Indian J. pure appl. Math., 32(5) : 677-693, May 2001

PERIODIC BOUNDARY VALUE PROBLEMS FOR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES*

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(*Received* 6 *June* 2000; *accepted* 17 *October* 2000)

In this paper, we use the monotone iterative technique and a comparison result to prove some existence theorems of minimal and maximal solutions of periodic boundary value problems for nonlinear first order impulsive functional differential equations in Banach spaces. Then, we give some applications to boundary value problems for second order functional differential equations in Banach spaces

Key Words : Impulsive Functional Differential Equations; Cone; Monotone Iterative Technique

1. INTRODUCTION

In the mathematical simulation in various fields of science and technology impulsive differential equations are often used[5, 8 & 9]. This leads to the necessity of justification of methods for their approximate solution.

It is well known that the method of upper and lower solutions coupled with the monotone iterative technique has been applied successfully to obtain results of existence and approximation of solutions for periodic boundary value problems (*PBVP*) for ordinary differential equations[1&9]. Some attempts have been made to extend these techniques to study problems of functional differential equations (*FDEs*)[4&5].

In [5], the boundary value problem for the system of impulsive differential-difference equations in the space R^n :

$$\left.\begin{array}{ll} x' = f(t, x, x-h), & t \in [0, 1], t \neq t_i, \\ \Delta x|_{t=t_i} = I_i(x(t_i - 0)), & i = 1, 2, \ldots, k, \\ x(0) = x(1), & \end{array}\right\}$$

is considered. By using the monotone iterative technique, some existence theorems are obtained.

In [9], Liu and Guo use the monotone Iterative technique and Banach contraction mapping principle to obtain some existence theorems of solutions of *PBVP* for first order nonlinear impulsive integro-differential equations of mixed type in Banach space.

In this paper, we consider the *PBVP* for nonlinear first order impulsive *FDEs* in a real Banach space E :

*Project supported by National Natural Science Foundation of China (19871048) and Natural Science Foundation of Shandong Province of China (Y97A12017).

$$\left.\begin{array}{ll} x' = f(t, x, x_t, Tx, Sx), & t \in [0, T], t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \ldots, m, \\ x(0) = x(T), & \end{array}\right\} \quad \ldots (1.1)$$

where $f \in C[J \times E \times C_0 \times E \times E, E], J = [0, T], C_0 = C[[-r, 0], E]$, $r \geq 0$, x_t be denoted by $x_t(s) = x(t+s), s \in [-r, 0], I_k \in C[E, E]$ $(k = 1, 2, \ldots, m)$,

$$(Tx)(t) = \int_0^t k(t, s)\, x(s)\, ds, \; (Sx)(t) = \int_0^T h(t, s)\, x(s)\, ds, \quad \ldots (1.2)$$

$$k \in C[D, R^+], D = \{(t, s) \in J \times J : t \geq s\}, \; h \in C[J \times J, R^+], R^+ = [0, +\infty),$$

and $$0 = t_0 < t_1 < \ldots < t_k < \ldots < t_m < T, \; \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-),$$

where $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$.

In section 2, we establish a comparison result, and then we state and prove the main theorem in section 3. Section 4 offers one example to illustrate our result. Finally, we give some applications to boundary value problems for second order functional differential equations in Banach space. The main results improve and generalize the related results in [5] [9].

2. Preliminaries

Let E be partially ordered by a cone P of E, i.e., $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E. P is normal $\Leftrightarrow$ every order interval $[x, y]$ is bounded in E; P is said to be regular if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$ implies $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in E$. Let E^* be the dual space of E, $P^* = \{\varphi \in E^* \mid \varphi(x) \geq 0, x \in P\}$ be the dual cone of P in E. P^* is said to be reproducing, if $E^* = P^* - P^*$.

In the following, let $J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_m = (t_m, T]$, $J' = J \setminus \{t_1, \ldots, t_m\}$, Δ = max $\{t_1, T - t_m, \max[t_{i+1} - t_i : i = 1, 2, \ldots, m-1]\}$. $I = [-r, T], PC[I, E] = \{x : I \to E \mid x(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and the right limit $x(t_k^+)$ exists for $k = 1, 2, \ldots, m\}$. Evidently, $PC[I, E]$ is a Banach space with $\|x\|_1 = \sup_{t \in I} \|x(t)\|$. Let $P_1 = \{x \in PC[I, E] \mid x(t) \geq \theta, t \in I\}$, then P_1 is a cone in space $PC[I, E]$, and so, $PC[I, E]$ is partial ordered by P_1. $x \in PC[I, E] \cap C^1[J', E]$ is called a solution of $PBVP$ (1.1), if it satisfies (1.1). the functions $v, w \in PC[J, E] \cap C^1[J', E]$ are said to be a lower and upper solutions of $PBVP$ (1.1), if the following inequalities hold:

$$\left.\begin{array}{ll} v'(t) \leq f(t, v(t), v_t, Tv(t), Sv(t)), & t \neq t_k, t \in J, \\ \Delta v|_{t=t_k} \leq I_k(v(t_k)), & k = 1, 2, \ldots, m, \\ v(0) \leq v(T); \text{ and} & \end{array}\right\}$$

$$\left.\begin{array}{l} w'(t) \geq f(t, w(t), w_t, Tw(t), Sw(t)), \quad t \neq t_k, t \in J, \\ \Delta w|_{t=t_k} \geq I_k(w(t_k)), \quad k = 1, 2, \ldots, m, \\ w(0) \geq w(T). \end{array}\right\}$$

Lemma 2.1[2] — Let E be a Banach space, $P \subset E$ a cone and P^* its dual cone. Then P is normal if and only if P^* is reproducing.

