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EXISTENCE RESULTS FOR FUNCTIONAL DIFFERENTIAL AND INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES

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In this paper, we investigate the existence of mild solutions on a compact interval to first order initial value problems for functional differential and integrodifferential inclusions in Banach spaces. We shall rely on a fixed point theorem for condensing maps due to Martelli.

Key Words and Phrases : Initial Value Problems; Convex Multivalued Map; Mild Solution; Functional Differential Inclusion; Existence; Fixed Point; Abstract Space

1. INTRODUCTION

In this paper we study the existence of mild solutions, defined on a compact interval, for initial value problem (IVP for short) for first order functional differential and integrodifferential inclusions. In section 3, we consider the following IVP

$$y' - A(t)\, y \in F(t, y_t), \text{ a.e. } t \in J = [0, b] \qquad \dots (1.1)$$

and
$$y_0 = \phi, \qquad \dots (1.2)$$

where $F : J \times C(J_0, E) \to 2^E$ (Here $J_0 = [-r, 0]$) is a bounded, closed, convex multivalued map, $\phi \in C(J_0, E)$, $A(t), t \in J$ a linear closed operator from a dense subspace $D(A(t))$ of E into E and E a real Banach space with the norm $|\cdot|$.

For any continuous function y defined on the interval $J_1 = [-r, b]$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

In Section 4, we investigate the existence of mild solutions, defined on a compact real interval, for the IVP for first order functional semilinear integrodifferential inclusions with the form

$$y' - A(t)\,y \in \int_0^t K(t,s)\,F(s, y_s 0)\,ds,\ t \in J = [0, b], \qquad \text{... (1.3)}$$

and
$$y_0 = \phi, \qquad \text{... (1.4)}$$

where F, ϕ are as in the problem (1.1)-(1.2) and $K : D \to I\!R$, $D = \{(t,s) \in J \times J : t \geq s\}$.

Existence results for differential inclusions on compact intervals, are given in the papers of Avgerinos and Papageorgiou[1], Papageorgiou[17&18], and Benchochra[3] for differential inclusions on noncompact intervals. Additional results for differential equations in abstract spaces, and properties of semigroup theory can be found in the book of Goldstein[7].

Recent results on existence of solutions on compact intervals for functional differential equations, may be found in Erbe, Kong and Zhang[6], Henderson[8], the survey paper of Ntouyas[16], Hristova and Bainov[9], Neito, Jiang and Jurang[15] and Liz and Nieto[13] and the references cited therein. The methods used are usually the topological transversality of Granas[5] and the monotone iterative method combined with upper and lower solutions[11].

The method we are going to use is to reduce the existence of mild solutions to problems (1.1)-(1.2) and (1.3)-(1.4) to the search for fixed points of a suitable multivalued map on the Banach space $C(J_1, E)$. In order to prove the existence of fixed points, we shall rely on a theorem due to Martelli[14].

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J, E)$ is the Banach space of continuous functions from J into E normed by

$$\| y \|_\infty = \sup\ \{ | y(t) | : t \in J \}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties) of the Bochner integral see Yosida[19]).

$L^1(J, E)$ denotes the linear space of equivalence classes of measurable functions $y : J \to E$ such that $\int_0^\infty | y(s) |\, ds < \infty$.

Let $(X, \| \cdot \|)$ be a Banach space. A multivalued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X, that is $\sup_{x \in B}\ \{\sup\ \{\| y \| : y \in G(x)\}\} < \infty$.

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_0)$, there exists an open neighbourhood A of x_0 such that $G(A) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_0, y_n \to y_0, y_n \in Gx_n$ imply $y_0 \in Gx_0$).

G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \to BCC(X)$ is said to be measurable if for each $x \in X$ the distance between x and $G(t)$ is a measurable function on J. For more details on multivalued maps see Deimling[4].

