pollutant concentration and grass biomass and decreases as pollutant concentration increases, but increases as the density of grass biomass increases. The dynamics of the system is assumed to be governed by the following system of differential equations :

$$\frac{\partial G}{\partial t} = r(C, U)\, G - \frac{r_0 G^2}{K(C)},$$

$$\frac{\partial U}{\partial t} = Q + U\,[s(C, G)] + D_1 \frac{\partial^2 U}{\partial x^2} \qquad \text{... (2.1)}$$

and
$$\frac{\partial C}{\partial t} = Q_1 - \alpha C + D_0 \frac{\partial^2 C}{\partial x^2},$$

where $G(x, t)$ and $U(x, t)$ are the densities of the grass biomass and cattle population respectively and $C(x, t)$ is the concentration of a pollutant discharged with a constant flux Q_1 in a one dimensional linear habitat given by $0 \le x \le L$. $r(C, U)$ and $s(C, G)$ are growth rates for grass biomass and cattle population respectively. $K(C)$ is the carrying capacity of the grass biomass and is a decreasing function of C. The constant r_0 represents the specific birth rate in the absence of pollutants and cattle. Q is the migration rate of cattle population from outside into the grassland. $Q_1 > 0$ is the constant determining the exogeneous rate of input of pollutants into the habitat. The constant α is the coefficient of the rate of decrease of pollutant due to various factors in the habitat and is assumed to be proportional to concentration of C. D_0 and D_1 are the diffusion coefficient of pollutant concentration C and cattle population U respectively.

Since the growth rate $r(C, U)$ of grass biomass decreases with both C and U it can be written in a simple linear form as follows :

$$r(C, U) = r_0 - r_1 C - r_2 U \qquad \text{... (2.2)}$$

such that

$$r(0, 0) = r_0 > 0, \frac{\partial r}{\partial u} < 0, \frac{\partial r}{\partial c} < 0 \text{ for } U \geq 0, C \geq 0.$$

Similarly since the growth rate of cattle population decreases as C increases and G decreases, we can take

$$s(C, G) = -s_0 - s_1 C + s_2 G \qquad \text{... (2.3)}$$

note that

$$s\ (0,\ 0) = -s_0 < 0, \frac{\partial s}{\partial G} > 0, \frac{\partial s}{\partial C} > 0 \text{ for } G \geq 0, C \geq 0.$$

In the present analysis we assume the following form of $K(C)$:

$$K(C) = K_{10} - K_{11} C \qquad \text{... (2.4)}$$

such that $K(0) = K_{10} > 0, K'(C) = -K_{11} < 0.$

Model (2.1) may be associated with the following initial and boundary conditions :

(*i*) Initial Conditions :

$$G(x, 0) = G_0 > 0, U(x, 0) = U_0 > 0, C(x, 0) = C_0 > 0 \qquad \text{... (2.5)}$$

(*ii*) Boundary Conditions

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \left.\frac{\partial C}{\partial x}\right|_{x=0} = 0, \left.\frac{\partial U}{\partial x}\right|_{x=L} = \left.\frac{\partial C}{\partial x}\right|_{x=L} = 0 \qquad \text{... (2.6)}$$

3. Mathematical Analysis

Now we analyse model (2.1) for two different cases :

(i) No diffusion,

(ii) With Diffusion,

(*i*) No Diffusion

In such a case our model reduces to

$$\dot{G} = r(C, U)\, G - \frac{r_0 G^2}{K(C)}$$

$$\dot{U} = Q + U[s, (C, G)] \qquad \text{... (3.1)}$$

and $$\dot{C} = Q_1 - \alpha C.$$

This model has two non negative equilibria $\overline{E}(0, \overline{U}, \overline{C})$ and $E^*(G^*, U^*, C^*)$ where,

$$\overline{U} = \frac{Q}{s_0 + s_1 \overline{C}} \qquad \text{... (3.2)}$$

and

$$\overline{C} = \frac{Q_1}{\alpha} \qquad \text{... (3.3)}$$

and G^*, U^* and C^* are the positive solutions of the system of algebraic equations given below :-

$$r_0 G = K(C)\,[(r_0 - r_1 C - r_2 U)]. \qquad \text{... (3.4)}$$

$$U = \frac{Q}{s_0 + s_1 C - s_2 G} = h(G). \qquad \text{... (3.5)}$$

$$C = \frac{Q_1}{\alpha}. \qquad \text{... (3.6)}$$

It may be noted here that for U to be positive we must have

$$s_2 G < s_0 + s_1 Q_1/\alpha. \qquad \text{... (3.7)}$$

Taking

$$F_1(G) = r_0 G - (r_0 - r_1 Q_1/\alpha - r_2 Q/s_0 + s_1 Q_1/\alpha - s_2 G))\, K(Q_1/\alpha) \qquad \text{... (3.8)}$$