Lemma 2.2 — Assume that $m \in PC[I, E] \cap C^1[J', E]$ satisfies

$$\left.\begin{array}{ll} m'(t) \leq -Mm(t) - M_1 \int_{-r}^{0} m_t(s)\,ds - N((Tm)(t)) - N_1((Sm)(t)), & t \in J, t \neq t_k, \\ \Delta m|_{t=t_k} \leq L_k m(t_k), & k = 1, 2, \ldots, m. \\ m(t) \equiv m(0), & t \in [-r, 0], \\ m(0) \leq m(T), & \end{array}\right] \quad \ldots (2.1)$$

where constants $M > 0, M_1 \geq 0, N \geq 0, N_1 \geq 0, 0 \leq L_k < 1$. Then $m(t) \leq \theta$, $\forall\ t \in I$ provided one of the following conditions holds :

$$\text{(a)}\ (M + M_1 r + N k_0 T + N_1 h_0 T)\,\Delta < \frac{\prod_{k=1}^{m} (1 - L_k)}{1 + \sum_{n=1}^{m} \prod_{k=n}^{m} (1 - L_k)}.$$

$$\text{(b)}\ M^{-1}[M_1(e^{Mr} - 1) + (Nk_0 + N_1 h_0)(e^{MT} - 1)]\,\Delta < \frac{\prod_{k=1}^{m} (1 - L_k)}{1 + \sum_{n=1}^{m} \prod_{k+n}^{m} (1 - L_k)}.$$

PROOF : Suppose the condition (a) is satisfied, for any $g \in P^*$, let $u(t) = g(m(t))$, then $u \in PC[I, R] \cap C^1[J', R]$ and $u'(t) = g(m'(t))$, $(Tu)(t) = g((Tm)(t))$, $(Su)(t) = g((Sm)(t))$, by (2.1), we have

$$\left.\begin{array}{l} u' \leq -Mu - M_1 \int_{-r}^{0} u_t(s)\,ds - NTu - N_1 Su, \ t \in J, t \neq k, \\ \Delta u|_{t=t_k} \leq -L_k u(t_k), \quad (k = 1, 2, \ldots, m), \\ u(t) = u(0), \quad t \in [-r, 0], \\ u(0) \leq u(T). \end{array}\right] \quad \ldots (2.2)$$

We now prove

$$u(t) \leq 0, \ \forall\ t \in I. \quad \ldots (2.3)$$

Suppose that (2.3) is not true. Then there are two cases: (1) there exists $t_1^* \in I$ such that $u(t_1^*) > 0$ and $u(t) \geq 0$, for $t \in I$; (2) there exist $t_1^*, t_2^* \in I$ such that $u(t_1^*) > 0$ and $u(t_2^*) < 0$.

If case (1) holds, it is easy to verify that (2.3) holds by referring to the paper [9].

In case (2), (i) let $\inf_{t \in I} u(t) = -\lambda$ and $u(T) \geq 0$, then $\lambda > 0$, and there exists $t_i < t_0^* \leq t_{i+1}$ for some i such that $u(t_0^*) = -\lambda$ or $u(t_i^+) = -\lambda$. We may assume that $u(t_0^*) = -\lambda$ (in case of $u(t_i^+) = -\lambda$, the proof is similar). From (2.2) it is easy to see that

$$u'(t) \leq \lambda (M + M_1 r + Nk_0 T + N_1 h_0 T) = \lambda M_0, \quad \forall\ t \in J, t \neq t_k, \qquad \text{... (2.4)}$$

where $M_0 = (M + M_1 r + Nk_0 T + N_1 h_0 T)$. By the mean value theorem, we have

$$\left.\begin{array}{l} u(T) - u(t_m^+) = u'(\eta_m)(T - t_m), (t_m < \eta_m < T), \\ u(t_m) - u(t_{m-1}^+) = u'(\eta_{m-1})(t_m - t_{m-1}), (t_{m-1} < \eta_{m-1} < t_m), \\ \cdots \quad \cdots \quad \cdots \\ u(t_{i+2}) - u(t_{i+1}^+) = u'(\eta_{i+1})(t_{i+2} - t_{i+1}), (t_{i+1} < \eta_{i+1} < t_{i+2}) \\ u(t_{i+1}) - u(t_0^*) = u'(\eta_i)(t_{i+1} - t_0^*), (t_0^* < \eta_i < t_{i+1}), \end{array}\right\} \qquad \text{... (2.5)}$$

and so, by (2.2) and (2.4),

$$\left.\begin{array}{l} u(T) - (1 - L_m) u(t_m) \leq \eta M_0 \Delta, \\ u(t_m) - (1 - L_{m-1} u(t_{m-1}) \leq \eta M_0 \Delta, \\ \cdots \quad \cdots \quad \cdots \\ u(t_{i+2}) - (1 - L_{i+1}) u(t_{i+1}) \leq \eta M_0 \Delta, \\ u(t_{i+1}) + \eta \leq M_0 \Delta, \end{array}\right\} \qquad \text{... (2.6)}$$

which implies

$$0 \leq u(T) \leq -\eta \prod_{k=i+1}^{m} (1 - L_k) + \eta M_0 \Delta \left\{ 1 + \sum_{n=i+1}^{m} \prod_{k=n}^{m} (1 - L_k) \right\}, \qquad \text{... (2.7)}$$

i.e.,

$$M_0 \Delta \geq \frac{\prod_{k=i+1}^{m} (1 - L_k)}{1 + \sum_{n=i+1}^{m} \prod_{k=n}^{m} (1 - L_k)},$$

and this contradicts (a). (ii) let $t_2^* \in J$ exist such that $u(t_2^*) < 0$ and $u(T) < 0$. Since $u(0) \leq u(T)$, hence $u(t) < 0$, $\forall\ t \in [-r, 0]$. By assumption there exists a point $\bar{t} \in [0, T]$ such that $u(t) \leq 0$, $\forall$

$t \in [-r, \bar{t})$ and $u(t) > 0$, $\forall\ t \in (\bar{t}, \bar{t}+\varepsilon)$, where $\varepsilon > 0$ is a sufficiently small number. Let inf $\{u(t) : t \in [-r, t_1^*]\} = -\lambda$ then $\lambda > 0$ and there exists $t_i < \eta < t_{i+1}$ for some i such that $u(\eta) = -\lambda$ or $u(t_i^*) = -\lambda$. We may assume that $u(\eta) = -\lambda$ (in case of $u(t_i^*) = -\lambda$, the proof is similar). Since $u(0) \le u(T) < 0$, then $0 < t_1^* < T$. Assume that $t_j < t_1^* \le t_{j+1}$, so $i \le j$. Similar to (2.5), we have

