*Indian J. pure appl. Math.*, 32(5) : 727-738, May 2001


# OPTIMALITY CONDITIONS AND DUALITY FOR A P-CONNECTED MINIMAX PROGRAMMING PROBLEM

DAVINDER BHATIA* AND APARNA MEHRA**

**Department of Operational Research, Faculty of Mathematical Sciences, University of Delhi, Delhi* 110 007, *India*

***Department of Mathematics, Acharya Narendra Dev College, Gobindpuri, Kalkaji, New Delhi* 110 019, *India*

(*Received* 23 *July* 1999; *after revision* 7 *March* 2000; *accepted* 17 *October* 2000)

Necessary optimality conditions are derived for a general minimax programming problem involving arcwise connected functions, defined on an arcwise connected set, in terms of directional derivatives of the functions with respect to the same arc. Sufficient optimality conditions and duality results are obtained under generalized arcwise connectedness assumptions on the objective and constraint functions.

**Key Words : Minimax Programming; Optimality Conditions; Directional Derivative of Function with Respect to An Arc; P-Connectedness, Q-connectedness; Duality Relations**

## 1. INTRODUCTION

Replacing a line segment joining two points by a continuous vector-valued function, Ortega and Rheinboldt[1] expanded the class of convex functions to the class of arcwise connected functions, which has been further generalized to $Q$-connected (QCN) functions and $P$-connected (PCN) functions by Avriel and Zang[2]. They have also discussed necessary and sufficient local-global minimum properties of these functions. Some elementary properties of these functions have been studied by Singh[3] and Yadav and Mukherjee[4].

Bector and Chandra[5] proved various duality results for a general minimax problem involving pseudolinear functions. More general results have been given by Mond *et al.*[6], under different convexity assumptions on the functions involved. In this set up, Weir and Mond[7] presented sufficient optimality conditions and duality theorems for a general minimax problem involving pseudoconvex functions. Using the notion of V-invexity, Bector *et al*[8] studied optimality conditions and duality results for a class of minimax programming problem. They have also applied these results to certain fractional and generalized fractional programming problems. Relaxing the differentiability conditions on the functions involved in a general minimax programming problem, Bhatia and Jain[9] obtained optimality conditions and duality results in terms of classical Dini derivatives and extended these results to generalized minimax fractional programming problem.

In the present paper, we have obtained the necessary optimality conditions, for a general minimax programming problem, in terms of directional derivatives of the functions involved with respect to the same arc. Sufficiency of these conditions have been proved under $P$-connectedness assumption on the objective functions. Duality results are established for Mond-Weir type dual to the minimax programming problem. Finally, the corresponding results for a generalized fractional minimax programming problem are presented.

## 2. Preliminaries

In this section, we provide some definitions and results that are used in the sequel. The following definitions are due to Avriel and Zang[2].

*Definition* 2.1 — The set $X \subset R^n$ is said to be an arcwise connected (AC) set if for every pair of points $x^1, x^2 \in X$, there exists a continuous vector-valued function $H_{x^1,x^2}$, called an arc, defined on the unit interval $[0, 1] \subset R$ and with values in $X$ such that

$$H_{x^1,x^2}(0) = x^1, \; H_{x^1,x^2}(1) = x^2.$$

*Definition* 2.2 — A real-valued function $f$, defined on an AC set $X \subset R^n$ is said to be arcwise connected (CN) function if for every pair of points $x^1, x^2 \in X$, there exists an arc $H_{x^1,x^2}$ in $X$ such that

$$f(H_{x^1,x^2}(\theta)) \leqq (1-\theta) f(x^1) + \theta f(x^2), \; \forall \; \theta \in [0, 1].$$

*Definition* 2.3 — A real-valued function $f$ defined on an AC set $X \subset R^n$ is said to be $Q$-connected (QCN) function if for every $x^1, x^2 \in X$ with $f(x^2) \leqq f(x^1)$ there exists an arc $H_{x^1,x^2}$ in $X$ such that $f(H_{x^1,x^2}(\theta)) \leqq f(x^1), \; \forall \; \theta \in [0, 1]$.