It is easy to see that $F_1(0) < 0, F_1(K_{10}) > 0$, showing the existence of G^* in the interval $0 < G^* < K_{10}$, here K_{10} is the carrying capacity of biomass in absence of pollutants. For G^* to be unique, the following conditions must be satisfied.

$$\frac{dF_1}{dG} = r_0 + \left[\frac{r_2 s_2 Q}{(s_0 + s_1 Q_1/\alpha - s_2 G)^2} K\left(\frac{Q_1}{\alpha}\right)\right] > 0 \qquad \text{... (3.9)}$$

Remark : Since $h'(G) > 0$, the in equality (3.9) satisfies automatically.

It can easily be checked that $\overline{E}$ is a saddle point whose unstabel manifold is locally in the G direction and the stable manifold is in $U - C$ plane.

In order to study the behaviour of E^*, we prove the following theorem :

Theorem 3.1 — *In addition to condition (3.7), let the following inequalities hold :*

$$(s_2 U^* + r_2)^2 < \frac{r_0}{K(C^*)}\left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 G^*\right) \qquad \text{... (3.10)}$$

$$\left(\frac{r_0 G^* K'(C^*)}{K^2(C^*)} + r_1\right)^2 < \frac{\alpha r_0}{K(C^*)} \qquad \text{... (3.11)}$$

$$(s_1 U^*)^2 < \alpha \left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 G^* \right) \qquad \text{... (3.12)}$$

Then equilibrium E^* is locally asymptotically stable.

PROOF : Consider the following positive definite function in the linearized system of model (3.1)

$$V_1 (G, U, C) = \frac{1}{2G^*} (G - G^*)^2 + \frac{1}{2} (U - U^2) + \frac{1}{2} (C - C^*)^2 \qquad \text{... (3.13)}$$

Differentiating $V_1 (G, U, C)$ with respect to t, we get :

$$\frac{dV_1 (G, U, C)}{dt} = -\frac{1}{2} a_{11} (G - G^*)^2 + a_{12} (G - G^*) (U - U^*) - \frac{1}{2} a_{22} (U - U^*)^2$$

$$- \frac{1}{2} a_{11} (G - G^*)^2 + a_{13} (G - G^*) (C - C^*) - \frac{1}{2} a_{33} (C - C^*)^2$$

$$- \frac{1}{2} a_{22} (U - U^*)^2 + a_{23} (U - U^*) (C - C^*) - \frac{1}{2} a_{33} (C - C^*)^2, \qquad \text{... (3.14)}$$

where
$$a_{11} = \frac{r_0}{K(C^*)}, \; a_{22} = \left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 G^* \right), \; a_{33} = \alpha,$$

$$a_{12} = s_2 U^* - r_2, \; a_{13} = \frac{r_0 G^* K' (C^*)}{K^2 (C^*)} - r_1, \; a_{23} = - s_1 U^*.$$

Sufficient conditions for dV_1/dt to be negative definite are that the following inequalities hold :

$$a_{12}^2 < a_{11} a_{22} \qquad \text{... (3.15)}$$

$$a_{13}^2 < a_{11} a_{33} \qquad \text{... (3.16)}$$

$$a_{23}^2 < a_{22} a_{33} \qquad \text{... (3.17)}$$

We note here that (3.10) implies (3.15), (3.11) implies (3.16) and (3.12) implies (3.17) and hence V_1 is a Liapunov function with respect to E^*, proving the theorem.

To study the global asymptotic stability of E^* in model (3.1), we first establish a region of attraction by proving the following lemma.

Lemma 3.2 — The region

$$R^* = \{(G, U, C);\ 0 \le G \le K_{10}, 0 \le U < \infty, 0 \le C \le Q_1/\alpha\}$$

is a region of attraction for all solutions initiating in the positive orthant.

PROOF : From (3.1) we have

$$G^* = r(C, U)\,G - \frac{r_0\,G^2}{K(C)}$$

$$\leq r_0\,G\left(1 - \frac{G}{K_{10}}\right).$$

Hence, $\lim_{t \to \infty} G(t) < K_{10}$. Again from (3.1) we have

$$\dot{U} = Q + U\,s(C, G)$$

$$\leq Q + U\,s(0, K_{10}).$$

Hence, $\lim_{t \to \infty} U(t) < \infty$

Again from (3.1) we have

$$\dot{C} = Q_1 - \alpha\,c$$

and $$\lim_{t \to \infty} C = \frac{Q_1}{\alpha}$$

i.e., $C \leq C^*$.