$$
\begin{aligned}
&u(t_1^*) - u(t_j^*) = u'(\xi_j)(t_1^* - t_j), (t_j < \xi_j < t_1^*),\\
&u(t_j) - u(t_{j-1}^+) = u'(\xi_{j-1})(t_j - t_{j-1}), (t_{j-1} < \xi_{j-1} < t_j),\\
&\dots \quad \dots \quad \dots \quad \dots\\
&u(t_{i+2}) - u(t_{i+1}^+) = u'(\xi_{i+1})(t_{i+2} - t_{i+1}), (t_{i+1} < \xi_{i+1} < t_{i+2})\\
&u(t_{i+1} - u(\eta) = u'(\xi_i)(t_{i+1} - \eta), (\eta < \xi_i < t_{i+1}),
\end{aligned}
$$

and so, as in (2.6) and(2.7), we get

$$
0 < u(t_1^*) = -\lambda \prod_{k=i+1}^{j} (1 - L_k) + \lambda M_0 \Delta \left\{ 1 + \sum_{n=i+1}^{j} \prod_{k=n}^{j} (1 - L_k), \right\}
$$

which implies

$$
M_0 \Delta > \frac{\prod_{k=i+1}^{j} (1 - L_k)}{1 + \sum_{n=i+1}^{j} \prod_{k=n}^{j} (1 - L_k)} = \frac{\prod_{k=i+1}^{m} (1 - L_k)}{\prod_{k=j+1}^{m} (1 + L_k) + \sum_{n=i+1}^{j} \prod_{k=n}^{m} (1 - L_k)}
$$

$$
\ge \frac{\prod_{k=1}^{m} (1 - L_k)}{1 + \sum_{n=1}^{m} \prod_{k=n}^{m} (1 - L_k)},
$$

and this contradicts (a). Since $g \in P^*$ is arbitrary, we conclude that $m(t) \le \theta, t \in I$.

Suppose the condition (b) is satisfied. For any $g \in P^*$, let $u(t) = g(m(t))$, then $u \in PC[I, R] \cap C_1[J', R]$ and $u'(t) = g(m'(t)), g((Tm)(t) = (Tu)(t), g((sm)(t)) = (Su)(t)$, by (2.1), we have

$$
\left.
\begin{aligned}
&u' \le -Mu - M_1 \int_{-r}^{0} u_t(s)ds - NTu - N_1 Su, t \in J, t \ne t_k,\\
&\Delta u|_{t=t_k} \le -L_k u(t_k), (k = 1, 2, \dots, m),\\
&u(t) = u(0), t \in [-r, 0]\\
&u(0) \le u(T).
\end{aligned}
\right] \quad \dots (2.8)
$$

Let $\upsilon(t) = u(t) e^{Mt}$, $\forall\, t \in I$. Then $\upsilon \in PC[I, R] \cap C^1[J', R]$ and (2.8) implies

$$\left.\begin{array}{l} v' \leq -M_1 \int_{-r}^{0} v_t(s) e^{-Ms} ds - N \int_{0}^{t} K^*(t, s) \upsilon(s) ds - N_1 \int_{0}^{T} h^*(t, s) \upsilon(s) ds, \\ \quad \forall\, t \in J, t \neq t_k, \\ \Delta \upsilon |_{t = t_k} \leq -L_k \upsilon(t_k), \quad k = 1, 2, \ldots, m \\ \upsilon(t) = \upsilon(0) e^{Mt}, \quad t \in [-r, 0], \\ \upsilon(0) \leq \upsilon(T) e^{-TM}, \end{array}\right] \quad \ldots (2.9)$$

where $k^*(t, s) = k(t, s) e^{M(t-s)}$, $h^*(t, s) = h(t, s) e^{M(t-s)}$.

We now prove

$$\upsilon(t) \leq 0, \; \forall\; t \in I. \quad \ldots (2.10)$$

We omit the following proof for it is similar to that of (a).

Remark 2.1 : The conditions in lemma 2.2 are much weaker than that used in comparison results.[5&9]

Consider the PBVP for the sacalar linear Impulsive FDEs :

$$x'(t) + Mx(t) + M_1 \int_{-r}^{0} x_t(s) ds + NTx(t) + N_1 Sx(t) = \sigma(t), t \in J, t \neq t_k, \quad \ldots (2.11)$$

$$\Delta x |_{t = t_k} = -L_k x(t_k) + \gamma_k, \; k = 1, 2, \ldots, m, \quad \ldots (2.12)$$

$$x(t) = \varphi(t), \quad t \in [-r, 0], \quad \ldots (2.13)$$

$$x(0) = x(T). \quad \ldots (2.14)$$

Lemma 2.3 — Let $\sigma \in PC[J, E]$, $\varphi(t) \in C[[-r, 0], E]$, then $x \in PC[I, E] \cap C^1[J', E]$ is a solution of the IVP for a linear impulsive differential eq. (2.11) - (2.13) if and only if $x \in PC[I, E]$ is a solution of the following impulsive integral equation

$$\left.\begin{array}{l} x(t) = \varphi(0) e^{-Mt} + \int_{0}^{t} e^{-M(t-s)} \Big[\sigma(s) - M_1 \int_{-r}^{0} x_s(\tau) d\tau - N(Tx)(s) \\ \qquad - N_1 (Sx)(s) \Big] ds + \sum_{0 < t < t} e^{-M(t-t_k)} (\gamma_k - L_k x(t_k)), t \in J, \\ x(t) = \varphi(t), t \in [-r, 0]. \end{array}\right] \quad \ldots (2.15)$$

PROOF : This lemma is easily obtained by the ordinary method, and so we omit the proof.

Lemma 2.4 — Assume that $\sum_{k=1}^{m} L_k < 1$, then eq. (2.15) has a unique solution in $PC[I, E]$

PROOF : Let $d \equiv (T+r) M_1 + NTk_0 + N_1 Th_0$ and take constant $R > 0$ sufficiently large such that

$$\lambda \equiv \frac{d}{R} + \sum_{k=1}^{m} L_k < 1. \qquad \text{... (2.16)}$$