*Definition* 2.4 — A real-valued function $f$ defined on an AC set $X \subset R^n$ is said to be $P$-connected (PCN) function if for every $x^1, x^2 \in X$ with $f(x^2) < f(x^1)$ there exist an arc $H_{x^1,x^2}$• in $X$ and a positive number $b\eta_{x^1,x^2}$ satisfying

$$f(H_{x^1,x^2}(\theta)) \leqq f(x^1) - \theta \beta_{x^1,x^2}, \; \forall \; \theta \in (0, 1).$$

If the above inequality is satisfied whenever $x^1 \neq x^2$ and $f(x^2) \leqq f(x^1)$ then $f$ is said to be strictly $P$-connected (STPCN) function.

*Definition* 2.5 — Let $X \subset R^n$ be an AC set and $f$ be a real-valued function defined on $X$. For each $x^1, x^2 \in X$, the right differential of $f$ with respect to the arc $H_{x^1,x^2}$, in $X$ at $th\eta = 0$ is defined as

$$\lim_{\theta \to 0_+} \frac{f(H_{x^1,x^2}(\theta)) - f(x^1)}{\theta}$$

provided the limit exists. This limit is denoted by $f^+(H_{x^1,x^2}(0))$ and is also called directional derivative of $f$ with respect to an arc $H_{x^1,x^2}$ at $\theta = 0$.

The following result can easily be proved on the lines of Mangasarian[10] and hence will be taken as definitions of CN, QCN, PCN and STPCN functions possessing directional derivatives with respect to an arc at $\theta = 0$.

*Lemma* 2.1 — Let $f$ be a real-valued function defined on an AC set $X \subset R^n$. Let for any $x^1, x^2 \in X$, $H_{x^1,x^2}$ be the corresponding arc connecting $x^1$ and $x^2$ in $X$. Suppose that the directional derivative of $f$ with respect to $H_{x^1,x^2}$ at $\theta = 0$ exists

(*a*) If $f$ is CN function then

$$f(x^2) - f(x^1) \geqq f^+(H_{x^1,x^2}(0)).$$

(*b*) If $f$ is QCN function then

$$f(x^2) \leqq f(x^1) \Rightarrow f^+(H_{x^1,x^2}(0)) \leqq 0.$$

(*c*) If $f$ is PCN function then

$$f(x^2) < f(x^1) \Rightarrow f^+(H_{x^1,x^2}(0)) < 0.$$

(*d*) If $f$ is STPCN function then

$$x^2 \neq x^1 \text{ and } f(x^2) \leqq f(x^1) \Rightarrow f^+(H_{x^1,x^2}(0)) < 0.$$

We are motivated to carry out this study because there exist functions that satisfy the definition (*a*) of Lemma 2.1 but are nondifferentiable and nonconvex as can be seen by the following non-trivial example.

*Example* 2.1*d* — Let $X = R^2$ and $f: X \rightarrow R$ be defined as

$$f(x) = f(x_1, x_2) = \begin{cases} x_2 - (x_1)^3, & \text{if } x_1 > 1 \text{ and } x_2 > 1 \\ 0, & \text{otherwise} \end{cases}$$

For $x^1 = (3,15)$, $x^2 = (2, 2)$ and $\theta = 1/2$, we have

$$f((1-\theta)x^1 + \theta x^2) > (1-\theta) f(x^1) + \theta f(x^2)$$

Thus, $f$ is not convex on $X$. Moreover, $f$ is neither quasiconvex nor pseudoconvex on $X$. However, $f$ is CN function on $X$ with respect to the arc

$$H_{x^1,x^2}(\theta) = \left( (1-\theta)x_1^1 + \theta x_1^2, (1-\theta)(x_2^1 - (x_1^1)^3 + \theta(x_2^2 - (x_1^2)^3) + ((1-\theta)x_1^1 + \theta x_1^2)^3) \right)$$

for $x^1 = (x_1^1, x_2^1) \in X, x^2 = (x_1^2, x_2^2) \in X$ and $\theta \in [0, 1]$.

