Indian J. pure appl. Math., **32**(5) : 649-656, May 2001

LIE IDEALS AND ANNIHILATOR CONDITIONS ON POWER VALUES OF COMMUTATORS WITH DERIVATION

VINCENZO DE FILIPPIS

Dipartimento di Mathematica, Università di Messina, Salita Sperone 31
98166 *Messina, Italy, (E-mail address: enzo@dipmat.unime.it)*

(*Received* 12 *January* 2000; *after revision* 11 *August* 2000; *accepted* 17 *October* 2000)

Let R be a prime ring of characteristic $\neq 2$ with a derivation $d \neq 0$, L a non-central Lie ideal of R, $a \in R$ such that $a\,[d(u), u]^n$ is central, for all $u \in L$. We prove that $a = 0$ unless R satisfies s_4 the standard identity in 4 variables.

Key Words : Prime Ring; Generalized Polynomial Identity; Differential Identity

Let R be a prime ring. For all $x, y \in R$ we will denote $[x, y] = xy - yx$ the commutator of x and y and let $[x, y]_2 = [[x, y], y] = [x, y]\,y - y\,[x, y]$. A well-known result of Posner says that if R is a prime ring and d a non-zero derivation of R such that $[d(x), x] \in Z(R)$, the centre of R, for any $x \in R$, then R is commutative[13]. Lanski[10] generalizes the result of Posner to Lie ideals. More precisely Lanski prove that if R is a prime ring, d a non-zero derivation of R, L a non-commutative Lie ideal of R, such that $[d(x), x] \in Z(R)$, for any $x \in L$, then *char* $(R) = 2$ and R satisfies s_4, the standard identity in 4 variables. Recently in [2], we studied a more general situation and we proved that if R is a prime ring of characteristic different from 2, d a non-zero derivation of R, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer such that $[d(x), x]^n \in Z(R)$, for any $x \in L$, then R satisfies s_4. The present paper continues the investigation on the set $S = \{[d(u), u]^n, u \in L\}$, where $n \geq 1$ is a fixed, integer and L is a non-central Lie ideal of the prime ring R. In particular, we will study the left annihilator of S defined as follows : $Ann_R(S) = \{x \in R : xS = (0)\}$. Explicitly we prove the following:

Theorem 1 — *Let R be a prime ring of characteristic different from 2, d a non-zero derivation of R, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer and $a \in R$. If $a\,[d(u), u]^n \in Z(R)$, for all $u \in L$, then either $a = 0$ or R satisfies s_4, the standard identity in 4 variables.*

We will proceed by first proving that :

Theorem 2 — *Let R be a prime ring of characteristic different from 2, d a non-zero derivation of R, L a non-central Lie idea of R, $n \geq 1$ a fixed integer and $a \in R$. If $a\,[d(u), u]^n = 0$, for all $u \in L$, then $a = 0$.*

In order to prove Theorems 1 and 2 we need to mention the following results contained in [5] and [2] :

***Theorem** A^5 — Let R be a prime ring of characteristic different from 2, d a non-zero derivation of R, L a non-central Lie ideal of R. Then the left annihilator of $\{[d(u), u], u \in L\}$ is zero.*

***Theorem** B^2 — Let R be a prime ring of characteristic different from 2, d a non-zero derivation of R, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer. If $[d(u), u]^n = 0$ for all $u \in L$, then R is commutative. If $[d(u), u]^n \in Z(R)$ for all $u \in L$, then R satisfies s_4, the standard identity in 4 variables.*

Notice that if a is a non-zero central element of R the results follow from [2]. In fact, since R is prime, any non-zero central element of R cannot be a zero-divisor. Therefore, if $a \in Z(R)$ and, for all $u \in L, a\,[d(u), u]^n = 0$ (respectively $a\,[d(u), u]^n \in Z(R)$), then we have that $[d(u), u]^n = 0$ (respectively $[d(u), u]^n \in Z(R)$). In these cases the conclusions are in Theorem B.

In all that follows R will be a prime ring of characteristic different from 2, L a non-central Lie ideal of R, d a non-zero derivation of R, $n \geq 1$ a fixed integer and $a \in R - Z(R)$. In addition C always denote the extended centroid of R, Q its two-sided Martindale quotient ring, $T = Q_{*C}\, C\{X\}$ the free product over C of the C-algebra Q and the free C-algegra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $x_1, \dots, x_n$... For more details about these objects we refer the reader to [1] and [3].