For any $x \in PC[J, E]$, let $\| x \|_0 = \sup_{t \in J} \left\{ e^{-Rt} \| x(t) \| \right\}$, it is obvious that $PC[J, E]$ is a Banach space with norm $\| \cdot \|_0$ and the norms $\| \cdot \|_0$ and $\| \cdot \|_1$ are equivalent. Define an operator F by

$$(Fx)(t) = \varphi(t) e^{-Mt} + \int_0^t e^{-M(t-s)}$$

$$\left[\sigma(s) - M_1 \int_{-r}^{0} x_s(\tau)\, d\tau - N(Tx)(s) - N_1 (Sx)(s) \right] ds \qquad \text{... (2.17)}$$

$$+ \sum_{0 < t_k < t} e^{-M(t-t_k)} \{ \gamma_k - L_k x(t_k) \}, \; t \in J,$$

where $x(t) = \varphi(t), t \in [-r, 0]$. Then F is an operator from $PC[J, E]$ into $PC[J, E]$. For any $x, y \in PC[J, E]$, by (2.17), we have

$$\| (Fx)(t) - (Fy)(t) \| \le \int_0^t M_1 \left\| \int_{s-r}^{s} (x(\tau) - y(\tau)\, d\tau \right\| + N \| (Tx)(s) - (Ty)(s) \|$$

$$+ N_1 \| (Sx)(s) - (Sy)(s) \| ds + \sum_{k=1}^{m} L_k \| x(t_k) - y(t_k) \|$$

$$\le \int_0^t [(T+r) M_1 + NTk_0 + N_1 Th_0] \| x(s) - y(s) \| ds$$

$$+ \sum_{k=1}^{m} L_k \| x(t_k) - y(t_k) \|$$

$$= \int_0^t e^{Rs} [(T+r) M_1 + NTk_0 + N_1 Th_0] \| x(s) - y(s) \| e^{-Rs} ds$$

$$+\sum_{k=1}^{m} L_k \| x(t_k) - y(t_k) \|$$

$$\leq \| x-y \|_0 \cdot \frac{d}{R} e^{Rt} + \sum_{k=1}^{m} L_k \| x(t_k) - y(t_k) \|.$$

So

$$e^{-Rt} \| (Fx)(t) - (Fy)(t) \| \leq \frac{d}{R} \| x-y \|_0 + \sum_{k=1}^{m} L_k \| x-y \|_0 = \left(\frac{d}{R} + \sum_{k=1}^{m} L_k \right) \| x-y \|_0$$

i.e., $\| Fx - Fy \|_0 \leq \lambda \| x-y \|_0$.

Hence, by (2.16) and the Banach contraction mapping principle, we see that F has a unique fixed point $x \in PC[J, E]$ and $x(0) = \varphi(0)$. Let $x(t) = \varphi(t)$, $t \in [-r, 0]$, then $x \in PC[I, E]$ is the unique solution of integral eq. (2.15).

Lemma 2.5 — Let the following conditions hold :

(1) $\sigma \in PC[J, E]$ and Cone $P \subset E$ be normal.

(2) $\upsilon, w \in PC[I, E] \cap C^1[J', E]$ are the lower and upper solutions of (2.11), (2.12), (2.14) and satisfy $\upsilon(t) \leq w(t)$ $\forall$ $t \in J \cdot \upsilon(t) \equiv \upsilon(0)$, $w(t) \equiv w(0)$, $\forall$ $t \in [-r, 0]$.

(3) the inequality (a) or (b) holds.

(4) $\sum_{k=1}^{m} L_k < 1$.

Then the following linear (PBVP) :

$$x'(t) + Mx(t) + M_1 \int_{-r}^{0} x_t(s)\,ds + N(Tx)(t) + N_1(Sx)(t) = \sigma(t), t \in J, t \neq t_k,$$

$$\Delta x|_{t=t_k} = -L_k x(t_k) + \gamma_k,\ k = 1, 2, ..., m, \qquad \text{... (2.18)}$$

$$x(t) = x(0), t \in [-r, 0],$$

$$x(0) = x(T),$$

has a unique solution $x \in [\upsilon, w]$.

PROOF : For any $g \in P^*$, let $\alpha(t) = g(\upsilon(t))$, $\beta(t) = g(w(t))$, $m(t) = g(x(t))$, $\delta(t) = g(\sigma(t))$, $\delta_k = g(\gamma_k)$, $k = 1, 2, ..., m$, then $\alpha, \beta, m \in PC[I, R] \cap C^1[J', R]$, $\delta \in PC[J, R]$, $\delta_k \in R$, and

$$m'(t)\, Mm(t) + M_1 \int_{-r}^{0} m_t(s)ds + N(Tm)(t) + N_1(Sm)(t) = \delta(t), t \in J, t \neq t_k,$$

$$\Delta m\,|_{t=t_k} = -L_k\, m(t_k) + \delta_k,$$

$$k = 1, 2, \ldots, m, \qquad \ldots (2.19)$$

$$m(t) = m(0), \; t \in [-r, 0]$$

$$m(0) = m(T),$$

and

$$\alpha \le \beta \Leftrightarrow v \le w. \qquad \ldots (2.20)$$

In fact, *PBVP* (2.18) is equivalent to the following *PBVP* (2.21) :

$$\varphi\,(x'(t) + Mx(t) + M_1 \int_{-r}^{0} x_t(s)ds + NTx(t) + N_1\, Sx(t)) = \varphi\,(\sigma(t)), \, t \in J, \, t \ne t_k,$$

$$\varphi\,(\Delta x\,|_{t=t_k}) = \varphi\,(-L_k\, x(t_k) + \gamma_k), \qquad \ldots (2.21)$$

$$\varphi\,(x(t)) = \varphi\,(x(0)), \, t \in [-r, 0],$$

$$\varphi\,(x(0)) = \varphi\, x(T)), \;\; \forall\;\; \varphi \in E^*,$$

we now prove that *PBVP* (2.21) is equivalent to *PBVP* (2.19). Suppose *PBVP* (2.19) holds, since $P \subset E$ be normal, by Lemma 2.1, P^* be reproducing, hence $E^* = P^* - P^*$, i.e., $\forall\; \varphi \in E^*$, $\exists\; \varphi_1, \varphi_2 \in P^*$, *s.t.* $\varphi = \varphi_1 - \varphi_2$, so (2.21) holds. Assume (2.21) holds, it is evident that (2.19) holds. So *PBVP* (2.18) is equivalent to *PBVP* (2.19) and α, β are the lower and upper solutions of (2.19). By lemma 2.3 and lemma 2.4, for each $a \in [\alpha(0), \beta(0)]$. Let $\varphi(t) = a, t \in [-r, 0]$, then the following linear *IVP*

$$m'(t) + Mm(t) + M_1 \int_{-r}^{0} m_t(s)ds + N(Tm)(t) + N_1\,(Sm)(t) = \delta(t), \, t \in J, \, t \ne t_k,$$

$$\Delta m\,|_{t=t_k} = -L_k\, m(t_k) + \delta_k, \;\; k = 1, 2, \ldots, m,$$

$$m(t) = a, \, t \in [-r, 0]$$

has a unique solution which is denoted by $m(\cdot; a)$. We now prove that there exists $c \in [\alpha(0), \beta(0)]$ such that $m(0;\, c) = m(T;, c)$.