The function $f$ is not differentiable at $x^* = (1, 1) \in X$, but for any $x = (x_1, x_2) \in X$, $f^+(H_{x^*,x}(0))$ exists and is given by

$$f^{+}(H_{x^*,x}(0)) = \begin{cases} x_2 - (x_1)^3, & \text{if both components of } H_{x^*,x}(\theta) \text{ are strictly greater than unity} \\ 0, & \text{otherwise} \end{cases}$$

and it satisfies

$$f(x) - f(x^*) \geqq f^{+}(H_{x^*,x}(0)).$$

Consider the following scalar-valued optimization problem :

(MP) Minimize $f(x)$

subject to $g(x) \leqq 0$,

$x \in X$.

where $f$ is a real-valued function and $g$ is an m-dimensional vector-valued function defined on an AC set $X \subset R^p$. Also for every $x^1, x^2 \in X$, $f$ and $g$ possess directional derivative with respect to the same arc $H_{x^1,x^2}$, consisting $x^1$ and $x^2$ in $X$, at $\theta = 0$.

The following theorem and its subsequent corollary giving necessary optimality conditions for (M) have been proved by Bhatia and Mehra[11].

***Theorem* 2.1** — *Let (MP) attain a minimum at $x = x^* \in X$, and let $f$ and $g$ be CN functions with respect to the same arc on X. Then there exist $\lambda_0^* \in R$ and $\lambda^* \in R^m$ such that $(x^*, \lambda_0^*, \lambda^*)$ satisfies*

$$\lambda_0^* f^{+}(H_{x^*,x}(0)) + \lambda^{*t} g^{+}(H_{x^*x}(0)) \geqq 0, \ \forall \ x \in X, \qquad \text{... (2.1)}$$

$$\lambda^{*t} g(x^*) = 0, \qquad \text{... (2.2)}$$

$$g(x^*) \leqq 0 \qquad \text{... (2.3)}$$

and
$$(\lambda_0^*, \lambda^*) \geq 0. \qquad \text{... (2.4)}$$

*Corollary* 2.1 — If in Theorem 2.1, we further assume that for some $\hat{x} \in X$, $g(\hat{x}) < 0$, then there exist $\lambda_0^* = 1$, $\lambda^* \in R^m$ such that $(x^*, \lambda_0^*, \lambda^*)$ satisfy conditions (2.1) to (2.4).

## 3. Optimality Conditions for Minimax Problem

Consider the following general minimax programming problem :

$$(P) \ \underset{x \in X}{\text{Minimize}} \ \underset{1 \leqq i \leqq n}{\text{Maximize}} \ (f_i(x))$$

subject to $g_j(x) \leqq 0, j = 1, 2, \ldots, m,$

where it is assumed that —

(H1) $X$ is an AC compact subset of $R^p$.

(H2) $f_i, i \in n = \{1, \ldots, n\}$, $g_j, j \in m = \{1, \ldots, m\}$ are realvalued functions defined on $X$.

(H3) For every $x^1, x^2 \in X, f_i, i \in n, g_j, j \in m$ possess directional derivative with respect to the same arc $H_{x^1, x^2}$, connecting $x^1$ and $x^2$ in $X$, at $\theta = 0$.

(H4) $f_i, i \in n, g_j, j \in m$ are CN functions with respect to the same arc $H_{x^1, x^2}$ on $X$ and there exists $\hat{x} \in X$ such that $g_j(\hat{x}) < 0, j \in m$.

If a general minimax problem (P) reaches a finite minimax then it can be equivalently expressed as

(EP) Minimize $q$

subject to $f_i(x) \leqq q, i \in n$

$g_j(x) \leqq 0, j \in m$

$q \in R, x \in X$

and (H1) - (H4) of (P) holds for (EP), where the equivalence of (P) and (EP) is in the sense of the following Lemmas, see Mond *et al.*[6].

*Lemma* 3.1 — (1) If $(x, q)$ is feasible for (EP) then $x$ is feasible for (P).

(2) If $x$ is feasible for (P) then there exist $q \in R$ such that $(x, q)$ is feasible for (EP).

*Lemma* 3.2 — $x^*$ is optimal for (P) with the corresponding optimal value of the (P) objective equal to $q^*$ if and only if $(x^*, q^*)$ is optimal for (EP) with the corresponding optimal value of (EP) objective equal to $q^*$.