We dedicate the first part of this note to prove Theorem 1. So we suppose that, for all $u \in L, a\,[d(u), u]^n = 0$. We being with :

Remark 1 : Since we assume that char $(R) \neq 2$ and L is not central, by a result of Herstein [7], $[I, R] \subseteq L$, for some $I \neq 0$ an ideal of R, and also L is not commutative. Therefore, we will assume, without loss of generality that $L = [I, I]$. Hence $a\,[d([r_1, r_2]), [r_1, r_2]]^n = 0$, for all $r_1, r_2 \in I$.

Moreover if R is a domain, it follows that either $a = 0$ or $[d(u), u]^n = 0$ and in this last case we are done by Theorem B. Thus we will assume that R is not a domain.

Lemma 1 — If d is not inner then $a = 0$.

PROOF : Suppose by contradiction $a \neq 0$. Since by [11] I and Q satisfy the same generalized differential identities, then $a\,[d([r_1, r_2]), [r_1, r_2]]^n = 0$, for all $r_1, r_2 \in Q$. Therefore, Q satisfies $a\,[[d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]]^n$. In particular, since d in not inner, by [8] Q satisfies the generalized polynomial identity $a\,[[y_1, x_2] + [x_1, y_2], [x_1, x_2]]^n$ and so also its blended component $a\,[[y_1, x_2], [x_1, x_2]]^n$. By Martindale's theorem in [12], it follows that Q is a ring dense of linear transformations of a vector space V over a division ring D. Suppose that $dim_D\, V \geq 3$. Let $v \in V$. We will prove first that v and av are linearly D-dependent. In fact, if v, av are linearly independent, then there exists $w \in V$ such that v, av, w are linearly independent. By the density of Q there exist $r_1, r_2, r_3 \in Q$ such that

$$r_1\, v = 0,\ r_1\, w = w,\ r_1\, (a\, v) = w$$

$$r_2\, v = a\, v,\ r_2 w = v,\ r_2\, (a\, v) = w$$

$$r_3\, v = 0,\ r_3\, w = 0,\ r_3\, (a\, v) = w.$$

This implies that

$$[r_1, r_2]\, v = w, \ [r_1, r_2]\, w = -\, v, \ [r_3, r_2]\, v = w, \ [r_3, r_2]\, w = 0$$

and we get the contradiction $0 = a\,[[\,r_1, r_2], [r_3, r_2]]^n\, v = (-1)^n\, a\, v \neq 0$. Therefore, v, av must be linearly D-dependent and in this case standard arguments show that $a \in Z(Q) = C$, which contradicts the non-centrality of a.

Therefore, $dim_D\, V \leq 2$, that is Q is a simple GPI-ring and so it is a central simple algebra finite dimensional over its center. From Lemma 2 in [9] it follows that there exists a suitable field F such that $Q \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies the same generalized polynomial identity as Q. Thus by the above argument $k \leq 2$ and, since R is not commutative, $k = 2$. Consider the following subring of $M_2(F)$:

$$A = \{b \in M_2(F) : b\,[[r_1, r_2], [r_3, r_2]]^n = 0, \text{ for all } r_i \in M_2(F)\}$$

Obviously A is invariant under the action of all automorphisms of $M_2(F)$, in the sense of [6], then one of the following holds: either $A \subseteq F$ or $A = M_2(F)$. In the first case, we have the contradiction $a \in F$; in the second $M_2(F)$ satisfies the polynomial identity $y_1\,[[x_1, x_2], [x_3, x_2]]^n$. In this last case, if e_{ij} denote the usual matrix units in $M_2(F)$, we obtain the following contradiction:

$$0 = e_{11}\,[e_{12}, e_{11}], [e_{21}, e_{11}]]^n = (-1)^n\, e_{11} \neq 0.$$

Therefore, a must be zero. □

Remark 2 : Since in the case d is not inner, by Lemma 1 we are done, then from now on we may consider the only case when d is an inner derivation induced by an element $q \in Q$ (see [8]). In this situation R satisfies the generalized polynomial identity $a\,([q, [x_1, x_2]]_2)^n$. Moreover, by [3] R and Q satisfy the same generalized polynomial identities, so we have that $a\,([q, [r_1, r_2]]_2)^n = 0$, for all $r_1, r_2 \in Q$, and we may replace R by Q by assuming $R = Q$, $C = Z(R)$ and R is a C-algebra centrally closed.