We first prove that $\alpha(0) \le m(T;\, \alpha(0)), \beta(0) \ge m\,(T;\, \beta(0))$.

Suppose $\alpha(0) > m(T; \alpha(0))$, let $u(t) = \alpha(t) - m(t; \alpha(0))$, then $u(t)$ satisfies

$$\left.\begin{array}{l} u'(t) + Mu(t) + M_1 \int_{-r}^{0} u_t(s)ds + N(Tu)(t) + N_1(Su)(t) \le 0, t \in J, t \ne t_k, \\ \Delta u|_{t=t_k} \le -L_k u(t_k), \\ u(t) \equiv 0 = u(0), t \in [-r, 0] \\ u(0) = \alpha(0) - \alpha(0)\ \alpha(T) - m(T; \alpha(0)) = u(T). \end{array}\right]$$

Thus Lemma 2.2 ensures that $u(t) \le 0, t \in I$, this implies that $u(T) = \alpha(T) - m(T; \alpha(0)) \le 0$, and therefore $\alpha(0) \le \alpha(T) \le m(T; \alpha(0))$. This is a contradiction and then $\alpha(0) \le m(T; \alpha(0))$. The same arguments show that $\beta(0) \ge m(T; \beta(0))$.

If $\alpha(0) = \beta(0)$, have that

$$\alpha(0) \le m(T; \alpha(0)) = m(T; \beta(0)) \le \beta(0) = \alpha(0).$$

Thus $m(T; \alpha(0)) = \alpha(0)$, and we may choose $c = \alpha(0)$.

If $\alpha(0) < \beta(0)$, we define the map $h : [\alpha(0)\ \beta(0)] \to R$ by $h(a) = a - m(T : a)$. Theorem 2.2.2 in [6] assures that h is continuous. Since $h(\alpha(0)) \le 0 \le h(\beta(0))$, then there must be one point $c \in [\alpha(0), \beta(0)]$ such that $h(c) = 0$, i.e., $c = m(T; c) = m(0; c)$. From the previous reasoning it is obvious that $m(\cdot; c)$ is a solution of *PBVP* (2.19). Since *PBVP* (2.18) is equivalent to *PBVP* (2.19), so *PBVP* (2.18) has at least one solution $x(t)$.

Let $p(t) = v(t) - x(t)$, we have

$$\begin{array}{l} p'(t) + M_P(t) + M_1 \int_{-r}^{0} p_t(s)ds + N(Tp)(t) + N_1(Sp)(t) \le 0, t \in J, t \ne t_k, \\ \Delta p|_{t=t_k} \le -L_k p(t_k),\ \ k = 1, 2, ..., m, \\ p(t) = p(0), t \in [-r, 0], \\ p(0) \le pT). \end{array}$$

Lemma 2.2 assures that $p(t) \le \theta, t \in I$, i.e., $v(t) \le x(t), t \in I$, The same arguments show that $w(t) \ge x(t), t \in I$, so $x \in [v, w]$.

Suppose x_1, x_2 are two solutions of *PBVP* (2.18) in $[v, w]$. By Lemma 2.2, we easily prove that $x_1 = x_2$, i.e., *PBVP* (2.18) has a unique solution $x \in [v, w]$.

Lemma 2.6[9] — Let $x_n \in PC[J, E]$ $(n = 1, 2, 3, ...)$. If functions $x_n(t)$ $(n = 1, 2, 3, ...)$ are equicontinuous on each J_k $(k = 0, 1, ..., m)$ and $\lim_{n\to\infty} x_n(t) = x(t)$, $\forall\ t \in J$, then $x \in PC[J, E]$ and $\|x_n - x\|_1 \to 0$ $(n \to \infty)$.

For convenience let us list some conditions for later use.

(H_1) Functions $v, w \in PC[I, E]$ are the lower and upper solutions of PBVP (1.1) such that $v(t) \le w(t), t \in I$ and $v(t) \equiv v(0), w(t) \equiv w(0), t \in [-r, 0]$.

(H_2) There exist constants M > 0, $M_1 \ge 0, N \ge 0, N_1 \ge 0$, which satisfy (a) or (b) in lemma 2.2 such that

$$f(t, x(t), x_t, (Tx)(t), (Sx)(t)) - f(t, y(t), y_t, (Ty)(t), (Sy)(t))$$

$$\geq - M(x(t) - y(t)) - M_1 \int_{-r}^{0} (x_t - y_t)(s)\, ds - N(T(x - y))(t) - N_1 (S(x - y))(t)$$

whenever $t \in J, \upsilon \leq y \leq x \leq w$.

(H_3) There exist constants $0 \leq L_k < 1 (k = 1, 2, \ldots, m)$ which satisfy $\sum_{k=1}^{m} L_k < 1$, such that

$$I_k(x) - I_k(y) \geq - L_k (x - y),$$

where $\upsilon(t_k) \leq x \leq y \leq w(t_k)$, $k = 1, 2, \ldots, m$.