The problem (EP) can be rewritten as follows

Minimize $\tau(x, q)$

subject to $\xi_i(x, q) \leqq 0, i \in n$

$z\eta_j(x, q) \leqq 0, j \in m,$

where $\tau(x, q) = q;\ \xi_i(x, q) = f_i(x) - q, i \in n;\ z\eta_j(x, q) = g_j(x), j \in m.$

*Remark* 3.1 : In view of the assumption (H4), the functions $\tau(\cdot, \cdot)$, $\xi_i(\cdot, \cdot), i \in n$, and $z\eta_j(\cdot, \cdot), j \in m$ are CN functions with respect to the arc $(H_{x^1, x^2}(\theta), (1-\theta) q^1 + \theta q^2)$ on $X \times R$, for every $x^1, x^2 \in X, q^1, q^2 \in R$ and $\theta \in [0, 1]$.

***Theorem*** 3.1 — *(Necessary Optimality Conditions) Let $x^* \in X$ be optimal for (P) with the corresponding optimal value $q^*$. Then there exist $\lambda^* \in R^n, \mu^* \in R^m$ such that*

$$\lambda^{*'} f^+(H_{x^*, x}(0)) + \mu^{*\,t} g^+(H_{x^*, x}(0)) \geqq 0,\ \forall\ x \in X \quad \ldots (3.1)$$

$$\lambda^{*t}(f(x^*) - q^* e) = 0, \qquad \dots (3.2)$$

$$\mu^{*t} g(x^*) = 0, \qquad \dots (3.3)$$

$$\lambda^* \geqq 0, \lambda^{*t} e = 1, \mu^* \geqq 0, \qquad \dots (3.4)$$

where $e = (1, \dots, 1) \in R^n$.

Proof follows from Theorem 2.1 and Corollary 2.1.

We now establish the sufficiency of these conditions under $P$-connectedness and $Q$-connectedness assumptions on the functions involved.

***Theorem* 3.2** — *Let $(x^*, q^*, \lambda^*, \mu^*)$ with $x^* \in X, q^* \in R, \lambda^* \in R^n$, $\mu^* \in R^m$ satisfy relations (3.1)-(3.4). Let for each $x \in X$, $\lambda^{*t} f$ be PCN and $\mu^{*t} g$ be QCN, with respect to the same arc $H_{x^*,x}$ in X, at $x^*$. Then $x^*$ is an optimal solution of (P) with corresponding objective value equal to $q^*$.*

PROOF : Suppose $(x^*, q^*)$ is not optimal for (EP) then there exists $(x, q)$ feasible for (EP) with $x \neq x^*$, and

$$q < q^*.$$

Feasibility of $(x, q)$ for (EP), implies

$$f(x) \leqq qe < q^* e.$$

Since $\lambda^* \geq 0$ hence

$$\lambda^{*t} f(x) < \lambda^{*t} q^* e,$$

which on using (3.2) implies

$$\lambda^{*t} f(x) < \lambda^{*t} f(x^*).$$

The above inequality along with the fact that $\lambda^{*t} f$ is PCN at $x^*$ yields

$$(\lambda^{*t} f)^+ (H_{x^*,x}(0)) < 0$$

that is,

$$\lambda^{*t} f^+ (H_{x^*,x}(0)) < 0.$$

It then follows from (3.1) that

$$\mu^{*t} g^+ (H_{x^*,x}(0)) > 0. \qquad \dots (3.5)$$

Now by feasibility of $x$ for (EP), (3.3) and (3.4), we have

$$\mu^{*t} g(x) \leqq 0 = \mu^{*t} g(x^*).$$

Q-connectedness of $\mu^{*t} g$ at $x^*$ gives

$$\mu^{*t} g^+ (H_{x^*, x} (0)) \leqq 0$$

which contradicts (3.5). Thus $(x^*, q^*)$ is optimal for (EP), and hence $x^*$ is optimal for (P) with corresponding objective value equal to $q^*$.

