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ON THE RATIONAL RECURSIVE SEQUENCE $x_{n+1} = \dfrac{bx_n^2}{1+x_{n-1}^2}$ *

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Consider the rational recursive sequence :

$$x_{n+1} = \frac{bx_n^2}{1+x_{n-1}^2} \text{ for } n = 0, 1, 2, ..., \qquad (*)$$

where $a, b \in [0, \infty)$. In this paper, we investigate the behaviour of solutions of eq (*) and obtain some new results.

Key Words : Rational Recursive Sequence; Monotonic Convergence; Oscillation

1. INTRODUCTION

In the monograph of Kocic and Ladas[1], they give an Open Problem (see [1, p 159]):

Investigate the global asymptotic stability of the rational recursive sequence :

$$x_{n+1} = \frac{bx_n^2}{1+x_{n-1}^2} \text{ for } n = 0, 1, 2, ..., \qquad ... (1)$$

where $b \in (0, \infty)$ and the initial values x_{-1} and x_0 are arbitrary positive numbers.

To this end, we consider the rational recursive sequence (1) and obtain some fascinating results.

2. SOME LEMMAS

Lemma 2.1 — Assume that $0 < b < 2$. Then eq. (1) has unique nonnegative equilibrium zero.

PROOF : The equilibrium equation of eq (1) can be written as $x(x^2 - bx + 1) = 0$. Hence, $x \equiv 0$. The proof is thus complete.

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Lemma 2.2 — Assume that $b > 2$. Then eq. (1) has three nonnegative equilibria, zero $\bar{x}_1 = \frac{b-\sqrt{b^2-4}}{2}$ and $\bar{x}_2 = \frac{b+\sqrt{b^2-4}}{2}$, where $\frac{1}{b} < \bar{x}_1 < 1, 1 < \bar{x}_2 < b$.

PROOF : From the equilibrium equation of eq. (1) in Lemma 2.1, it is easy to show that $\bar{x}_0 = 0, \bar{x}_1 = \frac{b-\sqrt{b^2-4}}{2}$ and $\bar{x}_2 = \frac{b+\sqrt{b^2-4}}{2}$, which satisfy the results of the lemma. This completes the proof.

Lemma 2.3 — Assume that $b \in (0, \infty)$. Then every solution of eq. (1) is bounded by $0 \le x_n \le b^5$ for $n \ge 3$.

PROOF : Eq (1) can be written as

$$x_{n+1} = \frac{b^3 x_{n-1}^4}{(1+x_{n-1}^2)(1+x_{n-2}^2)^2} = b^5 \left(\frac{x_{n-1}^2}{1+x_{n-1}^2}\right)\left(\frac{x_{n-2}^2}{1+x_{n-2}^2}\right)^2 \frac{1}{(1+x_{n-3}^2)^2},$$

$$n = 2, 3, \qquad \ldots (2)$$

hence, $0 \le x_{n+1} \le b^5$ for $n = 2, 3, \ldots$. This completes the proof.

Lemma 2.4 — (*a*) Assume that $x > 1$. Then $f(x) = \frac{bx}{1+x^2}$ is strictly decreasing; (b) Assume that $0 < x < 1$. Then $f(x) = \frac{bx}{1+x^2}$ is strictly increasing.

The proof of the Lemma above is very easy. So, we omit it.

3. THE CASE $0 < b \le 1$

In this section, we investigate the global asymptotic stability of eq. (1)

Theorem 3.1 — *Assume that* $0 < b \le 1, x_{-1}$ *and* x_0 *are arbitrary positive numbers. Then every positive solution* $\{x_n\}_{n=0}^{\infty}$ *of eq. (1) is strictly decreasing and converges to zero.*

PROOF : Eq. (1) can be written as

$$x_{n+1} = x_n b^2 \left(\frac{x_{n-1}^2}{1+x_{n-1}^2}\right)\left(\frac{1}{1+x_{n-2}^2}\right), n = 1, 2, \ldots .$$

Hence, $x_{n+1} < x_n$ for $n = 1, 2, \ldots$. From Lemma 2.3, we obtain $\lim_{n\to\infty} x_n = l \ge 0$. From Lemma 2.1, we know that $l = 0$. Thus, this completes the proof.

