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OPTIMIZATION AND IDENTIFICATION OF NONLINEAR SYSTEMS ON BANACH SPACE*

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In this paper, we consider the problem of identification of the operators for the systems governed by nonlinear evolution equations. For this identification problem, we show existence of solutions and present necessary condition of optimality.

Key Words : Optimal Control; Galerkin Method; Nonlinear Systems; Identification Problem; Necessary Condition

1. INTRODUCTION

Many physical systems arising from thermodynamics, electrodynamics, and population biology are modelled by differential equations, integrodifferential equations, and nonlinear evolution equations with uncertain parameters or undetermined operators.

In this paper, we consider the differential equations on Banach spaces as follows :

$$\begin{cases} \dot{x} + A(t, x) = g(t, x) \\ x(0) = x_0, \end{cases}$$

where A is a nonlinear monotone operator in a Banach space and $g(t, x)$ is a nonlinear but not a monotone operator.

An associated control system may be described as

$$(\text{CP}) \begin{cases} \dot{x} + A(t, x) + Bx = g(t, x) \\ x(0) = x_0, B \in \mathcal{P}_{a,b} \end{cases}$$

where $\mathcal{P}_{a,b}$ is a suitable subset of $\mathcal{L}(V, V^*)$. Define the error functional $J(\cdot)$ by the form

$$J(B) = \int_I f(t, x(B)(t))\, dt,$$

where $I = [0, T], T < \infty$. The problem is to find $B^0 \in \mathcal{P}_{a,b}$ (admissible set) so that

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(P) $J(B^0) \le J(B)$ for all $B \in \mathcal{P}_{a,b}$.

In recent years optimal control and identification problems have been extensively studied by many authors (see [5], [6] and the references therein) and more generally, functional differential inclusions has been studied by Ahmed and Papageorgiou and the references therein). These studies were mainly concerned with the question of existence of optimal controls in the uncertain systems.

In this paper, we study the existence and uniqueness of solutions for equation (CP) and obtain the optimal solution for an identification problem (P). We also derive necessary conditions of optimality for an identification problem (P).

2. Existence and Uniqueness of Solutions

Let H be a Hilbert space with a scalar product $(\cdot,\cdot)$ satisfying the embedding $V \hookrightarrow H$ being dense and continuous, and a reflexive Banach space. Let V be a subspace of H. Identifying H with its dual, we have $V \hookrightarrow H \hookrightarrow V^*$, where V^* is the topological dual of V. The system considered here is based on this evolution triple.

Let $\langle x, y \rangle$ denote the pairing of an element $x \in V$ and an element $y \in V^*$. If $x, y \in H$, then $\langle x, y \rangle = (x, y)$. The norm in any Banach space X will be denoted by $\|\cdot\|_X$.

Let $\{e_1, e_2, \dots\}$ be a basis of V and set

$$H_n = lin.\ span\ \{e_1, e_2, \dots, e_n\}.$$

We introduce in the n-dimensional space H_n the scalar product of Hilbert space H. Note $H_n \subset V \subset H$.

Let $0 < t \le T < \infty$, $I_t \equiv [0, t]$, $I \equiv [0, T]$, and let $p, q \ge 1$ such that

$$1/p + 1/q = 1 \text{ and } 2 \le p < \infty.$$

For simple notation, we write $L_p^t(V) \equiv L_p(I_t, V)$, $L_p(V) \equiv L_p(I, V)$, $L_q^t(V) \equiv L_q(I_t, V)$, $L_q(V^*) \equiv L_q(I, V^*)$. For p, q satisfying the preceding conditions, it follows from the reflexivity of V that both $L_p^t(V)$ and $L_q^t(V^*)$ are reflexive Banach spaces (see Theorem 1.1.17 of [3]). The pairing of $L_p^t(V)$ and $L_q^t(V^*)$ is denoted by $\langle\langle \cdot,\cdot \rangle\rangle_t$. In particular, we use $\langle\langle \cdot,\cdot \rangle\rangle \equiv \langle\langle \cdot,\cdot \rangle\rangle_T$. Clearly, for $u, v \in L_2(H)$, $\langle\langle u, v \rangle\rangle = ((u, v))$, where $((\cdot,\cdot))$ is the scalar product in Hilbert space $L_2(H)$. Define

$$W_{p,q} = \{x : x \in L_p(V), \dot{x} \in L_q(V^*)\}, \quad \|x\|^2_{W_{p,q}} = \|x\|^2_{P_p(V)} + \|\dot{x}\|^2_{L_q(V^*)}.$$

Then $\left\{ W_{p,q}, \|\cdot\|_{W_{p,q}} \right\}$ is a Banach space and the embedding $W_{p,q} \hookrightarrow C(I, H)$ is continuous. If $V \hookrightarrow H$ is compact, then $W_{p,q} \hookrightarrow L_p(H)$ is compact (see Proposition 23.23 and problem 23.13 of [10]). Let $\mathcal{L}(X, Z)$ denote the space of all bounded linear operators from X to Z and A^* the dual of the operator A. Let

$$\mathcal{P}_{a,b} = \{B \in \mathcal{L}(V, V^*) : \| B \|_{\mathcal{L}(V, V^*)} \leq b \text{ and } \langle B\xi, \xi \rangle + a \| \xi \|_H \geq 0, \text{ for all } \xi \in V\}.$$