## 4. Duality Results

We now consider the following program

$$\text{(D) Maximize}_{v \in R} \quad v$$

subject to

$$\lambda^t f^+ (H_{u,x} (0)) + \mu^t g^+ (H_{u,x} (0)) \geqq 0, \ \forall \ x \in X \qquad \text{... (4.1)}$$

$$\lambda^t (f(u) - ve) \geqq 0, \qquad \text{... (4.2)}$$

$$\mu^t g(u) \geqq 0, \qquad \text{... (4.3)}$$

$$\lambda \in R^n, \lambda \geq 0, \lambda^t e = 1 \qquad \text{... (4.4)}$$

and $$\mu \in R^m, \mu \geqq 0. \qquad \text{... (4.5)}$$

Under suitable arcwise connectedness assumptions on the functions, we now establish duality theorems relating (P) and (D).

***Theorem*** 4.1 — *(Weak Duality) Let $x$ be feasible for (P) with the corresponding objective value equal to $q$ and let $(u, v, \lambda, \mu)$ be feasible for (D). If $\lambda^t f$ is PCN and $\mu^t g$ is QCN with respect to the same arc $H_{u,x}$ on the sets of feasible solutions of (P) and (D), then $q \geqq v$.*

Proof : Suppose to the contrary,

$$q < v. \qquad \text{... (4.6)}$$

By feasibility of $(x, q)$ for (EP), we have

$$f(x) \leqq qe. \qquad \text{... (4.7)}$$

(4.6) and (4.7) together with $\lambda \geq 0$ imply

$$\lambda^t f(x) < \lambda^t ve.$$

The above inequality together with (4.2) yields

$$\lambda^t f(x) < \lambda^t f(u).$$

P-connectedness of $\lambda^t f$ gives

$$\lambda^t f^+ (H_{u,x}(0)) < 0.$$

which along with (4.1) implies

$$\mu^t g^+ (H_{u,x}(0)) > 0. \qquad \dots (4.8)$$

Now feasibility of $x$ for (P) and conditions (4.5) and (4.3) yields

$$\mu^t g(x) \leqq \mu^t g(u).$$

The above inequality along with the fact that $\mu^t g$ is QCN gives,

$$\mu^t g^+ (H_{u,x}(0)) \leqq 0$$

which contradicts (4.8). Hence the result.

***Theorem*** 4.2 — *(Strong Duality) Let $x^*$ be an optimal solution for (P). Then there exist $q^* \in R$, $\lambda^* \in R^n$, $\mu^* \in R^m$ such that $(x^*, q^*, \lambda^*, \mu^*)$ is feasible for (D) and the objective values of (P) and (D) are equal. If also, the hypotheses of Theorem 4.1 hold then $(x^*, q^*, \lambda^*, \mu^*)$ is optimal for (D).*

PROOF : Since $x^*$ is optimal for (P), it follows fromn Lemma 3.2 that there exist $q^* \in R$ such that $(x^*, q^*)$ is optimal for (EP). Hence by Theorem 3.1 there exist $(\lambda^*, \mu^*) \in R^{n+m}$ such that conditions (3.1)-(3.4) are satisfied. Hence $(x^*, q^*, \lambda^*, \mu^*)$ is feasible for (D) with $v = q^*$, and values of $(P)$ objective and (D)-objective are equal. Hence $(x^*, q^*, \lambda^*, \mu^*)$ is optimal for $(D)$.

***Theorem*** 4.3 — *(Strict Converse Duality) Let $x^*$ be optimal for (P) with the corresponding objective value equal to $q^*$ and $(u^*, v^*, \lambda^*, \mu^*)$ be optimal for (D). Let $\lambda^{*t} f$ be STPCN and $\mu^{*t} g$ be QCN with respect to the same arc $H_{u^*, x^*}$ on the sets of feasible solutions of (P) and (D), then $(u^*, v^*) = (x^*, q^*)$.*

PROOF : We assume that $(u^*, v^*) \neq (x^*, q^*)$ and exhibit a contradiction.

Since $x^*$ is optimal for (P) with the corresponding objective value equal to $q^*$ hence in view of Lemma 3.1, $(x^*, q^*)$ is optimal for (EP). Therefore, by Theorem 3.1 there exist $\lambda^* \in R^n$, $\mu^* \in R^m$ such that $(x^*, q^*, \lambda^*, \mu^*)$ is feasible for (D) with objective values equal. Hence $(x^*, q^*, \lambda^*, \mu^*)$ is optimal for (D).