Remark 3 : Recall that if B is a basis of Q over C, then any element of $T = Q_{*C}\, C\{x_1, ..., x_n\}$ can be written in the form $g = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B-monomials, that is $m_i = q_0 y_1 \cdots y_n q_n$, with $q_i \in B$ and $y_i \in \{x_1, ..., x_n\}$. In [3] it is showed that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if any α_i is zero. As a consequence, if $a_1, a_2 \in Q$ are linearly independent over C and $a_1\, g_1(x_1, \ldots, x_n) + a_2\, g_2(x_1, .., x_n) = 0 \in T$, for some $g_1, g_2 \in T$, then both $g_1(x_1, \ldots, x_n)$ and $g_2(x_1, \ldots, x_n)$ are the zero element of T.

This criterion is applied in the following lemma.

Lemma 2 — If R does not satisfy any non-trivial generalized polymonial identity, then $a = 0$.

PROOF : Since R does not satisfy any non-trivial generalized polynomial identities, we have that $a\,([q, [x_1, x_2]]_2)^n$ is the zero element in the free product $T = R_{*C}\, C\,\{x_1, x_2\}$ (see [3]), that is

$$a\,([x_1, x_2]^2\, q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\, q\,[x_1, x_2])^n = 0 \in T.$$

Suppose $aq \neq 0$ and a, aq linearly C-independent. We have

$$aq\,[x_1, x_2]^2\,([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^{n-1} = 0 \in T$$

and so

$$a\,([x_1, x_2]^2\,q - 2\,[x_1, x_2]\,q\,[x_1, x_2])\,([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^{n-1} = 0 \in T.$$

By Remark 3, it follows

$$[x_1, x_2]^2\,([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^{n-1} = 0$$

and also

$$([x_1, x_2]^2\,q - 2\,[x_1, x_2]\,q\,[x_1, x_2])\,\,([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^{n-1} = 0.$$

Therefore,

$$\begin{aligned}([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^n &= q\,[x_1, x_2]^2\,([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^{n-1}\\ &+ ([x_1, x_2]^2\,q - 2\,[x_1, x_2]\,q\,[x_1, x_2)]\,([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^{n-1} = 0,\end{aligned}$$

that is $[(q, [x_1, x_2]]_2)^n = 0 \in T$. Expanding this we see that

$$([q, [x_1, x_2]]_2)^{n-1}\,([q, [x_1, x_2]]_2) = 0$$

and since R has no non-zero GPI, we have, by Remark 3, that $([q, [x_1, x_2]]_2)^{n-1} = 0 \in T$. Continuing this process we will arrive at $[q, [x_1, x_2]]_2 = 0 \in T$. Therefore

$$q\,[x_1, x_2]^2 + [x_1, x_2]^2\,q - 2\,[x_1, x_2]\,q\,[x_1, x_2] = 0$$

and again by Remark 3, because R has no non-zero GPI, we get the contradiction that $q \in C$.

Thus we assume a, aq linearly C-dependent, that is $aq = \beta a$, with $\beta \in C$ and also $a(q - \beta) = 0$. Since q and $q - \beta$ induce the same inner derivation in R, without loss of generality, we may consider the case when $aq = 0$. Thus

$$\begin{aligned}0 = a\,([q, [x_1, x_2]]_2)^n &= a([q, [x_1, x_2]]_2)\,([q, x_2]]_2)^{n-1}\\ &= a\,([x_1, x_2]^2\,q - 2\,[x_1, x_2]\,q\,[x_1, x_2])\,([x_1, x_2]^2\,q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\,q\,[x_1, x_2])^{n-1}\end{aligned}$$

$$= a\,[x_1, x_2]^2\, q\,([x_1, x_2]^2\, q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\, q\,[x_1, x_2])^{n-1}$$

$$-\ 2a\,[x_1, x_2]\, q\,[x_1, x_2]\,([x_1, x_2]^2\, q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\, q\,[x_1, x_2])^{n-1}.$$