Consider the space of operators $\mathcal{L}(V, V^*)$ and suppose that it is given the strong (weak) operator topology which we denote by τ_{so} (τ_{wo}). Given this topology, $\mathcal{L}_s(V, V^*) \equiv (\mathcal{L}(V, V^*), \tau_{so})$ is a locally convex linear topological vector space which is sequentially complete. Similarly, $\mathcal{L}_w(V, V^*) \equiv (\mathcal{L}(V, V^*), \tau_{wo})$ with the weak operator topology τ_{wo} is also a sequentially completely locally convex topological space.

We introduce the following assumptions.

(A1) $A : I \times V \mapsto V^*$:

(1) $t \mapsto A(t, x)$ is measurable.

(2) $x \mapsto A(t, x)$ is uniformly monotone and hemicontinuous; i.e., there exists a constant $c > 0$ such that

$$\langle A(t, x_1) - A(t, x_2), x_1 - x_2 \rangle \geq c \| x_1 - x_2 \|_V^p,$$

for all $x, y \in V, t \in I$; $A(t, x + sy) \to A(t, x)$ weakly in V^*, for all $x, y \in V$ as $s \to 0$.

(3) There exist $c_1, c_2, c_3 > 0$ and $c_4 \in L_q(I, R_+)$ such that

$$\langle A(t, x), x \rangle \geq c_1 \| x \|_V^{p-1} - c_2, \text{ for all } x \in V, t \in I;$$

$$\| A(t, x) \|_{V^*} \leq c_4(t) + c_3 \| x \|_V^{p-1}, \text{ for all } x \in V, t \in I.$$

(G1) $g : I \times H \mapsto V$:

(1) g is measurable in the first variable and continuous in the second argument.

(2) there exist $\alpha \geq 0$ and $h \in L_q(I, R_+)$ such that

$$\| g(t, x) \|_{V^*} \leq h(t) + \alpha \| x \|_H^{2/q}, \text{ for all } x \in V, t \in I.$$

(3) g is locally Lipschitz continuous with respect to x, for any $b > 0$, there exists $L(b)$ such that $x_1, x_2 \in H, \| x_1 \|_H, \| x_2 \|_H \leq b$,

$$\| g(t, x_1) - g(t, x_2) \|_{V^*} \leq L(b) \| x_1 - x_2 \|_H \text{ for all } t \in I.$$

For $x \in L_p(V)$, we set

$$A(x)(t) = A(t, x(t)), G(x)(t) = g(t, x(t)), t \in I.$$

$A : L_p(v) \mapsto L_q(V^*)$ is bounded uniformly monotone hemicontinuous and coercive.

$G : L_p(V) \mapsto L_q(V^*)$ is bounded. Let $B \in \mathcal{P}_{a,b}$ be an arbitrary fixed operator.

$$\left.\begin{array}{l} \dot{x} + A(t, x) + Bx = g(t, x) \\ x(0) = x_0. \end{array}\right] \quad \ldots (2.1)$$

By Ahmed[4], we get the following lemma.

Lemma 2.1 — Suppose that the embedding $V \hookrightarrow H$ is compact. Then, whenever $x^n \to x^0$ weakly in $W_{p,q}$ $G(x^n) \to G(x^0)$ in $L_q(V^*)$. □

Remark : It is convenient to write system (2.1) as an operator equation in

$$W^0_{p,q} \equiv \{x \in W_{p,q}\,;\, x(0) = x_0\}$$

$$\left.\begin{aligned} &\dot{x} + A(x) + Bx = G(x), \\ &x \in W^0_{p,q}, \end{aligned}\right\}. \qquad \ldots (2.2)$$

The purpose of this section is to present an existence result for eq. (2.1) based on Galerkin approximation. At first, we give an *a priori* bound and prove the uniqueness of the solution.

Lemma 2.2 — There exists $b > 0$ such that

$$\| x \|_{C(I,H)} \le b, \| x \|_{L_p(V)} \le b, \| \dot{x} \|_{L_q(V^*)} \le b,$$

for any solution x if one exists of eq. (2.1).

