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GENERALIZED PSEUDO UNIVEXITY AND DUALITY IN MATHEMATICAL PROGRAMMING

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In the present paper, some of the earlier duality results of Nanda and Das have been generalized using more broader class of functions, which can be termed as univex functions. Also strong and converse duality theorems have been derived under the new setting.

Key Words : Nonlinear Programming; Duality Theory; Generalized Pseudo Univexity

1. INTRODUCTION

Duality in nonlinear programming is usually treated using the scheme of Wolfe[9] dual, in which the formation of a dual involves the introduction of new variables corresponding to primal constraints. First, Hanson and Mond[7] introduced Wolfe[9] type duality for invex functions with arbitrary cones and studied weak and strong duality theorems. Further, Mond and Wolfe suggested a number of duals for the nonlinear programming problem with different generalized convexity conditions. Consequently, Weir[15] studied Mond-Weir type dual for the strong pseudo-convexity case. In a subsequent work Hanson and Egudo[6] established Mond-Weir type duals for multiobjective programming problem under the invexity assumptions. Recently, Nanda and Das[14] studied four dual problems including the above two problems and established the weak and strong duality theorems for different non-convex conditions with arbitrary cones.

On the other hand, pseudo-invex and quasi-invex functions are the generalizations of invex functions, earlier introduced by Hanson[5]. Furthermore, several classes of functions have been defined for the purpose of weakening the limitations of invexity by some authors like Kaul and Kaur[8], Ben Israel and Mond[2], Craven[4], Chandra *et al.*[3] and they studied some properties and applications of these functions. More recently, Bector *et al.*[1] defined a further generalized form of invexity known as univexity. In this paper, we have generalized results of Nanda and Das[14]. We have further discussed four dual problems including Wolfe and Mond-Weir type dual problems and finally we have established the weak and strong duality theorems under univexity conditions with arbitrary cones. Also, these dual problems are similar to the dual problems which were earlier introduced by Mond-Weir[13] for non-negative variables.

It is to be noted in this context that the univexity concept is a more generalized concept and examples have already appeared (Mishra[11] and Mishra and Giorgia[12]). In the light of such generalization of convexity, our results derived in the paper are more interesting and significant. Moreover as has been proposed later in the conclusion part of the present paper, further extensions are possible in the area of symmetric duality results as suggested.

The paper is organized as follows. Section 2 gives preliminary notations and definitions. In section 3 we will state and prove different forms of weak, strong and converse duality theorems for different problems.

2. Preliminaries and Definitions

Consider the following nonlinear programming problem :

(P) Minimize $f(x)$

Subject to $g(x) \in C_2^*$

$X\ C_1,$

where $R \to R^n$ and $g : C_1 \to R^m$ are real and vector - valued functions;

C_1 and C_2 are closed convex cones with nonempty interiors in R^n and R^m, respectively.

C_i^*, $i = 1, 2$ is the polar cone to C_i, $i = 1, 2$ and is defined by

$C_i^* = \{z : x^t z \leq 0, \text{ for } x \in C_i\}$, $i = 1, 2$. The Wolfe type dual is

(D1) Maximize $f(u) + y^t g(u) - u^t [\nabla f(u) + \nabla y^t g(u)]$

Subject to $-[\nabla f(u) + \nabla y^t g(u)] \in C_1^*$ ... (1)

$u \in C_1, y \in C_2$ ... (2)

Nanda and Das[12] formulated the following dual problems.

(D2) Maximize $f(u)$

subject to $[\nabla f(u) + \nabla y^t g(u)] \in C_1^*$ ... (3)

$-g(u) \in C_2^*$ ... (4)

$u \in C_1, y \in C_2$ ... (5)

(D3) Maximize $f(u) - u^t \{\nabla f(u) + \nabla y^t g(u)\}$

Subject to - $[\nabla f(u) + \nabla y^t g(u)] \in C_1^*$, ... (6)

$- g(u) \in C_2^*$ ... (7)

and $u \in C_1, y \in C_2$ ... (8)

(D4) Maximize $(f(u) + y^t g(u))$,

Subject to $-[\nabla f(u) + \nabla y^t g(u)] \in C_1^*$ ... (9)

and $u \in C_1, y \in C_2$ ... (10)

The following definition is due to Bector *et al*[1]. Let X be a nonempty open set in $R^n, f: X \to R, \eta: X \times X \to R^n, \phi_0 : R \to R$ and $b_0 : X \times [0, 1] \to R_+, b_0 = b_0(x, u)$