An upper semi-continuous map $G : X \to X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kurotowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to Banas and Goebel[2].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

The consideration of this paper are based on the following fixed point theorem.

Lemma 2.1[14] — Let X be a Banach space and $N : X \to BCC(X)$ a condensing map. If the set

$$\Omega := \{y \in X : \lambda y \in Ny \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. First Order Differential Inclusions

In this section we give an existence result for the IVP (1.1)-(1.2). Let us list the following hypotheses:

(H1) $A(\cdot) : t \to A(t), t \in J$ is continuous such that

$$A(t)\, y = \lim_{h \to 0^+} \frac{T(t+h, t)\, y - y}{h}, \; y \in D(A(t)),$$

where $T(t, s) \in B(E)$ for each $(t, s) \in \gamma := \{(t, s); 0 \leq s \leq t < b\}$, satisfying

(i) $T(t, t) = I$(I is the identity operator in E),

(ii) $T(t, s)\, T(s, r) = T(t, r)$ for $0 \leq r \leq s \leq t \leq b$ and

(iii) the mapping $(t, s) \mapsto T(t, s)\, y$ is strongly continuous in γ for each $y \in E$;

(H2) $F : J \times C(J_0, E) \to BCC(E); (t, u) \mapsto F(t, u)$ is measurable with respect to t for each $u \in C(J_0 E)$, u.s.c. with respect to u for each $t \in J$ and for each fixed $u \in C(J_0, E)$ the set

$$S_{F,u} = \{g \in L^1(J, E) : g(t) \in F(t, u) \text{ for a.e. } t \in J\}$$

is nonempty;

(H3) $\| F(t, u) \| := \sup \{| v | : v \in F(t, u)\} \le p(t)\, \psi(\| u \|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, I\!R_+)$ and $\psi : I\!R_+ \to (0, \infty)$ is continuous and increasing with

$$M \int_0^b p(s) ds < \int_c^\infty \frac{d\tau}{\psi(\tau)};$$

where $c = M \| \phi \|$ and $M \ge 1$ be such that $| T(t, s) | \le M$ for $(t, s) \in \gamma$, and

(H4) for each bounded set $B \subset C(J, E)$, $y \in B$ and $t \in J$ the set

$$\left\{ T(t, 0)\, \phi(0) + \int_0^t T(t, s)\, g(s) ds : g\ S_{F, u} \right\}$$

is relatively compact.

Remark 3.1 : If dim $E < \infty$, then for each $u \in C(J_0, E)$, $S_{F, u} \neq 0$ (see Lasota and Opial)[12]

Definition 3.2 — A function $y \in C([-r, b], E)$ is said to be a mild solution of (1.1) – (1.2) if there exists a function $v \in L'(J, E)$ such that $v(t) \in F(t, y_t)$ a.e. on J, $y(t) = y(t)$, $t \in (-r, 0)$ and

$$y(t) = T(t, 0)\, \phi(0) + \int_0^t T(t - S)\, v(s)\, ds,\ t \in J.$$

with $y_0 = \phi$, is called a mild solution of (1.1)-(1.2) on J_1.

The following lemma is crucial in the proof of our main theorem.

Lemma 3.3[12] — Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H2) and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$, then the operator

$$\Gamma \circ S_F : C(I, X) \to BCC(C(I, X)),\ y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F, y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Now, we are able to state and prove our main theorem.

Theorem 3.4 — *Assume that hypotheses (H1)-(H4) are satisfied. Then the IVP (1.1)-(1.2) has at least one mild solution on* J_1.

PROOF : We transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J_1, E) \to 2^{C(J_1, E)}$ defined by :

$$N_y := \left\{ h \in C(J_1, E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ T(t, 0)\, \phi(0) + \int_0^t T(t, s)\, g(s) ds, & \text{if } t \in J \end{cases} \right\},$$

where

$$g \in S_{F, y} = \left\{ g \in L^1(J, E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J \right\}.$$

Remark 3.5 : It is clear that the fixed points of N are mild solutions to (1.1)-(1.2).