Suppose for the moment $a \neq 0$. Since R is not GPI, then

$$a\,[x_1, x_2]^2\, q\,(([x_1, x_2]^2\, q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\, q\,[x_1, x_2]^{n-1} = 0$$

$$-\ 2a\,[x_1, x_2]\, q\,[x_1, x_2]\,(([x_1, x_2]^2\, q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\, q\,[x_1, x_2]^{n-1} = 0.$$

In particular, since $q \notin C$, the last one implies

$$([x_1, x_2]^2\, q + q\,[x_1, x_2]^2 - 2\,[x_1, x_2]\, q\,[x_1, x_2])^{n-1} = 0.$$

As above this forces the contradiction that $q \in C$. □

Lemma 3 — Let R be a dense ring of linear transformations over a vector space V over a division ring D. If $dim_D\, V \geq 3$ then $a = 0$.

PROOF : Suppose a is not zero. If for every $v \in V$, v and qv are linearly D-independent, then there exists $w \in V$ such that $v, q\,v, w$ are linearly D-independent. By the density of R there exist $r_1, r_2 \in R$ such that

$$r_1\, v = 0\ ,\ r_1\, q\, v = 0,\ r_1\, w = w$$

$$r_2\, v = 0,\ r_2\, q\, v = w,\ r_2\, w = v.$$

Therefore, $[q, [r_1, r_2]]_2\, v = -\, v$ and if $a\, v \neq 0$, we get the contradiction

$$0 = a\,([q, [r_1, r_2]]_2)^n\, v = (-\,1)^n\, a\, v \neq 0.$$

This implies that if $a\, v \neq 0, q\, v = \alpha\, v$, for a suitable $\alpha \in D$.

Now suppose that $a\, v = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and $a\,(w - v) \neq 0$. By the above argument there exist $\beta, \gamma \in D$ such that $qw = \beta w$ and $q(w - v) = \gamma(w - v)$. Moreover, since v, w are linearly independent over D, there exists $u \in V$ such that u, v, w are linearly D-independent, moreover there exist $r_1, r_2 \in R$ such that

$$r_1\, u = w,\ r_1\, w = u,\ r_1\, v = u$$

$$r_2 w = u,\ r_2\, u = v,\ r_2\, v \quad 0.$$

By calculation we have

$$[r_1, r_2]\, w = w - v,\ [r_1, r_2]\, v = -\, v,\ [r_1, r_2]\,(w - v) = w,\ [r_1, r_2]^2\, w = w \text{ and also}$$

$$[q, [r_1, r_2]]_2\, w = 2\,(\beta - \gamma)\, w,\ ([q, [r_1, r_2]]_2)^n\, w = 2^n\,(\beta - \gamma)^n\, w.$$

This means that $0 = a([q, [r_1, r_2]]_2)^n w = 2^n (\gamma - \beta)^n aw$, and since $aw \neq 0$, it follows that $\gamma = \beta$ and $q\, v = \beta\, v$, that is in any case, for all $v \in V, q\, v$ and v are linearly D-dependent. As remarked in lemma 1, in this case standard arguments show that $q \in C$, which contradicts our hypothesis. □

Theorem 1 — *Let R be a prime ring of characteristc different from 2, d a non-zero derivation of R, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer, $a \in R$. If $a\,[d(u), u]^n = 0$, for all $u \in L$, then $a = 0$.*

PROOF : Suppose $a \neq 0$. By lemma 1 we assume that d is an inner derivation of R induced by $q \in Q$ and so R satisfies the generalized polynomial identity $a\,([q, [x_1, x_2]]_2)^n$. By lemma 2, R must satisfy a non-trivial generalized polynomial identity. Moreover, by Remark 2, we replace R by Q and assume $R = Q$, $C = Z(R)$ and R is a C-algebra centrally closed. Therefore, by Martindale's theorem[12], R is a primitive ring. Note that, by lemma 3 we may consider $R = M_2(D)$, the ring of 2×2 matrices over the division ring D. Since R is a simple GPI-ring, then there exists a suitable field F such that $R \subseteq M_l(F)$, for some $l \geq 1$, and as above $l = 2$. In this case $[r_1, r_2]^2 \in F$, for all $r_1, r_2 \in M_2(F)$, moreover $M_2(F)$ satisfies $a\,([q, [x_1, x_2]]_2)^n$. Since if $n = 1$, we are done by Theorem A, then we suppose $n \geq 2$.