Now we prove the following theorem for the global stability behaviour of E^*.

Theorem 3.2 — *In addition to conditions (2.2), (2.3), (2.4) and (3.7), let $U(t)$ and $K(C)$ satisfy in R^*.*

$$0 \leq U(t) \leq U_2,\ K_m \leq K(C) \leq K_0,\ 0 \leq -K'(C) \leq K_{11} \qquad \dots (3.18)$$

for some positive constants K_m and U_2. Then if the following inequalities hold :

$$(s_2\,U_2 + r_2)^2 < \frac{r_0}{K(C^*)}\left(s_0 + \frac{s_1\,Q_1}{\alpha} - s_2\,G^*\right), \qquad \dots (3.19)$$

$$\left(\frac{r_0\,K_{10}\,K_{11}}{K_m^2} + r_1\right)^2 < \frac{\alpha\,r_0}{K(C^*)} \qquad \dots (3.20)$$

and $$(s_1\,U_2)^2 < \alpha\left(s_0 + \frac{s_1\,Q_1}{\alpha} - s_2\,G^*\right), \qquad \dots (3.21)$$

E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

PROOF : We consider the following positive definite function about E^*

$$V_2(G, U, C) = (G - G^* - G^* \log G/G^*) + 1/2\,(U - U^*)^2 + 1/2\,(C - C^*)^2 \quad \text{... (3.22)}$$

The time derivative of V_2 along solution of (3.1) is,

$$\frac{dV_2(G, U, C)}{dt} = -\frac{r_0}{K(C^*)}(G - G^*)^2 - (r_0 G\,\xi(C) + r_1)(G - G^*)(C - C^*)$$
$$- (r_2 - s_2 U)(G - G^*)(U - U^*) - \left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 G^*\right)(U - U^*)^2$$
$$- s_1 U (C - C^*)(U - U^*) - \alpha (C - C^*)^2, \quad \text{... (3.23)}$$

where

$$\xi(C) = \begin{cases} \dfrac{\left[\dfrac{1}{K(C)} - \dfrac{1}{K(C^*)}\right]}{C - C^*} & C \neq C^* \\ \dfrac{K'(C^*)}{K^2(C^*)} & C = C^* \end{cases} \quad \text{... (3.24)}$$

From (3.18) and the mean value theorem, we note that

$$|\,\xi(C)\,| < K_{11}/K_m^2.$$

After some algebraic manipulations (3.23) can be written as :

$$\frac{dV_2(G, U, C)}{dt} = -\frac{1}{2} b_{11}(G - G^*)^2 + b_{12}(G - G^*)(U - U^*) - \frac{1}{2} b_{22}(U - U^*)^2$$
$$- \frac{1}{2} b_{11}(G - G^*) + b_{13}(G - G^*)(C - C^*) - \frac{1}{2} b_{33}(C - C^*)^2$$
$$- \frac{1}{2} b_{22}(U - U^*)^2 + b_{23}(U - U^*)(C - C^*) - \frac{1}{2} b_{33}(C - C^*)^2, \quad \text{... (3.25)}$$

where $b_{11} = \dfrac{r_0}{K(C^*)}$, $b_{22} = \left(s_0 + \dfrac{s_1 Q_1}{\alpha} - s_2 G^*\right)$, $b_{33} = \alpha$

$b_{12} = s_2 U - r_2$, $b_{13} = -(r_0 G\,\xi(C) + r_1)$, $b_{23} = -s_1 U$.

Then, sufficient conditions for V_2 to be negative definite are

$$b_{12}^2 < b_{11} b_{22}, \quad \text{... (3.26)}$$

$$b_{13}^2 < b_{11} b_{33} \quad \text{... (3.27)}$$

and $$b_{23}^2 < b_{22} b_{33}. \quad \text{... (3.28)}$$

However, (3.19) implies (3.26), (3.20) implies (3.27) and(3.21) implies (3.28). Hence, V_2 is a Liapunov function with respect to E^* whose domain constains R^*, proving the theorem.

The above analysis shows that under conditions (3.19), (3.20) and (3.21) the equilibrium E^* of the nonlinear model (3.1) is globally asymptotically stable which implies that the equilibrium of grass biomass can remain sustainable but at a recorded level due to pollution stress in the habitat and grazing by cattle from the plains. From eq. (3.4) it is clear that grass biomass decreases with increase in migration rate Q and pollutant concentration C as $r(C, U)$ is a decreasing function of C and U and $K(C)$ is a decreasing function of C (shown mathematically in section 5). It may be noted that if the pollutant concentration increases and a large cattle population is allowed to migrate into the grassland from plains for grazing the grass biomass would not remain sustainable for a long period.