Also, $\mu^{*t} g(x^*) \leqq 0$ and $\mu^{*t} g(u^*) \geqq 0$, therefore,

$$\mu^{*t} g(x^*) \leqq \mu^{*t} g(u^*).$$

$Q$-connectedness of $\mu^{*t} g$ implies

$$\mu^{*t} g^+ (H_{u^*, x^*}(0)) \leqq 0. \qquad \text{... (4.9)}$$

(4.9) together with (4.1) yields

$$\lambda^{*t} f^+ (H_{u^*, x^*}(0)) \geqq 0.$$

The above inequality along with the strict P-connectedness of $\lambda^{*t} f$, with respect to the same arc, implies

$$\lambda^{*t} f(x^*) > \lambda^{*t} f(u^*). \qquad \text{... (4.10)}$$

Further $(x^*, q^*)$ is feasible for (EP), therefore,

$$f(x^*) \leqq q^* e.$$

As $\lambda^* \geq 0$, hence,

$$\lambda^{*t} f(x^*) \leqq \lambda^{*t} q^* e. \qquad \text{... (4.11)}$$

Also, $(u^*, v^*, \lambda^*, \mu^*)$ is feasible for $(D)$ hence

$$\lambda^{*t} f(u^*) \geqq \lambda^{*t} v^* e. \qquad \text{... (4.12)}$$

It follows from (4.10), (4.11) and (4.12) that

$$\lambda^{*t} q^* e > \lambda^{*t} v^* e \qquad \text{... (4.13)}$$

As $\lambda^{*t} e = 1$, hence from (4.13), we have

$$q^* > v^*$$

which is a contradiction to the fact that $(u^*, v^*, \lambda^*, \mu^*)$ is optimal for $(D)$. Hence, $(u^*, v^*) = (x^*, q^*)$.

## 5. Generalized Fractional Minimax Programming Problem

In this section, we discuss the results obtained in Sections 3 and 4 for a generalized fractional minimax programming problem

$$\text{(GFP)} \ \eta^* = \underset{x \in X}{\text{Minimize}} \ \underset{1 \leqq i \leqq n}{\text{Maximize}} \left[ \frac{f_i(x)}{h_i(x)} \right]$$

subject to $g_j(x) \leqq 0, j \in m,$

$$x \in X,$$

where

(A1) $X$ is an AC compact set in $R^p$.

(A2) $f_i, h_i, i \in n = \{1, ..., n\}$, $g_j, j \in m = \{1, ..., m\}$ are real-valued functions.

(A3) $f_i(x) \geqq 0, h_i(x) > 0, i \in n$.

(A4) For every $x^1, x^2 \in X, f_i, h_i, i \in n, g_j, j \in m$ possess directional derivatives with respect to the same arc $H_{x^1, x^2}$ in X at $\theta = 0$.

(A5) $f_i, -h_i, i \in n, g_j, j \in m$ are CN functions with respect to the same arc.

(A6) There exists $\hat{x} \in X$ such that $g_j(\hat{x}) < 0, j \in m$.

The following Lemma relates the generalized fractional minimax programming problem (GFP) with a minimax nonlinear parametric programming problem $(P_\eta)$ where

$$(p_\eta) \quad \phi(\eta) = \underset{x \in X}{\text{Minimize}} \ \underset{1 \leqq i \leqq n}{\text{Maximize}} \ (f_i(x) - \eta h_i(x))$$

subject to $g_j(x) \leqq 0, j, \ m \ x \in X$.

*Lemma* 5.1 — (Crouzeix *et al.*[12]; Bector *et al.*[13]) - If (GFP) has an optimal solution $x^*$ with an optimal value of (GFP) objective equal to $\eta^*$ then $\phi(\eta^*) = 0$. Conversely, if $\phi(\eta^*) = 0$ for some $\eta^* \in R$ then (GFP) and $(P_{\eta^*})$ have same optimal solution set.