3. Main Theorem

***Theorem* 3.1** — *Let cone P be regular, conditions (H1) - (H3) be satisfied. Then there exist monotone sequences* $\{\phi_n\}$, $\{w_n\} \subset PC[I, E] \cap C^1[J', E]$ *which converge uniformly and monotonically on J to the minimal and maximal solutions* $x_*, x^* \in PC[I, E] \cap C^1[J', E]$ *of* PBVP *(1.1) in [v, w] respectively. That is, if* $x \in PC[I, E] \cap C^1[J', E]$ *is any solution of PBVP (1.1) in* $[\upsilon, w]$, *then*

$$\upsilon(t) \leq v_1(t) \leq \ldots \leq v_n(t) \leq \ldots \leq x_*(t) \leq x(t) \leq x^*(t)$$

$$\leq \ldots \leq w_n(t) \leq \ldots \leq w_1(t) \leq w(t), t \in I. \qquad \ldots (3.1)$$

PROOF : For any $\eta \in [\upsilon, w]$, consider the following linear *PBVP* :

$$\left.\begin{aligned} & x'(t) + Mx(t) + M_1 \int_{-r}^{0} x_t(s)ds + N(Tx)(t) + N_1 (Sx)(t) = \sigma(t, \eta)(t)), \\ & \qquad t \neq t_k, t \in J, \\ & \Delta_x|_{t = t_k} = - L_k x(t_k) + \gamma_k(\eta(t_k)), k = 1, 2, \ldots, m, \\ & x(t) = x(0), t \in [-r, 0], \\ & x(0) = x(T), \end{aligned}\right] \qquad \ldots (3.2)$$

where

$$\sigma(t, \eta(t)) = f(t, \eta(t), \eta_t \cdot (T\eta)(t), (S\eta)(t) + M\eta(t) + M_1 \int_{-r}^{0} \eta_t(s)ds$$

$$+ N(T\eta)(t) + N_1 (S\eta)(t),$$

$$\gamma_k(\eta(t_k)) = (I_k + L_k)(\eta(t_k)).$$

By Lemma 2.5, *PBVP* (3.2) has a unique solution $x \in [\upsilon, w]$. Let $x = A\,\eta$, then A is an operator from $[\upsilon, w]$ into $PC[I, E] \cap C^1[J', E]$. By lemma 2.2 and conditions (H_1)-(H_3), it is easy to verify that (a) $\upsilon \le A\,\upsilon, Aw \le w$ and (b) $Ah_1 \le Ah_2,\ \forall\, h_1, h_2 \in [\upsilon, w], h_1 \le h_2$.

Let $\upsilon_n = A\,\upsilon_{n-1}$ and $w_n = Aw_{n-1}$ $(n = 1, 2, ...)$, where $v_0 = \upsilon, w_0 = w$. By (a) (b), we have

$$\upsilon(t) \le \upsilon_1(t) \le \ldots \le \upsilon_n(t) \le \ldots \le w_n(t) \le \ldots \le w_1(t) \le w(t),\quad \forall\, t \in I. \qquad \text{... (3.3)}$$

and $A : [\upsilon, w] \to [\upsilon, w]$. So the regularity of P implies

$$\lim_{n \to \infty} v_n(t) = x_*(t),\quad \forall\, t \in I. \qquad \text{... (3.4)}$$

Since the normality of P implies the normality of P_1, so the order interval $[\upsilon, w]$ is bounded in $PC[I, E]$, thus, by (3.3), $B = \{\upsilon_n\}$ is bounded in $PC[I, E]$. For any $\eta \in [\upsilon, w]$, by (H_1) (H_2), we have

$$v' + M\,\upsilon + M_1 \int_{-r}^{0} v_t(s)ds + NT\upsilon + N_1 S\,\upsilon$$

$$\le f(t, \upsilon, v_t, T\,\upsilon, S\,\upsilon) + M\,\upsilon + M_1 \int_{-r}^{0} \upsilon_t(s)\,ds + NT\,\upsilon + N_1\,S\,\upsilon$$

$$\le f(t, \eta, \eta_t, T\,\eta, S\,\eta) + M\,\eta + M_1 \int_{-r}^{0} \eta_t(s)\,ds + NT\,\eta + N_1\,S\,\eta$$

$$\le f(t, w, w_t, T\,w, S\,w) + Mw + M_1 \int_{-r}^{0} w_t(s)\,ds + NTw + N_1\,Sw$$

$$\le w' + Mw + M_1 \int_{-r}^{0} w_t(s)ds + NTw + N_1\,Sw.$$

So, the normality of P_1 implies

$$\left\{ f(t, \eta, \eta_t, T\,\eta\, S\,\eta)\, M\,\eta + M_1 \int_{-r}^{0} \eta_t(s)\,ds + NT\,\eta + N_1\,S\,\eta : \eta \in [\upsilon, w] \right\}$$

is bounded in $PC[I, E]$. Therefore $\{\upsilon_n'(t) \mid n \in N\}$ is bounded in $PC[I, E]$. It is easy to follow from the mean value theorem that $\{v_n(t) \mid n \in N\}$ is equicontinuous on each J_k. By (3.4) and lemma 2.6, $x_* \in PC[I, E]$ and $\| \upsilon_n - x_* \|_1 \to 0$ $(n \to \infty)$. On account of the definition of υ_n and (3.2), we have

$$\left.\begin{array}{l} v_n'(t) + Mv_x(t) + M_1 \int_{-r}^{0} v_{n;t}(s)\,ds + N(Tv_n)(t) + N_1(Sv_n)(t) = \sigma(t, v_{n-1}(t)), \\ \quad t \neq t_k, t \in J, \\ \Delta v|_{t=t_k} = -L_k v_n(t_k) + \gamma_k(v_{n-1}(t_k)), k = 1, 2, \ldots, m, \\ v_n(t) = v_n(0), t \in [-r, 0], \\ v_n(0) = v_n(T), \end{array}\right] \quad \ldots (3.5)$$

where

$$\sigma(t, v_{n-1}(t)) = f(t, v_{n-1}(t), v_{n-1;t}, (Tv_{n-1}(t), S v_{n-1}(t) + M v_{n-1}(t)$$

$$+ M_1 \int_{-r}^{0} v_{n-1;t}(s)ds + N(Tv_{n-1})(t) + N_1(Sv_{n-1})(t),$$

and $\gamma_k(v_{n-1}(t_k)) = (I_k + L_k)(v_{n-1}(t_k))$.

Taking limits as $n \to \infty$ in (3.5), we get $x_*(t)$ is a solution of *PBVP* (1.1) and $x_* \in PC[I, E] \cap C^1[J', E]$.

In the same way, we can show that $\{w_n\}$ converes uniformly on I to some x^* and $x^* \in PC[I, E] \cap C^1[J, E]$ is a solution of *PBVP* (1.1).