Definition 2.1 — A differentiable function f is said to be univex at $u \in C$ with respect to η, ϕ_0 and b_0 if $\forall\ x \in X$ such that the following inequality is true :

$$b_0(x, u)\, \phi_0 [f(x) - f(u)] \geq [\nabla f(u)]^t\, \eta(x, u) \qquad \text{... (11)}$$

Definition 2.2 — Let $\eta : C_1 \times C_1 \to C_1$ be fixed. A differentiable function $f: C_1 \times C_1 \to R$ is said to be pseudo-univex with respect to

$\eta, b_0 \geq 0, \phi_0 > 0$ at u if $\forall\ x \in X$ such that

$$\eta^t(x, u)\, \nabla f(u) \geq 0 \Rightarrow b_0(x, u)\, \phi_0 [f(x) - f(u)] \geq 0 \qquad \text{... (12)}$$

for all $x, u \in C_1$.

Definition 2.3 — Let $\eta : C_1 \times C_1 \to C_1$. A differentiable function $f: C_1 \times C_1 \to R$ is said to be quasi-univex with respect to η and C_1 and also $b_0 \geq 0, \phi_0 > 0$ at $u \in C_1$ if $\forall\ x \in C_1$ such that

$$b_0(x, u)\, \phi_0 [f(x) - f(u)] \leq 0 \;\Rightarrow\; \eta^t(x, u)\, \nabla f(u) \leq 0 \qquad \text{... (13)}$$

for all $x, u \in C_1$.

3. Duality Theorems

Theorem 3.1 (*Weak duality for D1*) — *Let x be feasible for (P) and (u, y) be feasible solution for (D1). If for each $y \in C_2, u \in C_1$ the function $f + y^t g$ is pseudo-univex with respect to η on C_1 (that is equation of type (12) is satisfied) with $b_0 \geq 0, \phi_0 > 0$, then if $\phi_0(a) \geq 0 \Rightarrow a \geq 0$, then*

Inf (P) $\geq$ sup (D1).

PROOF : From the dual constraint, we have

$$-[\nabla f(u) + \nabla y^t g(t)] \in C_1^*$$

Also, it is given that $\eta(x, u) \in C_1$ with $b_0 \geq 0, \phi_0 > 0$, for all $x, u \in C_1$, we have

$$-\eta^t(x, u)[\nabla f(u) + \nabla y^t g(u)] \leq 0$$

$$\Rightarrow \eta^t(x, u)[\nabla f(u) + \nabla y^t g(u)] \geq 0 \quad \ldots (14)$$

Further, since it is given that $f(u) + y^t g(u)$ is pseudo-univex with respect to $\eta(x, u)$ on C_1, by the definition of pseudo-univexity we have

$$b_0(x, u)\, \phi_0\, [f(x) + y^t g(u) - f(u) - y^t g(u)] \geq 0 \quad \ldots (15)$$

As $u \in C_1$, because of (6)

$$-u^t[\nabla f(u) + \nabla^t g(u)] \leq 0 \quad \ldots (16)$$

we get $$f(x) + y^t g(x) \geq f(u) + y^t g(u). \quad \ldots (17)$$

But since $y^t g(x) \leq 0$, we have from (17)

$$f(x) \geq f(u) + y^t g(u). \quad \ldots (18)$$

From the inequalities (16) and (17), we have

$$f(x) \geq f(u) + y^t g(u) - u^t[\nabla f(u) + y^t g(u)].$$

Thus, inf $(P) \geq$ sup $(D1)$ which completes the proof.

Theorem 3.2 (*Weak duality for D2*) — *Let f be pseudo-univex with respect to η on C_1 and also with $b_0 \geq 0, \phi_0 > 0$ and for each $y \in C_2$, the function y^t g is pseudo-univex with respect to different $\phi_0 > 0$ and $b_0 \geq 0$. If* x *and (u, y) are feasible for (P) and $(D2)$ are respectively, then inf $(P) \geq$ sup $(D2)$, provided ϕ_0 and ϕ_1 satisfy (i) $\phi_0(a) \geq 0 \Rightarrow a \geq 0$ and (ii) $a \geq 0 \Rightarrow \phi_1(a) \geq 0$.*