PROOF : If x is any solution of (2.1), then for each $t \in I$,

$$\langle\langle \dot{x}, x \rangle\rangle_t + \langle\langle A(x), x \rangle\rangle_t + \langle\langle Bx, x \rangle\rangle_t = \langle\langle G(x), x \rangle\rangle_t.$$

Using the assumptions and the Cauchy inequality, for any $\varepsilon > 0$, we have

$$\frac{1}{2}\left(\| x(t) \|_H^2 - \| x(0) \|_H^2 \right) + \int_0^t \left(c_1 \| x(\sigma) \|_V^p - c_2 \right) d\sigma$$

$$\le a \int_0^t \| x(\sigma) \|_H^2 \, d\sigma + \int_0^t \| g(t, x(\sigma) \|_{V^*} \| x(\sigma) \|_V \, d\sigma. \qquad \ldots (2.3)$$

From (2.3), we have

$$\| x(t) \|_H^2 + 2c_1 \int_0^t \| x(\sigma) \|_V^p \, d\sigma \le 2c_2 T + \| x(0) \|_H^2 + 2a \int_0^t \| x(\sigma) \|_H^2 \, d\sigma$$

$$+ \; (2/q\, \varepsilon^q) \int_0^t (h(\sigma) + \alpha \| x(\sigma) \|_H^{2/q})^q \, d\sigma + (2\, \varepsilon^p / p) \int_0^t \| x(\sigma) \|_V^p \, d\sigma.$$

Choosing $\varepsilon > 0$ sufficiently small and $h \in L_q(I, R_+)$, one can easily verify that there exist positive constants c_5, c_6 and c_7 such that

$$\| x(t) \|_H^2 + c_5 \| x \|_{L_p(V)}^{p} \le c_6 + c_7 \int \| x(\sigma) \|_H^2 \, d\sigma. \qquad \ldots (2.4)$$

From the Gronwall's lemma it follows from the above inequality that

$$\| x(t) \|_H \leq c_s \text{ for all } t \in I,$$

for some constant c_3 depending on c_6 and c_7. Again, by virtue of assumptions (3) of (A1), (2) of (G1), definition of $\mathcal{P}_{a,b}$ and inequality (2.4), it is easy to verify that there exist positive constants c_9, c_{10} such that

$$\| x \|_{L_p(V)} \leq c_9, \| \dot{x} \|_{L_q(V^*)} \leq c_{10}.$$

Choosing $b = max\ \{c_8, c_9, c_{10}\}$ the assertion follows. □

Lemma 2.3 — Suppose that the embedding $V \hookrightarrow H$ is compact. The solution of (2.1), if one exists, is unique.

PROOF : Let $x_1, x_2 \in W^0_{p,q}$ be two solutions of (2.1). Using integration by parts and monotonicity of the operator A and also $B \in \mathcal{P}_{a,b}$, we obtain

$$\frac{1}{2} \| x_1(t) - x_2(t) \|_H^2 + c \| x_1 - x_2 \|^{p}_{L'_p(V)} \leq a \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H^2 \, d\sigma$$

$$+ \int_0^t \langle g(\sigma, x_1(\sigma)) - g(\sigma, x_2(\sigma)), x_1(\sigma) - x_2(\sigma) \rangle_{V^*, V} \, d\sigma.$$

By virtue of assumption (G1), Lemma 2.2, and the Cauchy inequality, for any $\varepsilon > 0$, we have

$$\frac{1}{2} \| x_1(t) - x_2(t) \|_H^2 + c \| x_1 - x_2 \|^{p}_{L'_p(V)} \leq | a | \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H^2 \, d\sigma$$

$$+ \int_0^t \| g(\sigma, x_1(\sigma)) - g(\sigma, x_2(\sigma)) \|_{V^*} \| x_1(\sigma) - x_2(\sigma) \|_V \, d\sigma$$

$$\leq | a | \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H^2 \, d\sigma + L(b) \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H \| x_1(\sigma) - x_2(\sigma) \|_V \, d\sigma$$

$$\leq | a | \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H^2 \, d\sigma + (L(b)/2\varepsilon) \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H^2 \, d\sigma$$

$$+ (L(b)\, \varepsilon/2) \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_V^2 \, d\sigma.$$

Using the continuous embedding $L_p^t(V) \hookrightarrow L_2^t(V)$, we obtain

$$\| x_1(t) - x_2(t) \|_H^2 + 2c \| x_1 - x_2 \|_{L_p^t(V)}^{p}$$

$$\leq (2|a| + L(b)/\varepsilon) \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H^2 \, d\sigma + L_1 \varepsilon \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_V^p \, d\sigma,$$

where L_1 is a constant depending on b and the embedding constant. Consequently, for sufficiently small $\varepsilon > 0$, there exists a constant $c' > 0$ such that

$$\| x_1(t) - x_2(t) \|_H^2 + c' \| x_1 - x_2 \|_{L_p^t(V)}^{p} \leq (2|a| + L(b)/\varepsilon) \int_0^t \| x_1(\sigma) - x_2(\sigma) \|_H^2 \, d\sigma$$

Using the Gronwall lemma, uniqueness follows from the above inequality. □

Theorem* 2.1 — *Under the assumptions (A1) and (G1), the evolution (2.1) has a unique solution.

PROOF : Let the sequence $\{x_0^n\}$ be an approximation of the given initial state $x_0 \in H$, i.e., $x_0^n \in H_n$, $x_0^n \to x_0$ in H as $n \to \infty$.