2. With Diffusion

In this case our model is same as (2.1). To discuss the linear stability of the model (2.1), we construct the following positive definite function about E^* in the linearlized system from (2.1) :

$$W_1 (G, U, C) = \int_0^L V_1 (G, U, C)\, dx,$$

where $V_1 (G, U, C)$ is given by (3.13). Using boundary conditions (2.6), the time derivative of $W_1 (G, U, C)$ is,

$$W_1^* (G, U, C) = \int_0^L \frac{dV_1 (G, U, C)}{dt} - \int_0^L \left[D_1 \left(\frac{\partial U}{\partial x}\right)^2 dx + D_0 \left(\frac{\partial C}{\partial x}\right)^2 \right] dx. \quad \text{... (3.29)}$$

Since $dV_1 (G, U, C)/dt$ is negative definite, W_1 is also negative definite. Then by Liapunov's stability theorem, we conclude that model (2.1) is linearly asymptotically stable.

In order to discuss the global asymptotic stability of model (2.1), we construct the following positive definitive function and E^*:

$$W_2 (G, U, C) = \int_0^L V_2 (G, U, C)\, dx, \quad \text{... (3.30)}$$

then $$\dot{W}_2 (G, U, C) = \int_0^L \frac{dV_2 (G, U, C)\, dx}{dt} - \int_0^L \left[D_1 \left(\frac{\partial U}{\partial x}\right)^2 dx + D_0 \left(\frac{\partial C}{\partial x}\right)^2 \right] dx. \quad \text{... (3.31)}$$

Since $dV_2 (G, U, C)/dt$ is negative definite, $\dot{W}_2$ is also negative definite. Then by Liapunov's stability theorem, we conclude that model (2.1) is globally asymptotically stable.

From above analysis it is clear that in this case both linear and global stability becomes stronger due to an extra negative term of diffusion (3.29) and (3.31). This implies that biomass will converge towards its carrying capacity at a faster rate than with no diffusion.

4. Conservation Model

To conserve the grass biomass various efforts such as fencing in the form of a green belt to check the migration cattle population as well as to control the pollution, irrigation of grasslands, use of fertilizers, plantation of fodder saplings etc. have to be applied in the forested grassland. In general such efforts should be related to the depleted level of grass biomass from its carrying capacity. Here we assume that the effort applied increases the grass biomass growth rate in a bilinear manner. The dynamics of the system in this case can be written as follows :

$$\left.\begin{aligned}
\frac{\partial G}{dt} &= r(C, U)\, G - \frac{r_0\, G^2}{K(C)} + (R_1 + R_2\, G)\, F, \\
\frac{\partial U}{\partial t} &= Q + U\,[s(C, G)] + D_1 \frac{\partial^2 U}{\partial x^2}, \\
\frac{\partial C}{\partial t} &= Q_1 - \alpha\, C + D_0 \frac{\partial^2 C}{\partial x^2} \\
\text{and} \quad \frac{\partial F}{\partial t} &= r_F\left(1 - \frac{G}{K_{10}}\right) - \gamma F,
\end{aligned}\right] \quad \text{... (4.1)}$$

where $r_F > 0$ denotes the constant growth rate coefficient of the effort applied, $R_1 > 0$ is the growth rate coefficient of the grass biomass due to effort, $R_2 > 0$ is the growth rate co-efficient of grass biomnass due to its interaction with effort, $\gamma > 0$ is the depreciation rate of the effort applied due to various causes and the rest have the same meaning as in the model (2.1).

Model (4.1) may be associated with the following initial conditions :

$$G(x, 0) = G_0 > 0, U(x, 0) = U_0 > 0, C(x, 0) = C_0 > 0, F(x, 0) = F_0 > 0 \quad \text{... (4.2)}$$

The boundary conditions are the same as in (2.6).

In this case model (4.1) has only one non negative equilibrium namely $\hat{E}(\hat{G}, \hat{U}, \hat{C}, \hat{F})$, where $\hat{G}, \hat{U}, \hat{C}$ and $\hat{F}$ are positive solutions of the following algebraic equations :

$$r_0\, G = K(C)\, [(r_0 - r_1\, C - r_2\, U) + (R_1/G + R_2)\, F], \quad \text{... (4.3)}$$

$$U = \frac{Q}{s_0 + \frac{s_1\, Q_1}{\alpha} - s_2\, G} = g(G), \quad \text{... (4.4)}$$

$$C = \frac{Q_1}{\alpha} \quad \text{... (4.5)}$$

and

$$F = \frac{r_F}{\gamma}\left(1 - \frac{G}{K_{10}}\right) \; (\gamma > 0). \quad \text{... (4.6)}$$