4. THE CASE $1 < b < 2$

In this section, we also investigate the global asymptotic stability of eq. (1).

***Theorem* 4.1** — *Assume that $1 < b < 2$, $\{x_n\}_{n=0}^{\infty}$ is an arbitrary positive solution of eq. (1). Then there exists an $n_0 > 0$ such that $\{x_n\}_{n=n_0}^{\infty}$ is decreasing and converges to zero.*

PROOF : Let $\{x_n\}_{n=0}^{\infty}$ be a positive solution of eq. (1). Then there must exist an $n_0 > 0$ such that $x_{n_0} \le x_{n_{0-1}}$. (Otherwise, $\{x_n\}$ would be strictly increasing to $l > 0$. By Lemma 2.1, we know that this is a contradiction.)

Now, we observe that

$$x_{n_0+1} = \frac{b x_{n_0}^2}{1 + x_{n_{0-1}}^2} \le \frac{b x_{n_0}^2}{1 + x_{n_0}^2} \le x_{n_0}$$

and by induction $x_{n+1} \le x_n$ for $n \ge n_0$. Hence, we know from Lemma 2.1 that $l = 0$. The proof is complete.

5. THE CASE $b = 2$

In this section, for the sake of completeness, we will list some results which were studied in Kocic and Ladas [1, pp 156-157].

Lemma 5.1 — Assume that $\{x_n\}_{n=0}^{\infty}$ is a solution of eq. (1), which is neither identically equal to 1 nor strictly increases to 1. Then $\lim_{n \to \infty} x_n = 0$.

Lemma 5.2 — Eq. (1) has a solution which is strictly increasing to 1.

6. THE CASE $b > 2$

Lemma 6.1 — Assume that $\{x_n\}_{n=0}^{\infty}$ is a positive solution of eq. (1), x_{-1} is an arbitrary positive number and $x_0 \le \frac{1}{b}$. Then $\{x_n\}_{n=0}^{\infty}$ is strictly decreasing and converges to zero.

PROOF : eq. (1) can be rewritten as $x_{n+1} = x_n \frac{bx_n}{1 + x_{n-1}^2}$, $n = 0, 1, \ldots$.

Hence, $x_1 = x_0 \frac{bx_0}{1 + x_{-1}^2} \le x_0 \le \frac{1}{b}$, $x_2 = x_1 \frac{bx_1}{1 + x_0^2} \le x_1 \le \frac{1}{b}$, ... By induction, we have $x_{n+1} < x_n < \frac{1}{b}$. Therefore, $\lim_{n \to \infty} x_n = l < \frac{1}{b}$. So, $l = 0$ from Lemma 2.2. This completes the proof.

Lemma 6.2 — Assume that $\left\{x_n\right\}_{n=0}^{\infty}$ is a positive solution of eq. (1), and there exists an $n_0 \geq 1$ such that $x_{n_0} \leq \frac{1}{b}$. Then $\left\{x_n\right\}_{n=0}^{\infty}$ is decreasing and converges to zero.

The proof of the above Lemma is just like that of Lemma 6.1. So we omit it.

Lemma 6.3 — Assume that $\left\{x_n\right\}_{n=0}^{\infty}$ is a positive solution of eq. (1), and there exists an $n_0 > 1$ such that $x_{n_0} < x_{n_{0-1}} < x_1$. Then $\left\{x_n\right\}_{n=0}^{\infty}$ is strictly decreasing and converges to zero.

PROOF : From eq. (1), we have $x_{n_0+1} = \dfrac{b\,x_{n_0}^2}{1+x_{n_{0-1}}^2} < x_{n_0}\dfrac{bx_{n_0}}{1+x_{n_0}^2}$. From Lemma 2.4, we know that $\dfrac{bx_{n_0}}{1+x_{n_0}^2} < \dfrac{bx_1}{1+x_1^2} = 1$. Thus, $x_{n_{0+1}} < x_{n_0} < \bar{x}_1$ and $x_{n+1} < x_n < \bar{x}_1$ for $n \geq n_0 - 1$ by induction. Hence, $\left\{x_n\right\}_{n_0-1}^{\infty}$ is strictly decreasing and $\lim_{n\to\infty} x_n = l < \bar{x}_1$. By Lemma 2.2, we get $l = 0$. This completes the proof.