We consider the sequence

$$x^n(t) = \sum_{k=1}^{n} C_{k,n}(t) \, e_k$$

and seek a function x^n such that

$$\left.\begin{aligned} &\langle \dot{x}^n(t), e_j \rangle + \langle A(t, x^n(t)), e_j \rangle + \langle Bx^n(t), e_j \rangle \\ &\qquad = \langle g(t, x^n(t)), e_j \rangle, j = 1, 2, \ldots, n; \\ &x^n(0) = x_0^n; \\ &x^n \in L_p(I, H_n), \dot{x}^n \in L_q(I, H_n). \end{aligned}\right] \qquad \ldots (2.5)$$

It follows from the existence theorem of Carathéodory for ordinary differential equation in R^n and Lemma 2.3 that, for each $n \in N$, the finite dimensional system (2.5) has a unique solution x^n. It can be seen from Lemma 2.2 that $\{x^n\}$ is contained in a bounded subset of $W_{p,q}$. Hence, by assumption (A1), $\{A(x^n)\}$ is bounded in $L_q(V^*)$. Since $B \in \mathcal{P}_{a,b}$, $\{B x^n\}$ is bounded in $L_q(V^*)$ and also since $L_p(V)$ and $L_q(V^*)$ are reflexive Banach spaces, there exists a subsequence, again denoted by $\{x^n\}$, an element $x \in L_p(V)$ with its distributional derivative $\dot{x} \in L_q(V^*)$ and $W \in L_q(V^*)$ such that

$$x^n \to x^0 \text{ weakly in } L_p(V),$$

$$\dot{x}^n \to \dot{x}^0 \text{ weakly in } L_q(V^*),$$

$$A(x^n) \to W \text{ weakly in } L_q(V^*),$$

$$B x^n \to B x^0 \text{ weakly in } L_q(V^*),$$

as $n \to \infty$.

Combining the assumptions with Lemma 2.1, we have

$$G(x^n) \to G(x^0) \text{ in } L_q(V^*),$$

$$x^n(0) \to x_0 \text{ in } H,$$

$$x^n(T) \to z \text{ weakly in } H,$$

as $n \to \infty$.

Let $\psi \in C^\infty(I, R)$ and $v \in H_n$. Using eq. (2.5) and integration by parts, one can obtain

$$(x^n(T), \psi(T) v) - (x^n(0), \psi(0) v)$$

$$= \int_0^T \langle x^n(t), \dot{\psi}(t) v \rangle dt + \int_0^T \langle g(t, x^n(t)) - A(t, x^n(t)) - Bx^n(t), \psi(t) v \rangle dt.$$

Letting $n \to \infty$, we have

$$(z, \psi(T) v) - (x_0, \psi(0) v) = \langle\langle G(x^0) - W - Bx^0, \psi v \rangle\rangle + \langle\langle \psi v, x^0 \rangle\rangle.$$

Using this, one can easily verify that the limit elements x^0, W, z, Bx^0 satisfy

$$\left\{ \begin{array}{l} \dot{x}^0 + W + Bx^0 = G(x^0), x^0 \in W_{p,q}, \\ x(0) = x_0, x^0(T) = z. \end{array} \right\}$$

Again us g eq. (2.5) and integration by parts, we have

$$\frac{1}{2}\left(\| x^n(T) \|_H^2 - \| x^n(0) \|_H^2 \right) = \langle\langle G(x^n) - A(x^n) - Bx^n, x^n \rangle\rangle.$$

By virtue of the fact that

$$\liminf_{n \to \infty} \| x^n(T) \|_H \geq \| x^0(T) \|_H,$$

we obtain

$$\langle\langle W, x^0 \rangle\rangle \leq \liminf_{n \to \infty} \langle\langle A(x^n), x^n \rangle\rangle$$

$$\leq \limsup_{n\to\infty} \langle\langle A(x^n), x^n \rangle\rangle$$

$$\leq \langle\langle G(x^0), x^0 \rangle\rangle - \langle\langle B(x^0), x^0 \rangle\rangle + \frac{1}{2}\left(\| x^0(0) \|_H^2 - \| x^0(T) \|_H^2 \right)$$

$$= \langle\langle W, x^0 \rangle\rangle.$$

Since A is monotone and hemicontinuous, $W = A(x^0)$. (see [10]). Thus the limit element x^0 satisfies eq. (2.2) and hence is a solution of (2.1). The uniqueness follows from Lemma 2.3. This finishes the proof of the theorem. □

3. Existence of Optimal Operators

In this section, we consider the identification problem (P) of nonlinear system (1.1) : Find a $B^0 \in \mathcal{P}_{a,b}$, so that $J(B^0) \leq J(B)$ for all $B \in \mathcal{P}_{a,b}$, where

$$J(B) = \int_0^T f(t, x(t))\, dt. \qquad \dots (3.1)$$

Here we assume $f : I \times H \to \overline{R}$.