It may be noted here from (4.4) and (4.6) that for U to be positive we must have

$$s_2\, G < s_0 + s_1\, Q_1/\alpha \quad \text{... (4.7)}$$

and for F to be positive we must have

$$G < K_{10}. \quad \text{... (4.8)}$$

Taking

$$F_2(G) = r_0\, G - K\left(\frac{Q_1}{\alpha}\right)$$

$$\times \left[(r_0 - r_1\, Q_1/\alpha - r_2\, Q/s_0 + s_1\, Q_1/\alpha - s_2\, G)\left(\frac{R_1}{G} + R_2\right)\frac{r_F}{\gamma}\left(1 - \frac{G}{K_{10}}\right)\right] \quad \text{... (4.9)}$$

It is easy to check that $F_2(0) < 0$ and $F_2(K_{10}) > 0$, showing the existence of $\hat{G}$ in the interval $0 < \hat{G} < K_{10}$. For $\hat{G}$ to be unique the following condition must be satisfied :

$$\frac{dF_2}{dG} = r_0 + \left[\frac{r_2 s_2\, Q}{(s_0 + s_1 Q_1/\alpha - s_2\, G)^2} + \frac{R_1\, r_F}{\gamma G^2} + \frac{R_2\, r_F}{\gamma K_{10}}\right] K\left(\frac{Q_1}{\alpha}\right) > 0. \quad \text{... (4.10)}$$

Now to determine the local stability of $\hat{E}$, we prove the following theorem :

Theorem 4.1 — *In addition to condition (4.7) let the following inequalities hold :-*

$$(s_2\, \hat{U} + r_2)^2 < \frac{r_0}{K(\hat{C})}\left(s_0 + \frac{s_1\, Q_1}{\alpha} - s_2\, \hat{G}\right). \quad \text{... (4.11)}$$

$$\left(\frac{r_0\, \hat{G}\, K'(\hat{C})}{K^2(\hat{C})} + r_1\right)^2 < \frac{\alpha\, r_0}{K(\hat{C})}. \quad \text{... (4.12)}$$

$$(s_1\, \hat{U})^2 < \alpha\left(s_0 + \frac{s_1\, Q_1}{\alpha} - s_2\, \hat{G}\right). \quad \text{... (4.13)}$$

Then equilibrium $\hat{E}$ *is locally asymptotically stable.*

PROOF : Consider the following positive definite function in the linearized system arising from model (4.1)

$$W_3(G, U, C) =$$

$$\int_0^L \left[\frac{1}{2\hat{G}}(G - \hat{G})^2 + \frac{1}{2}(U - \hat{U})^2 + \frac{1}{2}(C - \hat{C})^2 + \frac{1}{2}\frac{K_{10}}{r_F\, \hat{G}}(R_1 + R_2\, \hat{G})\,(F - \hat{F})^2\right] dx. \quad \text{... (4.14a)}$$

It can easily be checked that the derivative of W_3 with respect to t is negative definite under conditions (4.7), (4.11), (4.12) and (4.13) proving the theorem. Now we find out conditions under which $\hat{E}$ is globally asymptotically stable. First we prove the following lemma.

Lemma 4.1 — The region

$$\hat{R} = \left\{ (G, U, C, F); 0 \le G \le G_c, 0 \le U < \infty, 0 \le C \le \frac{Q_1}{\alpha}, 0 \le F \le \frac{r_F}{\gamma} \right\}$$

is a region of attraction for all solutions initiating in the positive orthant where

$$G_C = \frac{K_{10}}{2r_0}\left(r_0 + \frac{R_2 r_F}{\gamma} \right)\left[1 + \left\{ 1 + \frac{4r_0 r_F R_1}{\gamma K_{10}\left(r_0 + \frac{R_2 r_F}{\gamma} \right)^2} \right\}^{1/2} \right].$$

PROOF : From (4.1)

$$\frac{dF}{dt} = r_F\left(1 - \frac{G}{K_{10}} \right) - \gamma F$$

$$\le r_F - \gamma F$$

$$F \le \frac{r_F}{\gamma} - e^{-\gamma t}\left(\frac{r_F}{\gamma} - F_0 \right)$$

Thus if $F_0 \le r_{F/\gamma}$ we get $F \le r_F/\gamma$ for $t \le 0$.