PROOF : We now that the primal and dual constraints $g(x)$ and $-g(u)$ belong to C_2^*, we have

$$y^t g(x) \leq 0 \text{ and } -y^t g(x) \leq 0.$$

The above two inequalities imply because of (*ii*),

$$b_1(x, u)\, \phi_1\, [y^t g(x) - y^t g(u)] \leq 0.$$

Since $y^t g$ is quasi-convex with respect to η on C_1, we have for all $x, u \in C_1$,

$$\eta^t(x, u)\, \nabla y^t g(u) \leq 0. \quad \ldots (19)$$

Also, from the dual constraints, we have

$$-[\nabla f(u) + \nabla y^t g(u)] \in C_1^*.$$

Since $\eta(x, u) \in C_1$, we have

$$\Rightarrow \eta^t(x, u)[\nabla f(u) + \nabla y^t g(u)] \geq 0.$$

From (19) and the above inequality, we have

$$\eta^t(x, u)\ \nabla f(u) \geq 0.$$

Since f is pseudo-univex, we get

$$b_0(x, u)\ \phi_0[f(x) - f(u)] \geq 0,$$

which implies $f(x) - f(u) = 0$ because of (*i*), hence the proof is completed.

Theorem 3.3 (*Weak duality D3*) — *Let x be feasible for (P) and (u, y) be feasible for (D3). If f is pseudo-univex with respect to* η *on* C_1 *and also with* $b_0 \geq 0, \phi_0 > 0$ *and for each* $y \in C_2$, *the function* $y^t g$ *is quasi-unived with respect to the same* η *on* C_1 *but different* $\phi_1 > 0$ *then*

Inf (P) $\geq$ *sup (D3).*

When ϕ_0 *and* ϕ_1 *satisfy (i)* $\phi_0(a) \geq 0 \Rightarrow a \geq 0$, *(ii)* $a \geq 0, \Rightarrow \phi_1(a) \geq 0$.

PROOF : Proof can be given in a similar manner as above.

Theorem 3.4 (*Weak duality for D4*) — *Let x be feasible for (P) and (u, y) be feasible for (D4). If for each* $y \in C_2$, *the function* $f + y^t g$ *is pseudo-univex with respect to* η *on* C_1 *and* $b_0 \geq 0$, $\phi_0 > 0$, *then* $\phi_0(a) \geq 0 \Rightarrow a \geq 0$, *then*

Inf (P) $\geq$ *sup (D4).*

PROOF : From the dual constraints, we have

$$-[\nabla f(u) + \nabla y^t g(u)] \in C_1^*$$

For any $\eta(x, u) \in C_1$, we have

$$\eta^t(x, u)[\nabla f(u) + \nabla y^t g(u)] \geq 0$$

Since $f + y^t g$ is pseudo-univex with respect to η on C_1 and $b_0 \geq 0, \phi_0 > 0$, we have

$$b_0(x, y)\ \phi_0[f(x) + f(u)] + [y^t g(u) - y^t g(u)] \geq 0$$

$$\therefore\ f(x) + y^t g(x) \geq f(u) + y^t g(u). \qquad \text{... (20)}$$

But from the primal constraint $g(x) \in C_2^*$, we get for any $y \in C_2, y^t g(x) \leq 0$, hence $f(x) \geq f(u) + y^t g(u)$ which completes the proof.

Theorem 3.5 (*Strong Duality for D1*) — *If x is a local or global optimum of (P) at which a constraint qualification*[9] *is satisfied, then $\exists\, y \in C_2 \cdot (x, y)$ is feasible for (D1) and the corresponding values of (P) and (D1) are equal. If for each $y \in C_2$, the function $f + y^t g$ is pseudo-univex with respect to η on C_1 and also $b_0 \geq 0$, $\phi_0 > 0$, then x and (x, y) are global optimal solutions of (P) and (D), respectively.*

PROOF : Proof can follow easily from Theorem 3.5 of Kaul and Kaur[8] and weak duality Theorem 3.1.

Theorem 3.6 (*Strong duality for D2*) — *If x is a local or global optimum of (P) at which a constraint qualification is satisfied, then $\exists\, y \in C_2 \cdot (x, y)$ is feasible for (D2) and the corresponding values of (P) and (D2) are equal. If f is pseudo-univex with respect on η and also with $b_0 \geq \phi_0 > 0$, for each $y \in C_2$ the function $y^t g$ is quasi-univex with respect to same η on C_1 but different $\phi_0 > 0$ when $\phi_0 > 0$ when ϕ_0 and ϕ_1 satisfy (i) $\phi_0(a) \geq 0 \Rightarrow a \geq 0$, (ii) $a \geq 0 \Rightarrow \phi_0(a) \geq 0$, then x and (x, y) are global optimal solutions of (P) and (D2) respectively.*

PROOF : Suppose the following constraint qualification is satisfied at $x, \exists\, y \in C_2$, such that

$$\nabla f(x) + \nabla y^t g(x) = 0 \qquad \text{... (21)}$$

and

$$y^t g(x) = 0. \qquad \text{... (22)}$$

From the first equation, we obtain

$$-[\nabla f(x) + \nabla y^t g(x)] \in C_1^*,$$

which is also one of the constraints of (D2).