Finally, let $x \in PC[I, E] \cap C^1[J', E]$ be any solution of *PBVP* (1.1) in $[\upsilon, w]$. Assume that $v_{n-1}(t) \leq x(t) \leq w_{n-1}(t), t \in I$, and let $m = v_n - x$, then, by (3.2) and (H_2) (H_3), it is easy to verify that m satisfies (2.1), and so, lemma 2.2 implies that $m(t) \leq \theta, t \in I$, i.e., $v_n(t) \leq x(t)$, $\forall\ t \in I$. Similarly, we can show that $x(t) \leq w_n(t)$, $\forall\ t \in I$. Consequently, by induction, we have

$$x_n(t) \leq x(t) \leq w_n(t), \quad \forall\ t \in I\ (n = 0, 1, 2, \ldots),$$

and by taking limits, we get $x_*(t) \leq x(t) \leq x^*(t)$, $\forall\ t \in I$. Hence, (3.1) holds and the theorem is proved.

Remark 3.1 : In this paper, we not only cross out the conditions that f and I_k are bounded, but also weaken the constant restriction condition. Therefore, Theorem 3.1 in this paper improves and generalizes the related results in [9].

Remark 3.2 : Compared to papers [4, 5, 7 & 8], the results in this paper improve and generalize the related results in papers [4, 5, 7 & 8] to a certain extent.

Remark 3.3 : The condition that P is regular will be satisfied if E is weakly complete (reflexive in particular) and P is normal.

4. An Example

Example — Consider the *PBVP* of infinite system for scalar nonlinear impulsive *FDE* :

$$\left.\begin{aligned}
x_n' &= \frac{2}{\pi}\left(\frac{1}{16n^2} - x_n\right) + \frac{t}{2\pi^3 n^2}\left(\int_0^t e^{-ts} x_{n+1}(s)\,ds\right) \\
&- \frac{2t}{10^3 \pi^2 (n+1)^2}\int_{t-\frac{\pi}{4}}^{t} x_n(s)ds - \frac{2}{6\times 10^3 \pi^2 (n+1)^2}\left(\int_0^t e^{-ts} x_n(s)\,ds\right)^2 \\
&- \frac{1}{2\times 10^3 \pi^3 (n+2)^3}\left(\int_0^{2\pi} \frac{x_n(s)\,ds}{1+t^2+s^2}\right)^3, 0 \le t \le 2\pi, t \ne \pi, \\
\Delta x_n|_{t=\pi} &= -\frac{18}{19n} x_n(\pi) + x_{n+2}(\pi)
\end{aligned}\right] \quad \dots (4.1)$$

$$x_n(0) = x_n(2\pi), (n = 1, 2, 3, \dots).$$

Conclusion — *PBVP* (4.1) admits minimal and maximal solutions which are continuously differentiable on $[0, \pi) \cup (\pi, 2\pi]$ and satisfy

$$0 \le x_n(t) \le \begin{cases} \dfrac{1}{2n^2}, & \forall -\dfrac{\pi}{4} \le t \le \pi, \\ \dfrac{1}{2n^2}\left(3 - \dfrac{t}{\pi}\right), & \forall \pi < t \le 2\pi, \end{cases} \quad (n = 1, 2, 3, \dots)$$

PROOF : Let $E = l^1 = \left\{ x = (x_1, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n| < \infty \right\}$. With norm $|r| = \sum_{n=1}^{\infty}$ and $P = \{r = (r_1, \dots, r_n, \dots) \in L^1; r_n \ge 0, k = 1, 2, n\}$. Then P is a normal cone in E. Since l^1 is weakly complete, we know from Remark 3.3 that P is regular. (4.1) can be regarded as a *PBVP* of type (1.1) in E, where

$$k(t, s) = e^{-ts}, h(t, s) = (1 + t^2 + s^2)^{-1}, x = (x_1, \dots, x_n, \dots), \varphi = (\varphi_1, \dots, \varphi_n, \dots),$$

$$y = (y_1, \dots, y_n, \dots), z = (z_1, \dots, z_n, \dots), f = (f_1, \dots, f_n, \dots).$$

$$f_n(t, x, \varphi, y, z) = \frac{2}{\pi}\left(\frac{1}{16n^2} - x_n\right) + \frac{t}{2\pi^3 n^2} y_{n+1} - \frac{2t}{10^3 \pi^2 (n+1)^2}\int^0 \varphi_n(s)\,ds$$

$$- \frac{2}{6\times 10^3 \pi^2 (n+1)^2} y_n^2 - \frac{1}{2\times 10^3 \pi^3 (n+2)^3} z_n^3,$$

and $m = 1$, $t_1 = \pi$, $T = 2\pi$, $I_1 = (I_{11}, \dots, I_{1n}, \dots)$, $I_{1n}(x) = -\frac{18}{19n} x_n + x_{n+2}$. Let $\upsilon(t) = (0, \dots, 0, \dots)$, $\forall -\frac{\pi}{4} \le t \le 2\pi$,

$$w(t)=\begin{cases}\left(\frac{1}{2},\ldots,\frac{1}{2n^2},\ldots\right) & \forall -\frac{\pi}{4}\le t\le \pi,\\ \left(\frac{1}{2}\left(3-\frac{t}{\pi}\right),\ldots,\frac{1}{2n^2}\left(3-\frac{t}{\pi}\right),\ldots\right) & \forall \pi<t\le 2\pi\end{cases}$$

It is not difficult to verify that $\upsilon(t)$ and $w(t)$ satisfy condition (H_1). On the other hand, it is easy to see that conditions (H_2) and (H_3) are satisfied for $M=\frac{2}{\pi}$, $M_1=\frac{1}{10^3\pi}$, $N=\frac{1}{4\times10^3\pi}$, $N_1=\frac{1}{8\times10^3\pi}$ and $L_1=\frac{18}{19}$. Thus, our conclusion follows from the main theorem.

Remark 4.1 : Conclusion is still true in the special case $r = 0$. But we have

$$M^{-1}(Nk_0+N_1h_0)(e^{4\pi M}-1)\,\delta=\frac{\pi}{2}\left(\frac{3}{8\times10^3\pi}\right)(e^8-1)\,\pi>\frac{28084}{16000}>1>\frac{(1-L_1)^2}{1+1-L_1},$$

$$2\pi M^{-1}(Nk_0+N_1h_0(2-e^{-2\pi M})+\sum_{k=1}^{m}[1+(e^{2\pi M}-1)e^{Mt}]L_k$$

$$=\pi^2\left(\frac{1}{8\times10^3\pi}\right)(2-e^{-4})+\left(1+\frac{2^2}{e^4-1}\right)\cdot\frac{18}{19}>\frac{20}{19}>1.$$

that is, the inequalities (2.2) and (2.24) in [9] can't be satisfied. So, our conclusion can't follow from the main theorem in [9]. This shows that this paper improves and generalizes the related results in [9].