Again from (4.1)

$$\frac{dG}{dt} = r(C, U)\, G - \frac{r_0 G^2}{K(C)} + (R_1 + R_2 G)\, F$$

$$\le r_0 G - \frac{r_0 G^2}{K_{10}} + (R_1 + R_2 G)\frac{r_F}{\gamma}$$

$$\le \frac{dx}{dt}$$

where $$\frac{dx}{dt} = \frac{R_1 r_F}{\gamma} + \left(r_0 + \frac{R_2 r_F}{\gamma} \right) G - \frac{r_0 G^2}{K_{10}}.$$

We note that the solutions of this equation for $x(0) > 0$ are such that

$$\lim_{t \to \infty} x(t) = G_c = \frac{K_{10}}{2r_0}\left(r_0 + \frac{R_2 r_F}{\gamma} \right)\left[1 + \left\{ 1 + \frac{4r_0 r_F R_1}{\gamma K_{10}\left(r_0 + \frac{R_2 r_F}{\gamma} \right)^2} \right\}^{\frac{1}{2}} \right]$$

This implies that

$$\lim_{t \to \infty} Sup\ G(t) \le G_c \text{ for } G_0 \le G_c.$$

For $U(t)$ and $C(t)$ the proof is similar as in Lemma (3.2).

In the following theorem the global stability behaviour of $\hat{E}$ is studied.

Theorem 4.2 — *In addition to assumptions (2.2), (2.3), (2.4) and (4.7), let U(t) and K(C) satisfy in* $\hat{R}$,

$$0 \le U(t) \le U_2,\ K_m \le K(C) \le K_{10},\ 0 \le -K'(C) \le K_{11} \quad \text{... (4.14b)}$$

for some positive constants U_2 *and* K_m. *Then if the following inequalities hold*

$$(s_2 U_2 + r_2)^2 < \frac{r_0}{K(\hat{C})}\left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 \hat{G}\right) \quad \text{... (4.15)}$$

$$\left(\frac{r_0 G_c K_{11}}{K_m^2} + r_1\right)^2 < \frac{\alpha r_0}{K(\hat{C})} \quad \text{... (4.16)}$$

$$(s_1 U_2)^2 < \alpha\left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 \hat{G}\right), \quad \text{... (4.17)}$$

$\hat{E}$ *is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.*

PROOF : Consider the following positive definite function about $\hat{E}$.

$W_4 (G, U, C, F) =$

$$\int_0^L \left[\left(G - \hat{G} - \hat{G}\log\frac{G}{\hat{G}}\right) + \frac{1}{2}(U - \hat{U})^2 + \frac{1}{2}(C - \hat{C})^2 + \frac{1}{2}\frac{K_{10}}{r_F \hat{G}}(R_1 + R_2 \hat{G})(F - \hat{F})^2\right] dx \quad \text{... (4.18)}$$

Differentiating W_4 with respect to 't' along solution of (4.1), we obtain,

$$\dot{W}_4 (G, U, C, F) = \int_0^L -\frac{r_0}{K(\hat{C})}(G - \hat{G})^2 - \frac{R_1 F}{G\hat{G}}(G - \hat{G})^2 + (s_2 U - r_2)(G - \hat{G})(U - \hat{U})$$

$$- \left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 \hat{G}\right)(U - \hat{U})^2 - s_1 U (U - \hat{U})(C - \hat{C}) - \alpha (C - \hat{C})^2$$

$$- (r_1 + r_0 G \xi(C))(G - \hat{G})(C - \hat{C}) - \frac{\gamma K_{10}(R_1 + R_2 \hat{G})(F - \hat{F})^2}{r_F \hat{G}}$$

$$- \left[D_1\left(\frac{\partial U}{\partial x}\right)^2 + D_0\left(\frac{\partial C}{\partial x}\right)^2\right] dx, \quad \text{... (4.19)}$$

where
$$\xi(C)=\begin{cases}\dfrac{\left[\dfrac{1}{K(C)}-\dfrac{1}{K(\hat{C})}\right]}{(C-\hat{C})} & C\neq\hat{C}\\ \dfrac{K'(C)}{K^2(\hat{C})} & C=\hat{C}\end{cases} \quad \dots (4.20)$$