Lemma 3.1 — Consider the identification problem (P), and that the assumptions of Theorem 2.1 hold. Let the function $t \to f(t, x)$ be measurable for each $x \in H$, $x \to f(t, x)$ lower semicontinuous on H for almost all t. Then the functional $B \to J(B)$ is also lower semi continuous on $\mathcal{L}_s(V, V^*)$.

PROOF : Let $\{B^n\} \in \mathcal{L}_s(V, V^*)$ and suppose $B^n \to B^0\ (\tau_{so})$. Clearly $B^0 \in \mathcal{L}_s(V, V^*)$ and hence it follows from Theorem 2.1 that the system

$$\left.\begin{array}{l} \dot{x} + A(t, x) + B^0 x = g(t, x) \\ x(0) = x_0 \end{array}\right\}$$

has a unique sol ion $x^0 \in L_p(I, V) \cap C(I, H)$. Similarly, corresponding to each $B^n \in \mathcal{L}_s(V, V^*)$, the system

$$\left.\begin{array}{l} \dot{x} + A(t, x) + B^n x = g(t, x) \\ x(0) = x_0 \end{array}\right\}$$

has a unique solution $x^n \in L_p(I, V) \cap C(I, H)$. Defining $y^n = x^n - x^0$, one observes that y^n is the solution of the problem

$$\left.\begin{array}{l} \dot{y}^n + A(t, x^n) - A(t, x^0) + B^n y^n \\ \quad = (B^0 - B^n)\, x^0 + (g(t, x^n) - g(t, x^0)) \\ y^n(0) = 0. \end{array}\right] \qquad \dots (3.2)$$

Scalar multiplying the first eq. of (3.2) on either side by y^n and using Young's inequality, we have

$$\frac{1}{2}\| y^n(t)\|_H^2 + c\| y^n\|^{p}_{L_p^t(V)} \le |a| \int_0^t \| y^n(\sigma)\|_H^2 \, d\sigma$$

$$+ L(b) \int_0^t \| y^n(\sigma)\|_H \| y^n(\sigma)\|_{V^*} \, d\sigma \times \int_0^t \| (B^0 - B^n) x^0(\sigma)\|_V^* \| y^n(\sigma)\|_V \, d\sigma$$

$$\le |a| \int_0^t \| y^n(\sigma)\|_H^2 \, d\sigma + \frac{L(b)}{2\varepsilon} \int_0^t \| y^n(\sigma)\|_H^2 \, d\sigma + \frac{L(b)\,\varepsilon}{2} \int_0^t \| y^n(\sigma)\|_V^2 \, d\sigma$$

$$+ \frac{1}{2\varepsilon} \int_0^t \| (B^0 - B^n) x^0(\sigma)\|_{V^*}^2 \, d\sigma + \frac{\varepsilon}{2} \int_0^t \| y^n(\sigma)\|_V^2 \, d\sigma$$

$$\le \left(|a| + \frac{L(b)}{2\varepsilon} \right) \int_0^t \| y^n(\sigma)\|_H^2 \, d\sigma + \frac{\varepsilon (L(b)+1)}{2} \int_0^t \| y^n(\sigma)\|_V^2 \, d\sigma$$

$$+ \frac{1}{2\varepsilon} \int_0^t \| (B^0 - B^n) x^0(\sigma)\|_{V^*}^2 \, d\sigma.$$

Choosing $\varepsilon > 0$ sufficiently small and the continuous embedding $L_p^t(V) \hookrightarrow L_2^t(V)$ with the embedding constant b', we obtain

$$\| y^n(t)\|_H^2 + 2c\| y^n\|^{p}_{L_p^t()} \le \left(2|a| + \frac{L(b)}{\varepsilon} \right) \int_0^t \| y^n(\sigma)\|_H^2 \, d\sigma$$

$$+ L_2 \varepsilon \int_0^t \| y^n(\sigma)\|_V^p \, d\sigma + \frac{1}{\varepsilon} \int_0^t \| (B^0 - B^n) x^0(\sigma)\|_{V^*}^{2} \, d\sigma,$$

where $L_2 = (L(b) + 1)\, b'$.

Taking $\varepsilon = \dfrac{c}{L_2}$,

$$\| y^n(t)\|_H^2 + c\| y^n\|^{p}_{L_p^t(V)} \le \left(2|a| + \frac{L_2 L(b)}{c} \right) \int_0^t \| y^n(\sigma)\|_H^2 \, d\sigma$$

$$+ \frac{L_2}{c} \int_0^t \| (B^0 - B^n) x^0(\sigma)\|_{V^*}^2 \, d\sigma$$

Defining $\psi^n(t) = \| y^n(t) \|_H^2 + c \int_0^t \| y^n(\sigma) \|_V^p \, d\sigma$, it follows from the above inequality that

$$\psi^n(t) \leq \left(2|a| + \frac{L_2 L(b)}{c} \right) \int_0^t \psi^n(\sigma)\, d\sigma + \frac{L_2}{c} \int_0^t \| (B^0 - B^n)\, x^0(\sigma) \|_{V^*}^2 \, d\sigma.$$