If n is even then $([q, [r_1, r_2]]_2)^n$ is central, for all $r_1, r_2 \in M_2(F)$. In this case, since $a \neq 0$, $([q, [x_1, x_2]]_2)^n$ must be an identity for $M_2(F)$ and, by Theorem B, we get the contradiction that R is commutative.

On the other hand, if n is odd, then $([q, [x_1, x_2]]_2)^{n-1}$ is central in $M_2(F)$. In this case, for all $r_1, r_2 \in M_2(F)$, since $a\,[q, [r_1, r_2]]_2\,([q, [r_1, r_2]]_2)^{n-1} = 0$, we have that either $([q, [r_1, r_2]]_2)^{n-1} = 0$ or $a\,[q, [r_1, r_2]]_2 = 0$. In both cases it follows $a\,([q, [r_1, r_2]]_2)^{n-1} = 0$, where $n - 1$ is even, and so we conclude as above. □

Finally we are ready to prove the main result of this paper :

Theorem 2 — *Let R be a prime ring of characteristic different from 2, d a non-zero derivation of R, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer, $a \in R$. If $a\,[d(u), u]^n \in Z(R)$, for all $u \in L$, then either $a = 0$ or R satisfies s_4, the standard identity in 4 variables.*

PROOF : Assume $a \neq 0$. By remark 1 we assume $L = [I, R]$, where I is a non-zero two sided ideal of R. Let J any non-zero two-sided ideal of R. Then $V = [I, J^2] \subseteq L$ is a Lie ideal of R. If for any $v \in V, a[d(v), v]^n = 0$, then by Theorem 1, we are done. Otherwise, by our assumptions, $J \cap Z(R) \neq 0$. Let now K be a non-zero two-sided ideal of R_Z, the ring of central quotients of R. Since $K \cap R$ is a two-sided ideal of R, then $K \cap R \cap Z(R) \neq 0$, that is K contain an invertible element in R_Z and so R_Z is simple with 1. Moreover, without loss of generality, we assume $L = [I, I]$. Since I satisfies the generalized differential identity

$$[a\,[d([x_1, x_2]), [x_1, x_2]]^n, x_3]$$

then by [8] either d is inner induced by an element $q \in Q$ or I satisfies the generalized polynomial identity

$$[a\,[[y_1, x_2] + [x_1, y_2], [x_1, x_2]]^n, x_3].$$

In this last case, in particular I satisfies the blended component

$$[a\,[[y_1, x_2], [x_1, x_2]]^n, x_3].$$

By localizing R at R_Z it follows that $[\,a\,[[y_1, x_2], [x_1, x_2]]^n, x_3]$ is also an identity in R_Z satisfy the same polynomial identities, in order to prove that R satisfies s_4, we may assume that R is a simple ring with 1 and $L = [R, R]$. Thus R satisfies $[a\,[[y_1, x_2], [x_1, x_2]]^n, x_3]$, moreover $Q = RC = R$, R has a minimal right ideal and R is simple artinian. This means that $R = M_k(D)$, for a suitable division ring D and $k \geq 1$. In this case it is known that there exists a field F such that $R \subseteq M_l(F)$ and moreover $M_l(F)$ satisfies $[a\,[[y_1, x_2], [x_1, x_2]]^n, x_3]$. Since R is not commutative then $l \geq 2$. Suppose $l \geq 3$ and consider the following subgroup of $M_l(F)$:

$$G = \{b \in M_l(F) : b\,[[y_1, x_2], [x_1, x_2]]^n \in F\}$$

which is invariant under the action of all the automorphisms of $M_l(F)$. Of course $a \in G$. Since we may suppose $a \notin F$, then $[M_2(F), M_2(F)] \subseteq G$ [4], that is $[z_1, z_2]\,[[y_1, x_2], [x_1, x_2]]^n$ is a central polynomial for $M_2(F)$.

So, for all $t_1, t_2, s_1, r_1, r_2 \in M_2(F)$, the value $[t_1, t_2]\,[[s_1, r_2], [r_1, r_2]]^n$ is central. By choosing $[t_1, t_2] = e_{13}$, $[s_1, r_2] = e_{23}$, $[r_1, r_2] = -e_{32}$ we have the contradiction

$$[t_1, t_2]\,[[s_1, r_2], [r_1, r_2]]^n = e_{13} \notin F.$$

Therefore, l must be 2 and so R satisfies s_4.