From (4.14) and the mean value theorem, we note that

$$|\xi(G)|<\frac{K_{11}}{K_m^2}. \quad \dots (4.21)$$

After some algebraic manipulations (4.19) can be further written as

$$W_4=\int_0^L\Big[-\frac{1}{2}A_{11}(G-\hat{G})^2+A_{12}(G-\hat{G})(U-\hat{U})-\frac{1}{2}A_{22}(U-\hat{U})^2-\frac{1}{2}A_{11}(G-\hat{G})^2$$
$$+A_{13}(G-\hat{G})(C-\hat{C})-\frac{1}{2}A_{33}(C-\hat{C})^2-\frac{1}{2}A_{22}(U-\hat{U})^2+A_{23}(U-\hat{U})(C-\hat{C})$$
$$-\frac{1}{2}A_{33}(C-\hat{C})^2-\frac{R_1 F}{G\hat{G}}(G-\hat{G})^2-\frac{\gamma K_{10}(R_1+R_2\hat{G})}{r_F\hat{G}}(F-\hat{F})^2$$
$$-\left[D_1\left(\frac{\partial U}{\partial x}\right)^2+D_0\left(\frac{\partial C}{\partial x}\right)^2\right]\Big]dx, \quad \dots (4.22)$$

where
$$A_{11}=\frac{r_0}{K(\hat{C})},\ A_{22}=\left(s_0+\frac{s_1 Q_1}{\alpha}-s_2\hat{G}\right),\ A_{33}=\alpha,$$
$$A_{12}=s_2 U-r_2,\ A_{23}=-(r_1+r_0 G\xi(C)),\ A_{33}=-s_1 U.$$

Then sufficient conditions for W_4 to be negative definite are following :-

$$A_{12}^2<A_{11}A_{22} \quad \dots (4.23)$$

$$A_{13}^2<A_{11}A_{33} \quad \dots (4.24)$$

$$A_{23}^2<A_{22}A_{33}. \quad \dots (4.25)$$

We note that (4.15) implies (4.23), (4.16) implies (4.24) and (4.17) implies (4.25). Thus W_4 is negative definite under conditions (4.15), (4.16) and (4.17). Hence, W_4 is a Liapunov function with respect to $\hat{E}$ whose domain contains the Region $\hat{R}$ proving the theorem. Thus $\hat{E}(\hat{G},\hat{U},\hat{C},\hat{F})$ is asymptotically stable in region $\hat{R}$.

From the analysis of the conservation model it can be noted if effort applied is proportional to itself and to the difference between the forestry biomass and its natural carrying capacity, then the forestry biomass can be returned and maintained at an appropriate level in the presence of such effects.

Now we consider the case when $\gamma = 0$. In this case the positive uniform equilibrium point $(\overline{G}, \overline{U}, \overline{C}, \overline{F})$ of the model (4.1) can be obtained as,

$$\overline{G} = K_{10},$$

$$\overline{U} = \frac{Q}{-s\left(\frac{Q_1}{\alpha}, K_{10}\right)},$$

$$\overline{C} = \frac{Q_1}{\alpha}$$

and

$$\overline{F} = \frac{K_{10}}{(R_1 + R_2 K_{10}) K\left(\frac{Q_1}{\alpha}\right)} \left[r_0 K_{10} - r\left(\frac{Q_1}{\alpha}, \frac{Q}{-s\left(\frac{Q_1}{\alpha}, K_{10}\right)}\right) K\left(\frac{Q_1}{\alpha}\right)\right].$$

Similarly, as in the previous case, it can be proved if the following inequalities hold :

$$(s_2 \overline{U} + r_2)^2 < \frac{r_0}{K(\overline{C})}\left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 \overline{G}\right)\left(\frac{r_0 \overline{G} K'(\overline{C})}{K^2(\overline{C})} + r_1\right)^2$$

$$< \frac{\alpha r_0}{K(\overline{C})} (s_1 \overline{U})^2 < \left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 \overline{G}\right)$$

Then $\overline{E}$ is locally asymptotically stable. In this case the region

$$\overline{R}\left\{ (G, U, C, F); 0 \le G < \infty, 0 \le U < \infty, 0 \le C \le \frac{Q_1}{\alpha}, 0 \le F < \infty \right\}$$

is a region of attraction for all solutions initiating in the positive orthant.

In addition to assumptions (2.2), (2.3), (2.4) and (4.7), let $G(t)$, $U(t)$ and $K(C)$ satisfy in

$$\overline{R},\ 0 \le G(t) \le G_2 < \infty, 0 \le U(t) \le U_2 < \infty, \overline{K}_m \le K(C) \le K_{10}, 0 \le -K'(C) \le K_{11}$$

for some positive constants G_2, U_2 and $\overline{K}_m$, then the if the following inequalities hold :

$$(s_2 U_2 + r_2)^2 < \frac{r_0}{K(\overline{C})}\left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 \overline{G}\right)\left(\frac{r_0 G_2 K_{11}}{K_m^2} + r_1\right)^2$$

$$< \frac{\alpha r_0}{K(\overline{C})} (s_1 U_2)^2 < \left(s_0 + \frac{s_1 Q_1}{\alpha} - s_2 \overline{G}\right)$$

Then $\overline{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

The above analysis shows that more effort would be required to conserve the biomass at a level K_{10} if the migration rate Q and pollutant concentration Q_1/α increases. In general it is not possible to apply such an effort.