Lemma 6.4 — Eq. (1) has no solution which is oscillatory about x_1.

PROOF : Assume, for the sake of contradiction, that eq. (1) has a solution which is oscillatory about x_1. Then there exists an $n_0 > 0$ such that $x_{n_0-1} > \bar{x}_1$ and $x_{n_0-1} < \bar{x}_1$.

Now, we have from eq. (1) that

$$x_{n_0+1} = \frac{b\,x_{n_0}^2}{1+x_{n_{0-1}}^2} < x_{n_0}\frac{b\,x_{n_0}}{1+\bar{x}_1^2} < x_{n_0}\frac{b\,\bar{x}_1}{1+\bar{x}_1^2} < x_{n_0} < \bar{x}_1.$$

By Lemma 6.3, we know that $\left\{x_n\right\}_{n_{0-1}}^{\infty}$ is strictly decreasing and converges to zero. This is a contradiction and completes the proof.

Lemma 6.5 — Eq (1) has no solution which is decreasing and converges to $\bar{x}_2$.

PROOF : Assume, for the sake of contradiction, that eq. (1) has a solution which is decreasing and converges to x_2. Then there exists an $n_0 > 0$ such that $x_2 < x_{n+1} < x_n$ for $n \geq n_0$ and $\lim_{n\to\infty} x_n = x_2$.

Let $y_n = \dfrac{x_{n+1}}{x_n}$ for $n \geq n_0$. Then $0 < y_n < 1$ for . By eq (1), we obtain

$$y_{n+1} = \frac{x_{n+2}}{x_{n+1}} = \frac{b\,x_{n+1}^2}{(1+x_n^2)\,x_{n+1}} = y_n\frac{b\,x_n}{1+x_n^2}.$$

By Lemma 2.4, we know that $\frac{b x_n}{1+x_n^2} < \frac{b x_2}{1+x_2^2} = 1$. Thus, $y_{n+1} < y_n$ and $\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 1$. Hence, $y_n = 1$ for $n \geq n_0$ which implies that $x_{n+1} = x_n$ for $n \geq n_0$. This is a contradiction and completes the proof.

Lemma 6.6 — Eq. (1) has no solution which satisfies $x_n > x_2$ for $n \geq n_0$, where n_0 is a natural number.

PROOF : Assume, for the sake of contradiction, that eq. (1) has a solution which satisfies $x_n > x_2$ for $n \geq n_0$. Then there exists an $n_0 > 0$ such that $x_n > x_2$ for $n \geq n_0$. Hence, there exists an $n_0' > 0$ such that $x_{n_0'} \leq x_{n_0'-1}$. (Otherwise, $\left\{x_n\right\}_{n_0}^{\infty}$ would be strictly increasing to $l > x_2$, which is a contradiction.)

Now, observing that

$$x_{n_0'+1} = \frac{b x_{n_0'}}{1+x_{n_0'-1}} < \frac{b x_{n_0'}}{1+x_{n_0'}^2} < x_{n_0'} \frac{b x_{n_0'}}{1+x_{n_0'}^2}.$$

By Lemma 2.4, we know that $\frac{b x_{n_0'}}{1+x_{n_0'}^2} < \frac{b \bar{x}_2}{1+x_2^2} = 1$. Hence, $x_{n_0'+1} < x_{n_0'}$. By induction, we have $x_2 < x_{n+1} < x_n$ for $n \geq n_0 - 1$. Therefore, $\left\{x_n\right\}_{n_0}^{\infty}$ is decreasing and converges to x_2. This contradicts lemma 6.5. The proof is thus complete.

Lemma 6.7 — Eq (1) has no solution which is increasing and converges to x_2.