Now consider the case when d is an inner derivation induced by $q \in Q$, that is

$$[a([q, [r_1, r_2]]_2)^n, r_3] = 0 \text{ for all } r_1, r_2, r_3 \in I.$$

Since by [3] I and R satisfy the same generalized polynomial identities, then $a\,([q, [x_1, x_2]]_2)^n x_3 - x_3\, a\,([q, [x_1, x_2]]_2)^n$ is a generalized polynomial identity also for R.

If R does not satisfy any non-trivial GPI, we have that $a\,([q, [x_1, x_2]]_2)^n x_3 - x_3\, a\,([q, [x_1, x_2]]_2)^n$ is the zero element of $T' = R_{*_C} C\,\{x_1, x_2, x_3\}$ and so also $a\,([q, [x_1, x_2]]_2)^n$ is the zero element of T'. In this situation are done by Lemma 2.

Therefore, we suppose that R satisfies some non-trivial generalized polynomial identity. By proceeding as above, we may suppose that R is a simple ring with 1, $Q = RC = R$, $q \in R, R \subseteq M_l(F)$, for $l \geq 2$, and $[a\,([q\,[r_1, r_2]]_2)^n\, r_3] = 0$, for all $r_1, r_2, r_3 \in M_l(F)$.

Notice that if a is not invertible then $a\,([q, [x_1, x_2]\,]_2)^n$ is not invertible and so $a\,([q, [r_1, r_2]]_2)^n = 0$, for all $r_1, r_2 \in M_l(F)$. In this situation we may apply Theorem 1. Therefore, we assume that a is invertible and we will prove that in this last case we get a contradiction. In fact, since $([q,[x_1,x_2]]_2)^n \in Fa^{-1}$, then, for all $r_1, r_2 \in M_l(F)$, $[q,[r_1,r_2]]_2$ is zero or invertible. Choose $[r_1,r_2]=[e_{ij}, e_{jj}]=e_{ij}$, for any $i \neq j$, and denote $q=\Sigma\ q_{ij}e_{ij}$, with $q_{ij} \in F$. Then $[q,[r_1, r_2]]_2 = -2q_{ji}e_{ij}$, which is not invertible in $M_l(F)$, for $l \geq 2$. This implies that $q_{ji}=0$, for all $i \neq j$, that is q is a diagonal matrix in $M_l(F)$. For the same reason $\varphi(q)$ is a diagonal matrix, for any $\varphi \in Aut_F\ M_l(F)$. In particular, let $\varphi(x)=(1+e_{ij})x(1-e_{ij})$, for $i \neq j$. Thus $\varphi(q)=q+e_{ij}q-qe_{ij} = \sum_k q_{kk}e_{kk}+q_{jj}e_{ij}-q_{ii}e_{ij}$. Therefore, $q_{jj}=q_{ii}$, so we get the contradiction that q is a central matrix. □

ACKNOWLEDGEMENT

The author wishes to thank the referee for his helpful suggestions.

REFERENCES

1. K. I. Beidar, W. S. Martindale and V. Mikhalev, *Pure appl. Math.*, Dekker, New York, 1996.
2. L. Carini, V. De Filippis, *Pacific J. Math., to appear*
3. C. L. Chuang, *Proc. Amer. math. Soc.* **103** (1988) No. 3, 723-28.
4. C. L. Chuang, *Israel J. Math.* **57** (1987) 116-28.
5. V. De Filippis, *Red. Circ. Mat. Palermo, to appear*
6. I. N. Herstein, *J. Algebra* **83** (1983) 26-32.
7. I. N. Herstein, *Topics in Ring Theory*, Univ. Chicago Press, 1969.
8. V. K. Kharchenko, *Algebra Logic*, **17** (1978), 155-168.
9. C. Lanski, *Proc. Amer. math. Soc.* **118** No. 3, (1993) 731-34.
10. C. Lanski, *Pacific J. Math.* **134** No. 2 (1988) 275-297.
11. T. K. Lee, *Bull. Inst. Math. Acad. Sinica* **20** (1992) 127-38.
12. W. S. Martindale III, *J. Algebra* **12** (1969) 576-584.
13. E. C. Posner, *Proc. Amer. math. Soc.* **8** (1957) 1093-1100.