5. APPLICATION

There are all kinds of *FDEs* in physics. In elastic theory, the famous Vander Pol Equation.

$$x''(t)+\alpha x'(t)-f(x(t-r))\,x'(t-r)+x(t)=0$$

can be regarded as a special type of the following equation.

$$x''(t)=f(t,x(t),x_t,x'(t),x'_t).$$

Consequently, considering the solutions of this equation has important significance. (see [10]).

In this section, we apply the main theorem in this paper to the boundary value problem of the following second order *FDE*s:

$$\left.\begin{aligned}&x''(t)=f(t,x'(t),x'_t,x(t),x_t),\ t\in[0,T]\\&x_0=\varphi,\ x'(0)=x'(T).\end{aligned}\right] \quad \ldots (5.1)$$

$J=[0,T]$, $T>0$, $\varphi\in C_0$, $C_0=C^1[[-r,0],[E]$, $r>0$, $f\in C[J\times E\times C_0\times E\times C_0\,E]$. Let $x'=y$, then $x = \varphi(0) + \int_0^t y(s)ds$

$$x_t(s) = x(t+s) = \begin{cases} \int_0^{t+s} y(\xi)\,d\xi + \varphi(0), & t+s > 0, \\ \varphi(t+s), & t+s \le 0. \end{cases}$$

For convenience, let us list some conditions for later use

(A_1) There exist $\alpha, \beta \in C^2[J, E]$ such that $\alpha'(t) \le \beta'(t), t \in J, \alpha'(t) \equiv \alpha'(0), \beta'(t) \equiv \beta'(0), t \in [-r, 0]$ and

$$\left.\begin{array}{l} \alpha''(t) \le f(t, \alpha'(t), \alpha'_t, (T_1\alpha')(t), (T_2\alpha')(t)), \\ \alpha'(0) \le \alpha'(T), \\ \beta''(t) \le f(t, \beta'(t), \beta'_t, (T_1\beta')(t), (T_2\beta')(t)), \\ \beta'(0) \le \beta'(T), \end{array}\right\}$$

(A_2) There exist constants $M > 0$, $M_1 \ge 0, N \ge 0, N_1 \ge 0$, which satisfy (i) or (ii) such that

$$f(t, x(t), x_t, (T_1 x)(t), (T_2 x)(t)) - f(t, y(t), y_t, (T_1 y)(t), (T_2 y)(t))$$

$$\ge -M(x-y) - M_1 \int_{-r}^{0} (x_t - y_t)(s)ds - NT_1(x-y) - N_1 T_2(x-y)$$

whenever $t \in J, x, y \in D = \{x(t) \in C^1[J, E], \alpha' \le x \le \beta'\}$, $x \ge y$. where

$$(T_1 x)(t) = \varphi(0) + \int_0^t x(s)ds, \quad (T_2 x)(t) = \begin{cases} \varphi(0) + \int_0^{t+s} x(\xi)\,d\xi, & t+s > 0, \\ \varphi(t+s), & t+s \le 0, \end{cases}$$

(*i*) $(M + M_1^r + NT + N_1 T)\,T < 1$.

(ii) $[M_1(e^{Mr} - 1) + (N + N_1)(e^{MT} - 1)]\,T < M$.

Theorem 5.1 — *Let cone P be regular, conditions (A_1), (A_2) are satisfied, then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\} \subset C^1[I, E], \cap C^2[J, E]$, which converge uniformly and monotonically on J to the minimal and maximal solutions $x_*, x^* \in [I, E] \cap C^2[J, E]$.*

PROOF : Let $y = x'$, then, by (5.1), we get

and
$$\left.\begin{array}{l} y'(t) = f(t, y(t), y_t(T_1 y)(t), (T_2 y)(t)), \quad t \in J \\ y(0) = y(T). \end{array}\right] \qquad \text{... (5.2)}$$

Condition (A_1) assures that α', β' are the lower and upper solutions of (5.2). From the condition (A_2), PBV (5.2) satisfies the condition (H2) in theorem 3.1. Therefore, from Theorem 3.1, there exist monotone sequences $\{v_n\}$, $\{w_n\} \subset C[I, E] \cap C^1[J, E]$ and $\{v_n\}$, $\{w_n\}$ which converge uniformly and monotoniclaly on I to the minimal and maximal solutions f_*, w_* $C[I, E] \cap C^1[J, E]$ of PBVP (5.2). Let $\alpha_n(t) = \varphi(0) + \int_0^t v_n(s)ds$, $\beta_n(t) = \varphi(0) + \int_0^t w_n(s)ds$, then it is easy to verify that $\{\alpha_n\}$, $\{\beta_n\} \subset C^1[I, E] \cap C^2[J, E]$ are also monotone sequences and converge uniformly and monotonically on I to the minimal and maximal solutions $x_* = \varphi(0) + \int_0^t v_*(s)ds$, $x^* = \varphi(0) + \int_0^t w_*(s)ds$ of PBVP (5.1).

ACKNOWLEDGEMENT

The author would like to thank Professor Dajun Guo and Professor Sun Jingxian for their guidance and encouragement.

REFERENCES

1. Y. Chen and W. Zhuang, *Nonlinear Anal.* **22** (1994), 295-303.
2. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
3. Guo Dajun and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
4. J. R. Haddock and M. N. Nkashama, *Nonlinear Anal.* **22** (1994), 267-76.
5. S. G. Hristova and D. D. Bainov, *J. math. Anal. Appl.* **197** (1996), 1-13.
6. J. K. Hale, *Theory of Functional Differential Equations*, Springer, New York, 1977.
7. Lishan Liu and Hongqian Zhang, *Far East. J. Math. Sci.* **43** (1996), 59-73.
8. Xinzhi Liu, *Nonlinear Times and Digest* **2** (1995), 69-82.
9. Xinzhi Liu and Dajun Guo, *Chin. Ann. Math.* **19B** (1998), 517-28.
10. Zuxiu Zheng, *Theory of Functional Differential Equations*, Anhui Education Publishing House, Hefei, 1994.