PROOF : Assume, for the sake of contradiction, that eq. (1) has a solution which is increasing and converges to x_2. Then there exists an $n_0 > 0$ such that $x_n < x_{n+1} < x_2$ for $n \geq n_0$ and $\lim_{n \to \infty} x_n = x_2$.

Let $y_n = \frac{x_n}{x_{n+1}}$ for $n \geq n_0$. Then $0 < y_n < 1$ for $n \geq n_0$. By eq (1), we obtain

$$y_{n+1} = \frac{x_{n+1}}{x_{n+2}} = \frac{(1+x_n^{2)}}{b x_{n+1}} = y_n \frac{1+x_n^2}{b x_n}.$$

By Lemma 2.4, we know that $\frac{1+x_n^2}{b x_n} \leq \frac{1+\bar{x}_2^2}{\bar{x}_2} = 1$. Then, $y_{n+1} < y_n$ and $\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 1$. Hence, $y_n = 1$ for $n \geq n_0$ and $\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 1$. Hence, $y_n = 1$ for $n \geq n_0$ which implies that $x_{n+1} = x_n$ for $n \geq n_0$. This is a contradiction and completes the proof.

Lemma 6.8 — Assume that $\left\{x_n\right\}_{n_0}^{\infty}$ is a positive solution of eq. (1), and there exists an n_0 such that $\bar{x}_1 \le x_{n_{0-1}} \le x_{n_0} \le \bar{x}_2$. Then $x_{n_{0+1}} \ge x_{n_0}$.

PROOF : (*a*) Assume that $\bar{x}_1 \le x_{n_{0-1}} \le x_{n_0} \le 1$. Then

$$x_{n_0+1} = \frac{b\,x_{n_0}^2}{1+x_{n_{0-1}}^2} \ge \frac{b\,x_n^2}{1+x_{n_0}^2} = x_{n_0}\frac{b\,x_{n_0}}{1+x_{n_0}^2}.$$

By Lemma 2.4, we obtain $x_{n_0+1} > x_{n_0}\dfrac{b\,\bar{x}_1}{1+\bar{x}_1^2} = x_{n_0}$.

(*b*) Assume that $1 \le x_{n_0-1} \le x_{n_0} \le \bar{x}_2$. Then

$$x_{n_0+1} = \frac{b\,x_{n_0}^2}{1+x_{n_{0-1}}^2} \ge \frac{b\,x_{n_0}^2}{1+x_{n_0}^2} = x_{n_0}\frac{b\,x_{n_0}}{1+x_{n_0}^2}.$$

By Lemma 2.4, we obtain $\dfrac{b\,x_{n_0}}{1+x_{n_0}^2} > \dfrac{b\,\bar{x}_2}{1+\bar{x}_2^2}$. So, $x_{n_0+1} \ge x_{n_0}$.

(*c*) Assume that $\bar{x}_1 \le x_{n_0-1} < 1 < x_{n_0} \le \bar{x}_2$. Then

$$x_{n_0+1} = \frac{b\,x_{n_0}^2}{1+x_{n_{0-1}}^2} \ge \frac{b\,x_{n_0}^2}{1+x_{n_0}^2} > x_{n_0}.$$

The proof is complete.

From the above Lemmas 6.1-6.8, we have the following theorem :

Theorem 6.1 — *Assume that* $\left\{x_n\right\}_{n_0}^{\infty}$ *is a nontrivial solution of eq. (1). Then it may be*

i) strictly decreasing and converges to zero or $\bar{x}_1$;

ii) strictly oscillatory about $\bar{x}_2$;

iii) strictly increasing and converges to zero or $\bar{x}_1$.

7. A Remark

From Theorem 6.1, we see that the solutions may be oscillatory about x_2 and decreasing or increasing and converges to $\bar{x}_1$. Can one find a solution of eq (1) which is strictly oscillatory about $\bar{x}_2$? Can one find a solution of eq. (1) which is strictly decreasing or increasing and converges to $\bar{x}_1$?

References

1. V. L. Kocic and G. Ladas, *Global Behaviour of Nonlinear Difference Equations of Higher Order and Applications*, Kluwer Academic Publishers, Dordrecht, 1993.