Using Gronwall's lemma, one conclude that

$$\psi^n(t) \leq \exp\left(\left(2|a| + \frac{L_2 L(b)}{c} \right) T \right) \frac{L_2}{c} \int_0^t \| (B^0 - B^n)\, x^0(\sigma) \|_{V^*}^2 \, d\sigma, \qquad \ldots (3.3)$$

for all $t \in I$. Since $B^n \to B^0$ in the strong operator topology in $\mathcal{L}_s(V, V^*)$ and $x^0 \in L_p(I, V)$, it is clear that $\| (B^0 - B^n)\, x^0 \|_{V^*} \to 0$ almost everywhere on I and also there exists a finite number γ such that

$$\| (B^0 - B^n)\, x^0(t) \|_{V^*} \leq \gamma \| x^0(t) \|_V \text{ for all } t \in I.$$

Hence by the Lebesque dominated convergence theorem, it follows that $\psi^n(t) \to 0$ as $n \to \infty$ uniformly on I. Thus one may conclude from (3.3) that $x^n \to x^0$ in $C(I, H)$ as well as in $L_p(I, V)$, and in particular $x^n(t) \to x^0(t)$ in H for all $t \in I$.

Define

$$J(B^n) = \int_I f(t, x^n(t))\, dt, \text{ and } J(B^0) = \int_I f(t, x^0(t))\, dt,$$

where x^n and x^0 are the solutions of the system (2.1) corresponding to B^n and B^0, respectively. Since, by assumption, for almost all $t \in I, x \to f(t, x)$ is lower semicontinuous on H, we have

$$f(t, x^0(t)) \leq \liminf_n f(t, x^n(t)) \text{ almost all on } I,$$

and consequently, by Fatou's lemma

$$\int_I f(t, x^0(t))\, dt \leq \liminf_n \int_I f(t, x^n(t))\, dt.$$

Clearly, this is equivalent to

$$J(B^0) \leq \liminf_n J(B^n).$$

This completes the proof of the lemma. □

Lemma 3.2 — The set $\mathcal{P}_{a,b}$ considered as a subset of $\mathcal{L}(V, V^*)$ is sequentially compact in the strong operator topology τ_{so}.

PROOF : For proof, see Lemma 1.2 of [3]. □

Theorem 3.1 — *Suppose $f(t, x) > -\infty$ for $(t, x) \in I \times H$. Then under the assumptions of Theorem 2.1 and Lemma 2.1, 2.2, there exists a $B^0 \in \mathcal{P}_{a,b}$ such that*

$$J(B^0) \leq J(B) \text{ for all } B \in \mathcal{P}_{a,b}.$$

PROOF : Define $l = \inf \{J(B), B \in \mathcal{P}_{a,b}\}$. Since $f(t, x) > -\infty$ for $(t, x) \in I \times H$, the infimum is well defined and $l > -\infty$. Let $\{B^k\}$ be a minimizing sequence from $\mathcal{P}_{a,b}$, i.e., $\lim_k J(B^k) = l$. Then by Lemma 3.2, there exists $\{B^{k_i}\} \subset \{B^k\}$ relabeled as $\{B^k\}$ and a $B^0 \in \mathcal{P}_{a,b}$ such that $B^k \to B^0\ (\tau_{so})$. Since $B \to J(B)$ is lower semicontinuous with respect to the topology τ_{so} (see Lemma 3.1) and $B^k \to B^0\ (\tau_{so})$, we have

$$l \leq J(B^0) \leq \liminf_k J(B^k) \leq \lim_k J(B^k) = l.$$

Hence, $J(B^0) = l$ implying that $J(\cdot)$ attains its infimum on $\mathcal{P}_{a,b}$. This completes the proof. □

4. Necessary Conditions of Optimality

We consider the necessary conditions of optimality for the identification problem (P). First, we note that usually mapping $B \to x(B)$ from $\mathcal{L}(V, V^*)$ to $L_p(I, V)$ is unique. In this section we assume that $p = q = 2$. In order to derive the necessary optimality conditions, we need some additional assumptions

(A) $A : I \times V \to V^*$;

(1) A satisfies condition (A1).

(2) A is Fréachet-differentiable with respect to $x \in V$ and for each $\xi \in W_{2,2}$ the Nemytski operator A_x defined by

$$A_x(\xi)(t) \equiv A_x(t, \xi(t)) \in \mathcal{L}(L_2(V), L_2(V^*)).$$

Further, the map $\xi \to A_x(\xi)$ is continuous and bounded on bounded subset of $W_{2,2}$.

(G) $g : I \times H \to V^*$;

(1) g satisfies condition ($G1$).

(2) g is measurable in the first variable and continuous in the second arguments.