Since $x \in C_1$ we, have

$$-x^t [\nabla f(x) + \nabla y^t g(x)] \leq 0$$

$$\Rightarrow x^t [\nabla f(x) + \nabla y^t g(x)] \geq 0 \qquad \text{... (23)}$$

From the primal constraint $g(x) \in C_2^*$ we have

$$y^t g(x) \leq 0 \text{ for all } y \in C_2.$$

Eq. (10) gives

$$y^t g(x) - y^t g(x) \leq 0.$$

From condition (ii), we have

$$b1\,(x, x)\,\phi_1\,[y^t g(x) - y^t g(x)] \leq 0. \qquad \text{... (24)}$$

Since $y^t g(x)$ is quasi-univex with respect to and $b_1 \geq 0$, $\phi_1 > 0$, we have

$$\eta^t (x, x)\, \nabla y^t g(x) < 0. \qquad \text{... (25)}$$

From (18), for each $\eta\, C_1 \times C_1 \to C_1$ we have

$$\eta^t(x,\bar{x})\ \nabla f(x) + \eta\, y^t(x,\bar{x})\ \nabla y^t g(\bar{x}) \geq 0.$$

Using the inequality (25) in the above inequality, we get

$$\eta^t(x,\bar{x})\ \nabla f(\bar{x}) \geq 0.$$

But f is pseudo-univex on C_1 with respect to η and $b_0 \geq 0, \phi_0 > 0$, therefore, we have

$$b_0(x,x)\ \phi_0\,[f(x) - f(\bar{x})] \geq 0.$$

Thus because of condition (*i*) x is an optimal solution to the primal problem. By the Kuhn-tucker necessary optimality conditions of Hanson and Egudo[6] $\exists\, y \in C_2$ such that (x, y) is a solution to (D2). Let (u, y) be a feasible solution to (D2). Hence, $-g(u) \in C_2^*$ which means for $y \in C_2, y^t g(u) \leq 0$. But eq. (23) now gives

$$b_1(x,u)\ \phi_0\,[y^t g(x) - y^t g(u)] \leq 0.$$

Since $y^t g$ is quasi-univex, we have

$$\eta^t(x,u)\ \nabla y^t g(u) \leq 0. \qquad \text{... (26)}$$

But from dual constraint, we get

$$-[\nabla f(u) + \nabla y^t g(u)] \in C_1^*$$

$$\Rightarrow -x^t[\nabla f(u) + \nabla y^t g(u)] \leq 0, \text{ for all } x \in C_1^*$$

$$\Rightarrow -x^t[\nabla f(u) + \nabla y^t g(u)] \geq 0$$

$$\eta^t(x,u)\ \nabla f(u) + \eta^t(x,u)\ \nabla y^t g(u) \geq 0.$$

Using (26) is the above inequality, we get

$$\eta^t(x,u)\ \nabla f(u) \geq 0.$$

Since f is pseudo-univex with respect to $\eta^t(x,u)$, we have

$$b_0(x,\bar{x})\ \phi_0\,[f(x) - f(\bar{x})] \geq 0. \qquad \text{... (27)}$$

Thus x and (x, y) are the global optimal solution of (P) and $(D2)$ respectively. The equality holds as the objective functions of both problems are the same.

This completes the proof.

Theorem 3.7 (*Strong duality for D3*) — *If x is a local or global optimum of (P) at which a constraint qualification is satisfied, then $\exists\ y \in C_2$ such that (x, y) is feasible for (D3) and the* *corresponding values of (P) and (D3) are equal. If f is pseudo-univex for each $y \in C_2$ with respect to η and $b_0 \geq 0, \phi_0 > 0$ and also the function $y^t g$ is quasi-univex with respect to same η but different $b_0 \geq 0, \phi_0 > 0$, where ϕ_0 and ϕ_1 satisfy (i) $\phi_0(a) \geq 0 \Rightarrow a \geq 0$, (ii) $a \geq 0 \Rightarrow \phi_1(a) \geq 0$. Then x and (x, y) and global optimal solutions of (P) and (D3), respectively.*

PROOF : Proof can follow easily from the above.

