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A CLASS OF UNIFORMLY CONVEX FUNCTIONS OF ORDER α WITH NEGATIVE AND FIXED FINITELY MANY COEFFICIENTS

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The purpose of the present paper is to obtain several interesting properties of the class UCT (α, c_k), consisting of uniformly convex functions of order α with negative and fixed finitely many coefficients. These properties include coefficient estimates, radius of convexity and closure theorems for the functions belonging to the class UCT (α, c_k).

Key Words : Univalent; Convex; Starlike; Uniformly Convex

1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{... (1.1)}$$

that are analytic and univalent in the unit disk $D = \{z : |z| < 1\}$. A function $f \in S$ is said to be starlike of order $\alpha, 0 \le \alpha < 1$, denoted by $f \in S^*(\alpha)$, if $Re\,(zf'(z)/f(z) > \alpha$ for $z \in D$ and is said to be convex of order $\alpha, 0 \le \alpha < 1$, denoted by $K(\alpha)$, if $Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha$ for $z \in D$. $S^*(0) = S^*$ and $K(0) = K$ are, respectively, the classes of starlike and convex functions in S. The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson[10] and were studied subsequently by Schild[12], MacGregor[6], Pinchuk[9], Jack[4] and others. (see, for example, [15]).

Let UCV (α) be the subclass of S consisting of functions satisfying the inequality

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \ge \alpha \left| \frac{zf''(z)}{f'(z)} \right|, z \in D. \qquad \text{... (1.2)}$$

We note that UCV (1) = UCV, the class of uniformly convex functions. This familiar class was studied by Goodman[3], Ma and Minda[5] and Ronning[11]. ALso UCV(0) = K (0) = K.

Further let T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \; (a_n \geq 0). \qquad \text{... (1.3)}$$

This useful class was introduced by Silverman[13].

Let UCT(α) = UCV(α) $\cap T$. We begin by recalling the following lemma due to Murugusundaramoorthy[7].

Lemma 1 — A function $f(z)$ of the form (1.3) is in UCT (α) if and only if

$$\sum_{n=2}^{\infty} n\,[n\,(\alpha+1)-\alpha]\,a_n \leq 1. \qquad \text{... (1.4)}$$

Further if $f(z) \in UCT(\alpha)$, then

$$a_n \leq \frac{1}{n\,(n\,(\alpha+1)-\alpha]} \qquad \text{... (1.5)}$$

with equality for the function $f_n(z) = z - \dfrac{z^n}{n\,[n\,(\alpha+1)-\alpha]}$.

Since (to a certain extent) the work in the class of uniformly convex functions with negative and fixed finitely many coefficients has paralleled that of the analytic case[8], one is tempted to search results analogous to those of Owa and Srivastava[8] for uniformly convex functions in D. Thus, making use of (1.5), we now introduce the following class of functions.

Let UCT (α, c_k) denote the subclass of UCT(α) consisting of functions of the form

$$f(z) = z - \sum_{i=2}^{k} \frac{c_i}{i\,[i\,(\alpha+1)-\alpha]} z^i - \sum_{n=k+1}^{\infty} a_n z^n \qquad \text{... (1.6)}$$

where $\quad a_n \geq 0, 0 \leq c_i \leq 1$ and $0 \leq \sum_{i=2}^{k} c_i \leq 1.$

In the present paper we prove several interesting and useful results for functions belonging to the class UCT (α, c_k). These results include coefficient estimates, radius of convexity and closure theorems. Techniques used here are similar to those Silverman and Silvia[14] and Owa and Srivastava[8]

2. Coefficient Estimates

We state our first result as

Theorem 1 — *Let the function $f(z)$ be defined by (1.6), then $f(z)$ is in the class UCT (α, c_k) if and only if*

$$\sum_{n=k+1}^{\infty} n\,[n\,(\alpha+1)-\alpha]\,a_n \leq \left(1 - \sum_{i=2}^{k} c_i\right), \qquad \text{... (2.1)}$$

where $0 \leq c_i \leq 1$ and $0 \leq \sum_{i=2}^{k} c_i \leq 1$. The result is sharp.

PROOF : Putting

$$a_i = \frac{c_i}{i\,[i\,(\alpha+1)-\alpha]} \quad (i = 2, 3, ..., k) \qquad ... (2.2)$$

in lemma 1, we have

$$\sum_{i=2}^{k} c_i + \sum_{n=k+1}^{\infty} n\,[n(\alpha+1)-\alpha]\,a_n \le 1, \qquad ... (2.3)$$

which clearly implies (2.1). Further, by taking the function $f\,(z)$ of the form

$$f(z) = z - \sum_{i=2}^{k} \frac{c_i}{i\,[i\,(\alpha+1)-\alpha]} z^i - \frac{\left(1 - \sum_{i=2}^{k} c_i\right)}{n\,[n\,(\alpha+1)-\alpha]} z^n \qquad ... (2.4)$$

for $n \ge k+1$, we can easily verify that the result (2.1) is sharp.

Corollary 1 — Let the function $f(z)$ defined by (1.6) be in the class UCT (α, c_k). Then

$$a_n \le \frac{\left(1 - \sum_{i=2}^{k} c_i\right)}{n\,[n\,(\alpha+1)-\alpha]} \; (n \ge k+1). \qquad ... (2.5)$$

The result (2.5) is sharp for the function $f\,(z)$ given by (2.4).