(3) g is Fréchet-differentiable with the respect to $x \in H$ and the mapping

$$g_x(x) : L_2(H) \to \mathcal{L}(L_2(H), L_2(V^*))$$

are bounded and continuous, where

$$g_x(x)(t) \equiv g_x(t, x(t)).$$

(F) $f : I \times H \to R \cup \{\infty\}$ is continuous and Fréchet-differentiable in $x \in H$, so that $f_x \in L_2(I, H)$.

For the proof of necessary conditions of optimality, we shall make use of the Gâteaux differential of x at B^0 in the direction B, defined by

$$\hat{x}(B^0, B) = w - \lim_{\varepsilon \to 0} \left(\frac{x(B^0 + \varepsilon B) - x(B^0)}{\varepsilon} \right)$$

exists and that it is the solution of a related differential equation.

We present in the following lemma the Gâteaux differentiability of the mapping $B \to x(B)$ in the weak sense.

Lemma 4.1 — Under the assumptions of Theorem 3.1, (*A*), (*G*) and (*F*). Let $x(B)$ denote the (weak) solution of the Cauchy problem (2.1) corresponding to $B \in \mathcal{P}_{a,b}$. Then at each point $B \in \mathcal{P}_{a,b}$ the function $B \to x(B)$ has a weak Gâteaux differential in the direction $B - B^0$ denoted $\hat{x}(B^0, B - B^0)$, and it is the solution of the Cauchy problem

$$\left\{ \begin{array}{l} \dot{e} + A_x(x)\, e + B^0 e = (B^0 - B)\, x + g_x(t, x)\, e \\ e(0) = 0 \end{array} \right\} \qquad \text{... (4.1)}$$

satisfying $\hat{x} \in L_2(I, V) \cap L_\infty(I, H)$, where $x(B^0)$ is the solution of (2.1) corresponding to $B = B^0$.

PROOF Let $B^0, B \in \mathcal{P}_{a,b}$. Since $\mathcal{P}_{a,b}$ is a closed convex subset of $\mathcal{L}_w(V, V^*)$, $B^\varepsilon \equiv B^0 + \varepsilon(B - i \quad) \in \mathcal{P}_{a,b}, x^\varepsilon \equiv x(B^\varepsilon), x \equiv x(B^0)$ for $0 \le \varepsilon \le 1$.

Define
$$\varphi^\varepsilon \equiv \frac{(x(B^\varepsilon) - x(B^0))}{\varepsilon}.$$

Then using the differential eq. (2.1), one obtains

$$\left. \begin{array}{l} \dot{\varphi}^\varepsilon + \dfrac{A(t, x^\varepsilon) - A(t, x)}{x^\varepsilon - x} \varphi^\varepsilon + B^\varepsilon \varphi^\varepsilon \\ = (B^0 - B)\, x + \dfrac{g(t, x^\varepsilon) - g(t, x)}{x^\varepsilon - x} \varphi^\varepsilon \\ \varphi^\varepsilon(0) = 0. \end{array} \right] \qquad \text{... (4.2)}$$

We show that the Gâteaux differential given by the weak limit (in $L_2(I, V)$) a subsequence thereof exists; and it is the (weak) solution of (4.1). Since $(B^0 - B)\, x \in L_2(I, V^*), A_x(t, x)$

$\in L_2(I, V^*)$ and $g_x(t, x) \in L_2(I, V^*)$, the set $\{\varphi^\varepsilon : \varepsilon \in [0, 1]\}$ is contained in a bounded subset of $L_2(I, V) \cap L_\infty(I, H)$. Hence from every sequence $\varphi^\varepsilon \equiv \varphi^{\varepsilon_n}$, with $\varepsilon_n \in [0, 1]$ and $\varepsilon_n \to 0$ one can extract a subsequence relabeled as $\{\varphi^n\}$ and a $\varphi^0 \in L_2(I, V) \cap L_\infty(I, H)$ such that $\varphi^n \to \varphi^0$ weakly in $L_2(I, V)$. Hence, the Gâteaux differential of x exists and it is given by $\hat{x}(B^0, B - B^0) = \varphi^0$. It remains to show that φ^0 is a solution of (4.1). Indeed, since $\frac{A(x^\varepsilon) - A(x)}{\varepsilon} = \frac{A(x^\varepsilon) - A(x)}{x^\varepsilon - x} \frac{x^\varepsilon - x}{\varepsilon} \to$ $A_x(x)\varphi^0$ weakly in $L_2(I, V^*)$, (weak and weak* convergence being equivalent in reflexive Banach spaces), $B^n \varphi^n = B^0 \varphi^n + \varepsilon_n (B - B^0)\varphi^n \to B^0 \varphi^0$ weakly in $L_2(I, V^*)$ and $\frac{g(t, x^\varepsilon) - g(t, x)}{\varepsilon} =$ $\frac{g(t, x^\varepsilon) - g(t, x)}{x^\varepsilon - x} \frac{x^\varepsilon - x}{\varepsilon} \to g_x(t, x)\varphi^0$ weakly in $L_2(I, V^*)$, it follows from (4.2) that $\dot{\varphi}^n \in L_2(I, V^*)$ and $\dot{\varphi}^n \to \eta$ weakly in $L_2(I, V^*)$ for a suitable $\eta \in L_2(I, V^*)$, and that η is the distributional derivative of φ^0. Hence, φ^0 satisfies the equality