Theorem 3.8 (*Strong duality for D4*) — *If x is a local or global optimum of (P) at which a constraint qualification is satisfied, then $\exists$ $y \in C_2$ such that (x, y) is feasible for (D4) and the corresponding values of (P) and (D4) are equal. If for each $y \in C_2$, the function $f + y^t g$ is pseudo-univex with respect to η on C_1 and $b_0 \geq 0, \phi_0 > 0$ then x and (x, y) are global optimal solutions of (P) and (D4), respectively.*

PROOF : Proof can easily begin on similar lines as in Theorem 3.5

Theorem 3.9 (*Converse duality for D1*) — *Let (x, y) be a local or global optimum of (D1). Also let $f + y^t g$ be pseudo-univex with respect to η on C_1 and $b_0 \geq 0, \phi_0 > 0$ then x and (x, y) are global optimal solutions of (P) and (D4), respectively.*

PROOF : To prove this theorem, we use generalized Fritz-John theorem of Mangasarian and Fromovitz[10], By this theorem, for $\alpha \in R, q \in C_1$ and $r \in C_1$, the following equations and inequalities hold :-

$$[\alpha[\nabla f(x) + \nabla y^t g(x)] + [\nabla^2 f(x) + \nabla^2 y^t g(x)]\, q] = 0. \quad \text{... (28)}$$

$$[\alpha\, g(x) + \nabla g(x)\, q] + r = 0. \quad \text{... (29)}$$

$$y^t g(x) = 0. \quad \text{... (30)}$$

$$r^t y = 0. \quad \text{... (31)}$$

$$(\alpha, q, r) \neq 0. \quad \text{... (32)}$$

$$(\alpha, q, r_0) \geq 0 \quad \text{... (33)}$$

Multiplying (26) by y^t and using (31), we get

$$[\alpha y^t g(x) + y^t g(x)\, q] = 0$$

$$\Rightarrow y^t g(x)\, q = y^t g(x)\, q = 0$$

$$\Rightarrow q = 0 \text{ as } \nabla y^t g(x) \neq 0 \quad \text{... (34)}$$

Now substituting this value in (28), we get

$$[\alpha[\nabla f(x) + \nabla y^t g(x)]] = 0 \Rightarrow \alpha \neq 0.$$

Again we prove this by contradiction.

Suppose $\alpha = 0$, then assuming $q = 0$ and substituting these values in (29), we have $r = 0$, which is contradiction to (29).

$\therefore\ \alpha > 0$ and (18) gives $g(x) \in C_2$.

Thus x is a feasible solution for (P). If $f + y^t g$ is pseudo-univex, then by weak duality Theorem 3.1, x is an optimal solution for (P), hence the proof is completed.

Theorem 3.10 *(Converse duality for D2) — Let (x, y) be a local or global optimum of (D2). If f is pseudo-univex with respect to η on C_1 and $b_0 \geq 0, \phi_0 > 0$ and $y^t\, g$ is quasi-univex with respect to same η on C_1 but with different $\phi_0 > 0$ where ϕ_0 and ϕ_1 must satisfy (i) ϕ_0 (a) $\geq 0 \Rightarrow a \geq 0$, (ii) $a \geq 0 \Rightarrow \phi_1$ (a) ≥ 0. If $\nabla y^t g(x) \neq 0$, then x is a global solutions for (P).*

Theorem 3.11 *(Converse duality for D3) — Let (x, y) be a local or global optimum of (D3). If f be pseudo-univex with respect to η on C_1 and $b_0 \geq 0, \phi_0 > 0$ and $y^t g$ is quasi-univex with respect to same η on C_1 and also $b_0 \geq 0, \phi_0 > 0$, where ϕ_0 and ϕ_1 must satisfy the conditions (i) ϕ_0 (a) $\geq 0 \Rightarrow a \geq 0$, (ii) $a \geq 0 \Rightarrow \phi_1$ (a) ≥ 0. If $\nabla y^t g(x) \neq 0$, then x is a global solutions for (P).*

Remark 3.1 : Proofs of Theorem 3.10 and 3.11 are similar to that given by Mond and Weir[11].

Remark 3.2 : If the constraint $[\nabla f(u) + \nabla y^t g(u)]$ lies on the boundary of the polar cone C_1^*, then the problems (D1) and (D2) are equal to (D4) and (D3), respectively.

Conclusion 3.1 — In the present paper, we have thus derived weak, strong and converse duality theorems pertaining to four dual problems which were earlier introduced by Nanda and Das[14] under more general invexity type of conditions. It is our strong feeling that under such conditions some of the older formulations symmetric duality can be given in a more general setting. We propose on to do this in a forthcoming work.

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