3. RADIUS OF CONVEXITY

Theorem 2 — *Let the function $f\,(z)$ defined by (1.6) be in the class UCT (α, c_k). Then $f\,(z)$ is convex of order $\rho\,(0 \le \rho < 1)$ in the disc $0 < |z| < r_0$, where r_0 is the largest value for which*

$$\sum_{i=2}^{k} \frac{(1-\rho)\,c_i\,r_0^{\,i-1}}{i\,(\alpha+1)-\alpha} + \frac{(n-\rho)\left(1 - \sum_{i=2}^{k} c_i\right) r_0^{\,n-1}}{n\,(\alpha+1)-\alpha} \le 1-\rho, \qquad ... (3.1)$$

for $n \ge k+1$.

The result is sharp for the function $f(z)$ given by (2.4).

PROOF : In order to prove the theorem, it is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \le 1-\rho \; (|z| < r_0),$$

for a function $f\,(z)$ belonging to the class UCT (α, c_k), where r_0 is defined by (3.1). The details involved are fairly straight-forward and may be omitted.

4. CLOSURE THEOREM

The following inclusion properties are an easy consequences of theorem 1.

Theorem 3 — *Let the functions*

$$f_j(z) = z - \sum_{i=2}^{k} \frac{c_i}{i\,[i\,(\alpha+1)-\alpha]} z^i - \sum_{n=k+1}^{\infty} a_{n,j} z^n \, (a_{n,j} \geq 0) \qquad \text{... (4.1)}$$

be in the class UCT (α, c_k) *for every* $j = 1, 2, ..., m$. *Then the function* $f(z)$ *defined by*

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) \, (d_j \geq 0) \qquad \text{... (4.2)}$$

is also in the same class UCT (α, c_k), *where*

$$\sum_{i=1}^{m} d_j = 1.$$

Theorem 4 — *Let the function* $f_j(z)$ *defined by (4.1) be in the class UCT* (α, c_k) *for each* $j = 1, 2, ..., m$, *then the function* $h(z)$ *defined by*

$$h(z) = z - \sum_{i=2}^{k} \frac{c_i}{i\,[i\,(\alpha+1)-\alpha]} z^i - \sum_{n=k+1}^{\infty} b_n z^n, \, (b_n \geq 0) \qquad \text{... (4.4)}$$

is also in the same class UCT (α, c_k), *where*

$$b_n = \frac{1}{m} \sum_{j=1}^{m} a_{n,j}. \qquad \text{... (4.5)}$$

With a view to determine the extreme points of the class UCT (α, c_k), we first prove the following theorem.

Theorem 5 — *The class UCT* (α, c_k) *is closed under 'convex linear combination'.*

PROOF : The proof of theorem 5 is much akin to that of theorem 2 of [8], and therefore we omit the details involved.

Since the class UCT (α, c_k) is convex, as we have shown in theorem 5, it does have some 'extreme points' given by theorem 6 and corollary 2.

Theorem 6 — *Let*

$$f_k(z) = z - \sum_{i=2}^{k} \frac{c_i}{i\,[i\,(\alpha+1)-\alpha]} z^i \qquad \text{... (4.6)}$$

and $$f_n(z) = z - \sum_{i=2}^{k} \frac{c_i}{i[i(\alpha+1)-\alpha]} z^i - \sum_{n=k+1}^{\infty} \frac{\left(1-\sum_{i=2}^{k} c_i\right)}{n[n(\alpha+1)-\alpha]} z^n, \; n \geq k+1, \qquad \dots (4.7)$$

Then the function $f(z)$ is in the class $UCT(\alpha, c_k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z), \text{ where } \lambda_n \geq 0 \; (n \geq k) \qquad \dots (4.8)$$

and $$\sum_{n=k}^{\infty} \lambda_n = 1. \qquad \dots (4.9)$$

PROOF : We suppose that $f(z)$ can be expressed in the form (4.8). Then it follows from (4.6), (4.7) and (4.9) that

$$f(z) = z - \sum_{i=2}^{k} \frac{c_i}{i[i(\alpha+1)-\alpha]} z^i - \sum_{n=k+1}^{\infty} \frac{\left(1-\sum_{i=2}^{k} c_i\right)\lambda_n z^n}{n[n(\alpha+1)-\alpha]}.$$

We note that

$$\sum_{n=k+1}^{\infty} n[n(\alpha+1)-\alpha] \cdot \frac{\left(1-\sum_{i=2}^{k} c_i\right)\lambda_n}{n[n(\alpha+1)-\alpha]}$$

$$= \left(1-\sum_{i=2}^{k} c_i\right) \sum_{n=k+1}^{\infty} \lambda_n$$

$$= \left(1-\sum_{i=2}^{k} c_i\right)(1-\lambda_k)$$

$$\leq \left(1-\sum_{i=2}^{k} c_i\right)$$

which implies that $f(z) \in UCT(\alpha, c_k)$.

For the converse, assume that the function $f(z)$ of the form (1.6) belongs to the class UCT (α, c_k). Since $f(z)$ satisfies (2.5) for $n \geq k+1$, we may set

$$\lambda_n = \frac{n\,[n\,(\alpha+1)-\alpha]}{\left(1-\sum_{i=2}^{k} c_i\right)}\, a_n \leq 1, \; (n \geq k+1)$$

and

$$\lambda_k = 1 - \sum_{n=k+1}^{\infty} \lambda_n.$$

Hence $f(z)$ has the representation (4.8). This clearly completes the proof of theorem 6.

Corollary 2 — The extreme points of the class UCT (α, c_k) are the functions $f_n(z)$ $(n \geq k)$ given by (4.6) and (4.7).

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