$$\dot{\varphi}^0 + A_x(t, x)\varphi^0 + B^0 \varphi^0 = (B^0 - B)x + g_x(t, x)\varphi^0$$

in the sense of vector-valued distribution in V^*. Since $\varphi^0 \in L_2(I, V)$ and $\dot{\varphi}^0 \in L_2(I, V^*)$, it is clear that $\varphi^0 \in C(I, H)$ and $\varphi^0(0)$ is well defined and equals $\varphi^n(0) = 0$ for all n. Hence, φ^0 satisfies the differential eq. (4.1) and one may identify φ^0 as e. This completes the proof. □

With the help of the above lemma, we prove the following necessary conditions for optimality of the operator B^0.

Theorem 4.1 — *Under the assumptions of Lemma 4.1, consider the system (2.1) and the identification problem (P) with $J(B) = \int_I f(t, x(B)(t))\,dt$. Then the best approximation B^0 for the unknown operator is determined by the simultaneous solution of the system equation :*

$$\left.\begin{aligned} &\dot{x} + A(t, x) + B^0 x = g(t, x), \\ &x(0) = x_0, B^0 \in \mathcal{P}_{a,b} \end{aligned}\right\}$$

the adjoint equation

$$\left.\begin{aligned} &-\dot{z}(t) + (A_x(t, x(t)))^* z(t) + (B^0)^* z(t) - (g_x(t, x(t)))^* z(t) \\ &\quad = f_x(t, x(t)), \text{ for } all\ t \in [0, T) \\ &z(T) = 0, \end{aligned}\right\} \qquad \dots (4.3)$$

and the inequality

$$\int_I \langle B^0 x(t), z(t) \rangle_{V, V^*} dt \geq \int_I \langle Bx(t), z(t) \rangle_{V, V^*} dt \qquad \dots (4.4)$$

for all $B \in \mathcal{P}_{a,b}$.

PROOF : Since $B \to x(B)$ has (weak) Gâteaux differential on $\mathcal{P}_{a,b}$, it follows that J as defined above also has a Gâteaux differential. Denote $x \equiv x(B^0)$. Then in order that J attains its minimum at $B^0 \in \mathcal{P}_{a,b}$, it is necessary that

$$J'_{B^0}(B - B^0) \equiv \lim_{\varepsilon \to 0} \left(\frac{J(B^\varepsilon) - J(B^0)}{\varepsilon} \right) \geq 0 \qquad \text{... (4.5)}$$

for all $B \in \mathcal{P}_{a,b}$. Using the result of Lemma 4.1, it follows from the above that

$$J'_{B^0}(B - B^0) = \int_I \langle f_x(t, x(t)), \varphi^0(t) \rangle_H \, dt \geq 0 \qquad \text{... (4.6)}$$

for all $B \in \mathcal{P}_{a,b}$, where $\varphi^0(t)$ is the Gâteaux differential as given by Lemma 4.1. Using (4.1) and (4.6), we obtain the adjoint equation :

$$\left.\begin{aligned} &-\dot{z} + (A_x(t,x))^* z + (B^0)^* z - (g_x(t,x))^* z \\ &\quad = f_x(t,x), \text{ for all } t \in [0, T), \\ &z(T) = 0. \end{aligned}\right] \qquad \text{... (4.7)}$$

Reversing the flow of time $t \to T - t$, it follows from Theorem 2.1 that the system (4.7) also has a unique weak solution $z \in L_2(I, V) \cap C(I, H)$. Utilizing (4.7) into the inequality (4.6) and integrating by parts, we obtain

$$\int_I \langle \dot{\varphi}^0 + A_x(t, x(t))\, \varphi^0 + B^0 \varphi^0 - g_x(t, x(t))\, \varphi^0, z(t) \rangle_{V^*, V} \, dt \geq 0. \qquad \text{... (4.8)}$$

From (4.1) and (4.8), we obtain

$$\begin{aligned} &\int_I \langle f_x(t, x(t)), \varphi^0(t) \rangle_H \, dt \\ &\quad = \int_I \langle \dot{\varphi}^0 + A_x(t, x(t))\, \varphi^0(t) + B^0 \varphi^0(t) - g_x(t, x(t))\, \varphi^0(t), z(t) \rangle_{V^*, V} \, dt \\ &\quad = \int_I \langle (B - B^0)\, x(t), z(t) \rangle_{V^*, V} \, dt \geq 0. \end{aligned}$$

for all $B \in \mathcal{P}_{a,b}$. Hence we obtain (4.4). This completes the proof. □

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