5. NUMERICAL EXAMPLE

A numerical example is presented in this section to explain the applicability of the results discussed in the previous sections.

Taking the following values of the various parameters in model (2.1),

$$r_0 = 4.0, \; r_1 = 0.2, \; r_2 = 0.01, \; K_{10} = 12.0,$$

$$K_{11} = 1.2, \; s_0 = 3.0, \; s_1 = 0.3, \; s_2 = 0.02,$$

$$Q = 20.0, \; Q_1 = 40.0, \; \alpha = 15.0. \qquad \ldots (5.1)$$

It can be checked that under the above set of parameters the condition (3.9) for the existence of E^* is satisfied and E^* is given by :

$$G^* \sim 7.5057, \; U^* \sim 5.4796, \; C^* \sim 2.6667 \qquad \ldots (5.2)$$

It can also be verified that condition (3.10), (3.11) and (3.12) in Theorem (3.1) are satisfied. This shows that E^* is locally asymptotically stable.

By choosing $K_m = 6.5$ and $U_2 = 24.0$ in Theorem (3.2), it can be checked that conditions (3.19), (3.20) and (3.21) are satisfied which shows that E^* is globally asymptotically stable.

Again taking the same values of the various parameters as given an (5.1) for the model (2.1) and the following values of the remaining parameters in the model (4.1),

$$R_1 = 0.5, \; R_2 = 0.6, \; r_F = 0.5, \; Y = 0.15 \qquad \ldots (5.3)$$

it can be checked that condition (4.10) for the existence of $\hat{E}$ is satisfied and $\hat{E}$ is given by

$$\hat{G} \sim 8.7929, \; 0 \sim 5.5185, \; \hat{C} \sim 2.6667, \; \hat{F} \sim 0.8899 \qquad \ldots (5.4)$$

It can be verified again that conditions in Theorem (4.1) and Theorem (4.2) are satisfied for the same values of parameters given in (5.1) and the same values of K_m and U_2.

Taking $Q = 40$ and the same values of the remaining parameters as given in (5.1) and (5.3), it can be checked that all the conditions are satisfied and E^* is given by

$$G^* \sim 7.3850, \; U^* \sim 10.9520, \; C^* \sim 2.6667 \qquad \ldots (5.5)$$

and $\hat{E}$ is given by

$$\hat{G} \sim 8.7057, \; 0 \sim 11.0317, \; \hat{C} \sim 2.6667, \; \hat{F} \sim 0.9142 \qquad \ldots (5.6)$$

Again taking $Q_1 = 50$ and the same values of the remaining parameters as given in (5.1) and (5.3) it can be checked that all conditions are satisfied and E^* is given by

$$G^* \sim 6.5634,\ U^* \sim 5.1698\ C^* \sim 3.3333 \qquad \text{... (5.7)}$$

and $\hat{E}$ is given by

$$\hat{G} \sim 7.4488,\ 0 \sim 5.1934,\ \hat{C} \sim 3.3333,\ \hat{F}\ 1.2630 \qquad \text{... (5.8)}$$

Comparing the values of the equilibrium in (5.2) and (5.4), it is noted that by applying suitable effort, the level of biomass can be maintained at an appropriate level even under continued grazing.

Comparing the values of the equilibria in (5.2) and (5.5), it is noted that equilibrium of the grass biomass can remain sustainable but at a reduced level due to an increase in Q, the migration of cattle population from the plains into the grassland, while the equilibrium level of the cattle population becomes higher. From (5.4) and (5.6) it is clear that more effort would be required to maintain the biomass at an appropriate level.

Comparing the values of the equilibria in (5.2) and (5.7), it is noted that if the concentration of pollutant in the biomass increases, the equilibrium level of grass biomass and cattle population decreases. It may be noted also from (5.4) and (5.8) that in this case more effort would be required to conserve the biomass at an appropriate level.

6. Conclusion

In this paper we have considered a model with diffusion for the depletion of grass biomass due to grazing by animal populations and atmospheric acid rain was shown that if the pressure due to these factors increase the biomass in the region under consideration may not last long.

From the analysis of the conservation model it is clear that with sufficient restoration effort, extinction of the grass biomass can be averted and brought back to and maintained at an appropriate level. It has also been shown that if no effort were made to conserve the grass biomass, it could eventually diminish to the point of extinction.

In order to discuss the applicability of the model, a numerical example is also presented. This example shows that if suitable efforts are not applied to conserve the grass biomass, then the density of grazing population and concentration of pollutants increases, and consequently, the density of the grass biomass decreases.

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