We shall show that N is completely continuous with bounded, closed, convex values and it is upper semicontinuous. The proof will be given in several steps.

Step 1 — Ny is convex for each $y \in C(J_1, E)$.

Indeed, if h_1, h_2 belong to Ny, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_1(t) = T(t,0)\,\phi(0) + \int_0^t T(t,s)\,g_1(s)\,ds$$

and

$$h_2(t) = T(t,0)\,\phi(0) + \int_0^t T(t,s)\,g_2(s)\,ds.$$

Let $0 \le k \le 1$. Then for each $t \in J$ we have

$$(kh_1 + (1-k)\,h_2)(t) = T(t,0)\,\phi(0) + \int_0^t T(t,s)\,[kg_1(s) + (1-k)\,g_2(s)]\,ds.$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$kh_1 + (1-k)\,h_2 \in Ny.$$

Step 2 — N is bounded on bounded sets of $C(J_1, E)$.

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in Ny, y \in B_q$ $B_q = (y \in C(J_1, F) : \| y \|_\infty \le a)$ one has $\| h \|_\infty \le l$. If $h \in Ny$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = T(t,0)\,\phi(0) + \int_0^t T(t,s)\,g(s)\,ds.$$

By (H3) we have for each $t \in J$

$$|\,h(t)\,| \le |\,T(t,0)\,\phi(0)\,| + \int_0^t \|\,T(t,s)\,g(s)\,\|\,ds$$

$$\le M\,\|\,\phi\,\| + M \cdot \sup_{t \in [0,b]} \left(\int_0^t p(s)ds \right) \max_{y \in B_r} \sup_{y \in [0,q]} \psi(y) := l.$$

Step 3 — N-sends bounded sets into equicontinuous sets of $C(J_1, E)$.

Let $t_1, t_2 \in J, t_1 < t_2$ and B_q be a bounded set in $C(J_1, E)$.

For each $y \in B_q$ and $h \in Ny$, there exists $g \in Ny$, there exists $g \in S_{F,y}$ such that

$$h(t) = T(t,0)\,\phi(0) + \int_0^t T(t,s)\,g(s)ds.$$

Thus

$$\begin{aligned}
|h(t_2) - h(t_1)\| &\le |T(t_2,0)\,\phi(0) - T(t_1,0)\,\phi(0)| + \left\| \int_0^{t_2} [T(t_2,s) - T(t_1,s)]\,g(s)\,ds \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} T(t_1,s)\,g(s)\,ds \right\| \\
&\le |T(t_2,0)\,\phi(0) - T(t_1,0)\,\phi(0)| + \left\| \int_0^{t_2} [T(t_2,s) - T(t_1,s)]\,g(s)\,ds \right\| \\
&\quad + M \int_{t_1}^{t_2} \|g(s)\|\,ds \\
&\le |T(t_2,0)\,\phi(0) - T(t_1,0)\,\phi(0)| + \left\| \int_0^{t_2} [T(t_2,s) - T(t_1,s)]\,g(s)\,ds \right\| \\
&\quad + M \cdot \sup_{y \in [0,q]} \psi(y) \left(\int_{t_1}^{t_2} p(s)ds \right).
\end{aligned}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $t_1 < t_2 \le 0$ and $t_1 \le 0 \le t_2$ follows from the uniform continuity of ϕ on the interval J_0 and from the relation

$$|h(t_2) - h(t_1)| = \|h(t_2) - \phi(t_1)\| \le \|h(t_2) - h(0)\| + \|\ (0) - \phi(t_1)\|$$

respectively.

As a consequence of Step 2, Step 3 and (H4) together with the Ascoli-Arzela theorem we can conclude that N is completely continuous, and therefore, a condensing map.

Step 4 — N has a closed graph.