In relation to $(P_\eta)$ we have an equivalent programming problem, for a given $\eta$ ·

$(E(P_\eta))$ Minimize $q$

subject to $f(x) - \eta h(x) \leqq qe$,

$$g(x) \leqq 0,$$

$$x \in X, q \in R,$$

where $e = (1, ..., 1) \in R^n$, and $f = (f_1, \ldots, f_n), h = (h_1, \ldots, h_n), \ g = (g_1, \ldots, g_m)$.

Mond-Weir type dual to $(E(P_\eta))$ (or equivalently to $(P_\eta)$) is given by

$(D_\eta)$ Maximize $\lambda^T (f(u) - \eta h(u))$

subject to

$$\lambda^t f^+(H_{u,x}(0)) - \lambda^t \eta h^+(H_{u,x}(0)) + \mu^t g^+(H_{u,x}(0)) \geqq 0, \ \forall \ x \in X,$$

$$\mu^t g(u) \geqq 0,$$

$u \in X, \lambda \in R^n, \lambda \geq 0, \lambda^t e = 1,$

$\mu \in R^m, \mu \geqq 0, \eta \in R.$

Now the results obtained in Sections 3 and 4 can easily be generalized for fractional minimax problem (GFP).

## 6. Concluding Remarks

The results proved in the present paper require the functions involved to possess only directional derivatives with respect to the same arc, the functions themselves need not be differentiable Moreover the arc $H_{x^1,x^2}$, connecting $x^1$ and $x^2$ in $X$ need not be subdifferentiable i.e,

$$H^{+}_{x^1,x^2}(0) = \lim_{th\eta \to 0^+} \frac{H_{x^1,x^2}(th\eta) - H_{x^1,x^2}(0)}{th\eta}$$

need not exist. For if the functions $f$ and $g$ involved in (P), considered in Section 3 of this paper, are differentiable arc $H_{x^1,x^2}$, then

$$f^{+}(H_{x^1,x^2}(0)) = [H^{+}_{x^1,x^2}(0)]^t \nabla f(x^1)$$

and

$$g^{+}(H_{x^1,x^2}(0)) = [H^{+}_{x^1,x^2}(0)]^t \nabla g(x^1).$$

Therefore, $f$ and $g$ are invex functions on $X$ with respect to same $\eta(x^1, x^2)$ given by

$$\eta(x^1, x^2) = H^{+}_{x^1,x^2}(0)$$

and hence the optimality conditions and duality results studied by Bector *et al.*[8] follows.

## Acknowledgement

The authors are indebted to Professor (Retd.) R.N. Kaul, Department of Mathematics, University of Delhi, for his constant encouragement in preparing this paper. The authors are also thankful to the referees for their valuable suggestions.

## References

1. J. M. Ortega and W. C. Rheinboldt, *Iterative Solutions of Nonlinear Equations in Several Variables*, Academic Press, New York (1970).
2. M. Avriel and I. Zang, *J. Optimization Theory Appl.* **32** (1980), 403-25.
3. C. Singh, *J. Optimization Theory Appl.* **41** (1983) 377-87.
4. S. R. Yadav and R. N. Mukherjee, *Bull. Austral. math. Soc.* **31** (1985) 369-75.
5. C. R. Bector and S. Chandra, *Asia-pacific J. OR* **3** (1986) 86-94.
6. B. Mond, T. Weir and B. D. Craven, *Asia-Pacific J. OR* **7** (1990) 55-62.

7. T. Weir, and B. Mond, *Cashiers du CERO*, **33** (1991) 123-28.
8. C. R. Bector, S. Chandra and V. Kumar, *Optimization*, **30** (1994), 93-103.
9. D. Bhatia and P. Jain, *Optimization*, **35** (1995), 207-14.
10. O. L. Mangasarian, *Nonlinear Programming*, McGraw Hill, New York (1969).
11. D. Bhatia and A. Mehra, *J. Optimization Theory Appl.* **100** (1999) 181-94.
12. J. P. Crouzeix, J. A. Ferland and S. Schaible, *J. Optimization Theory Appl.* **47** (1985) 35-49.
13. C. R. Bector, S. Chandra and M. K. Bector, *J. Optimization Theor. Appl.* **60** (1989), 243-60.