Let $y_n \to y^*$, $h_n \in Ny_n$, and $h_n \to h^*$. We shall prove that $h^* \in Ny^*$. $h_n \in Ny_n$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t, 0)\,\phi(0) + \int_0^t T(t, s)\, g_n(s)ds.$$

We must prove that there exists $g^* \in S_{F, y^*}$ such that

$$h^*(t) = T(t, 0)\,\phi(0) + \int_0^t T(t, s)\, g^*(s)ds. \qquad \text{... (3.1)}$$

Consider the linear continuous operator

$$\Gamma : L^1(J, E) \to C(J, E)$$

$$g \mapsto \Gamma(g)(t) = \int_0^t T(t, s)\, g(s)ds.$$

Clearly we have that

$$\| (h_n - T(t, 0)\,\phi(0)) - (h^* - T(t, 0)\,\phi(0)) \|_\infty \to 0, \text{ as } n \to \infty.$$

From Lemma 3.3, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - T(t, 0)\,\phi(0) \in \Gamma(S_{F, y_n}).$$

Since $y_n \to y^*$, it follows from Lemma 3.3 that

$$h^*(t) - T(t, 0)\,\phi(0) = \int_0^t T(t, s)\, g^*(s)\, ds$$

for some $g^* \in S_{F, y^*}$.

Step 5 — The set

$$\Omega := \{y \in C(J_1, E) : \lambda y \in N(y), \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F, y}$ such that

$$y(t) = \lambda^{-1}\, T(t, 0)\,\phi(0) + \lambda^{-1} \int_0^t T(t, s)\, g(s)\, ds, \; t \in J.$$

This implies by (H3) that for each $t \in J$ we have

$$|y(t)| \leq M\|\phi\| + M \int_0^t p(s)\,\psi(\|y_s\|)\,ds.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\},\ 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, b]$, by the previous inequality we have for $t \in [0, b]$

$$\mu(t) \leq M\|\phi\| + M \int_0^t p(s)\,\psi(\|y_s\|)\,ds$$

$$\leq M\|\phi\| + M \int_0^t p(s)\,\psi(\mu(s))\,ds.$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M \geq 1$.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M\|\phi\| \text{ and } \mu(t) \leq v(t),\ t \in [0, b].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq Mp(t)\,\psi(v(t)),\ t \in [0, b].$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq M \int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant L such that $v(t) \leq L,\ t \in J$, and hence $\mu(t) \leq L,\ t \in J$. Since for every $t \in [0, b]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty := \sup\{|y(t)| : -r \leq t \leq b\} \leq L,$$

where L depends on b and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a mild solution of (1.1)-(1.2) □

4. First Order Integrodifferential Inclusions

In this section we consider the solvability of the IVP (1.3)-(1.4). We need the following assumptions :

(H5) for each $t \in J$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t, s)|,\ 0 \leq s \leq t\},$$

is bounded on J;

(H6) the map $t \mapsto K_t$ is continuous from J to $L^\infty(J, I\!R)$; here $K_t(s) = K(t, s)$;

(H7) $\| F(t, u) \| := \sup \{ |v| : v \in F(t, u) \} \leq p(t)\, \psi(\| u \|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, I\!R_+)$ and $\psi : I\!R_+ \to (0, \infty)$ is continuous and increasing with

$$Mb \sup_{t \in J} K(t) \int_0^b p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)};$$

where $c = M \| \phi \|$ and $M \geq 1$ be such that $\| T(t, s) \| \leq M$ for $(t, s) \in \gamma$,

(H8) for each bounded set $B \subset C(J, E), y \in B$ and $t \in J$ the set

$$\left\{ T(t, 0)\, \phi(0) + \int_0^t T(t, s) \int_0^s K(s, u)\, g(u) du ds : g \in S_{F, y} \right\}$$

is relatively compact.

Definition 4.1 — A function on $y \in C([-r, b], F)$ is said to be a mild solution of (1.3)-(1.4) if there exists on function $v \in L'(J, E)$ see that $v(t) \in F(t, y_t)$ a.e. on $J, y(t) = \phi(t), t \in [-r, 0]$, and

$$y(t) \in T(t, 0)\, \phi(0) + \int_0^t T(t, s) \int_0^s K(s, u)\, V(u), du\, ds, t \in J1$$

with $y_0 = \phi$, is called a mild solution of (1.3)-(1.4) on J_1.

Theorem 4.2 — *Assume that hypotheses (H1), (H2), (H5)-(H8) are satisfied. Then the IVP (1.3)-(1.4) has at least one mild solution on* J_1.

PROOF : We transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J_1, E) \to 2^{C(J_1, E)}$ defined by :

$$N_y := \left\{ h \in C(J_1, E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ T(t, 0)\, \phi(0) \\ \quad + \int_0^t T(t, s) \int_0^s K(s, u)\, g(u) du ds, & \text{if } t \in J \end{cases} \right\}$$

where $g \in S_{F, y} = \left\{ g \in L^1(J, E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J \right\}$.

Remark 4.3 : It is clear that the fixed points of N are mild solutions to (1.3)-(1.4).

As in Theorem 3.4 we can show that N is completely continuous with bounded, closed, convex values and it is upper semicontinuous. We repeat only the step 5, i.e. we show that the set

$$\Omega := \{ y \in C(J_1, E) : \lambda y \in Ny, \text{ for some } \lambda > 1 \}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F, y}$ such that

$$y(t) = \lambda^{-1} T(t, 0) \phi(0) + \lambda^{-1} \int_0^t T(t, s) \int_0^s K(s, u) g(u)\, du ds, \; t \in J.$$

This implies by (H7) that for each $t \in J$ we have

$$| y(t) | \leq M \| \phi \| + M \left\| \int_0^t \int_0^s K(s, u) g(u)\, du ds \right\|$$

$$\leq M \| \phi \| + M \int_0^t \int_0^s | K(s, u) | p(u) \psi(\| y_u \|)\, du ds$$

$$\leq M \| \phi \| + Mb \sup_{t \in J} K(t) \int_0^t p(s) \psi(\| y_s \|)\, ds.$$

We consider the function μ defined by

$$\mu(t) = \sup \{ | y(s) | : -r \leq s \leq t \}, \; 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = | y(t^*) |$. If $t^* \in [0, b]$, by the previous inequality we have for $t \in [0, b]$

$$\mu(t) \leq M \| \phi \| + Mb \sup_{t \in J} K(t) \int_0^t p(s) \psi(\| y_s \|)\, ds$$

$$\leq M \| \phi \| + Mb \sup_{t \in J} K(t) \int_0^t p(s) \psi(\mu(s))\, ds.$$

If $t^* \in J_0$ then $\mu(t) = \| \phi \|$ and the previous inequality holds, since $M \geq 1$.

Let us take the right-hand side of the above inequality as $\upsilon(t)$, then we have

$$\upsilon(0) = M \| \phi \| \text{ and } \mu(t) \leq \upsilon(t), \; t \in J.$$

Using the nondecreasing character of ψ we get

$$\upsilon'(t) \leq Mb \sup_{t \in J} K(t) p(t) \psi(\upsilon(t)), \; t \in J.$$

This implies for each $t \in J$ that

$$\int_{\upsilon(0)}^{\upsilon(t)} \frac{du}{\psi(u)} \leq Mb \sup_{t \in J} K(t) \int_0^b p(s) ds < \int_{\upsilon(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant L such that $\upsilon(t) \leq L, t \in J$, and hence $\mu(t) \leq L, t \in J$. Since for every $t \in [0, b]$, $\| y_t \| \leq \mu(t)$, we have

$$\|y\|_{\infty} := \sup \{|y(t)| : -r \le t \le b\} \le L,$$

where L depends only b and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a mild solution of (1.3)-(1.4).

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